Dissertation Proposal

Dabeen Lee

"Structure and Complexity in Integer Programming"

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Integer Linear Programming is a general framework for modeling a wide range of practical problems in operations research. Software for solving integer linear programming has been developed and improved over the last decades, and the current state-of-the-art solvers work well in practice. Mathematically, integer linear programming is defined as the problem of optimizing a linear function over the set of integer points contained in a rational polyhedron, which is described by a system of linear inequalities. Despite its success in practice, integer linear programming is a class of NP-hard problems, so it is believed that we cannot find a polynomial-time algorithm for solving the problem in general. However, the question about which structural properties of an integer linear programming instance determine its computational complexity still remains widely open. In this dissertation, we explore how certain structures affect the complexity of integer linear programming.

An important idea in integer programming is the concept of *Chvatal rank*, which was introduced by Chvatal in 1973. It is believed that integer programs with high Chvatal rank are often harder to solve. One of the directions that we investigate in this dissertation is to study conditions under which one can guarantee that the Chvatal rank is small.

Many combinatorial optimization problems can be formulated as binary linear programs. One of the simplest inequalities for such problems are the *generalized set covering inequalities*. Another direction of our work in this dissertation is exploring when generalized set covering inequalities are sufficient to describe the convex hull of integer feasible solutions. This question also motivated us to study when a given system of *set covering inequalities* and non-negativity constraints which appear in an integer programming formulation for the set covering problem defines an integral polyhedron.

Integer programming problems that arise in practice often involve nonnegative or bounded decision variables. Using information about bounds on variables, one can generate, possibly, stronger cuts valid for all integer feasible solutions. Our last direction is to investigate a generalization of Chvatal closure for integer programs where bounds on some variables are present.

In the first chapter, we study integer linear programming over a rational polytope with Chvatal rank 1. It is known that the problem belongs to complexity class NP intersect co-NP, an indication that it is probably not NP-complete, so we seek to find an efficient algorithm to solve it. We prove that the integer width of a compact convex set with empty Chvatal closure is at most the ambient dimension, improving the flatness theorem in this special case. This leads to a simpler Lenstra-type algorithm for the problem. However, providing a simple characterization of the polytopes with Chvatal rank 1 is indeed a difficult task. We prove that even when a polytope is contained in the unit hypercube, it is NP-hard to decide whether the polytope has Chvatal rank 1.

In the second chapter, we look at a polytope contained in the unit hypercube and provide some sufficient conditions for it to have small Chvatal rank. The convex hull of a set of binary integer points is obtained by applying some inequalities that cut off the other binary integer points. Hence, it is natural to think that the structure of the set of binary points not in the polytope determines the structure of its integer hull. We analyze the skeleton graph of the hypercube and its subgraph induced by the set of infeasible binary points, and we give some conditions under which one can guarantee that the Chvatal rank is small, in terms of the structure of the subgraph.

In the third chapter, we consider generalized set covering inequalities for binary integer linear programs and try to understand when they are sufficient to describe the integer hull of a polytope contained in the unit hypercube. We call a set S of binary integer points *q-ideal* if its convex hull can be described by generalized set covering inequalities. In this chapter, we refrain from analyzing the structure of the subgraph of the skeleton graph induced by S. We rather investigate the *cuboid* of S, a clutter whose members correspond to the points in S, and it turns out that working with cuboids has many advantages in studying q-idealness. We start by showing that a set of binary points is q-ideal if and only if its cuboid is an ideal clutter. We also show that q-idealness is a local property, which means local structures of a set of binary points determine whether the set is q-ideal. Providing a complete characterization of q-ideal sets might be difficult, but we study *resistant sets*, a special class of q-ideal sets. We study their structures and give its polynomial time recognition algorithm, based on its equivalence to cuboid idealness and the locality of q-idealness.

In the fourth chapter, we study ideal clutters. A clutter is ideal if the corresponding set covering polyhedron is integral, so understanding ideal clutters enlightens another perspective to analyze integrality of a polyhedron with simple inequalities. Besides, as we mentioned earlier, understanding q-idealness is equivalent to understanding idealness. In this chapter, we look at some special classes of ideal clutters, clutters with the *MFMC property* and the clutters with the *packing property*. We say that a clutter has the MFMC property if its corresponding set covering polyhedron is totally dual integral. We say that a clutter has the packing property if its corresponding set covering polyhedron is dual integral for all objective coefficients whose values are among 0, 1, and infinity. It is known that the MFMC property implies the packing property and the packing property implies idealness. It is conjectured that the MFMC property is in fact equivalent to the packing property. Although it is still open, we provide some tools and ideas of how to prove or disprove this conjecture.

In the last chapter, we discuss a generalization of Chvatal closure. A Chvatal-Gomory inequality is obtained by rounding down the right and side value of a valid inequality (when it is of cx<=d form). In fact, it is the *intersection* obtained from a lattice-free strip (or split set), a convex set containing no integer point in its interior. In many integer programming instances that arise from practice, we have some bounds on variables including non-negativity constraints and binary conditions. Then the set of integer points satisfying some conditions on bounds is strictly contained in the set of all lattice points, and this leads to a stronger intersection cut from a wider strip. In other words, we can decrease the right hand side value of a valid inequality more, to obtain a stronger inequality, if we use some information about the bounds on variables. We prove that the closure of a rational polyhedron obtained by applying all the generalized Chvatal-Gomory inequalities is also polyhedral, meaning that there are only finitely many non-redundant inequalities defining the closure.