Tax Design for the Long Run*

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Abstract

Costs of adjustment delay and complicate behavioral response to tax change. To accommodate such response, we integrate a dynamic discrete choice framework into optimal tax theory. We identify long run outcomes with stationary distributions of workers over income-generating states and formulate optimal tax equations in terms of the sensitivity of such distributions to consumption variation. We obtain formulas for these sensitivities that facilitate quantitative evaluation of long run substitution patterns. Novel "inverted" optimal tax equations are derived that establish marginal costs of inducing long run population movements to states as sufficient statistics for optimal taxes. The optimal tax implications of a dynamic quantitative model of occupational choice are analyzed. **JEL Codes:** H21, H24, H31. **Key Words:** Optimal Taxation, Occupational Choice

1 Introduction

While some responses to tax reform are fast, others take time, because they require costly adjustment of an occupation, location, skill, or other income-generating work state. To accommodate tax-induced substitution that is delayed and complicated by costs of adjustment, we integrate a dynamic discrete choice model into optimal tax analysis. In this setting, we analyze optimal long run tax designs that reach beneath incomes and tax underlying work states.

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Dynamic discrete choice models relate flows of workers across landscapes of income-generating states and, hence, stationary distributions of workers over these states ("work state distributions") to consumption allocations. We identify sensitivities to consumption of such distributions as central components of long run optimal tax analyses. We then make three contributions. First, we provide formulas for such sensitivities that elucidate the structure of long run substitution and facilitate the quantitative evaluation of long run tax designs. Second, we "invert" a classical discrete choice optimal tax formula to obtain a novel expression that identifies the marginal cost of inducing a long run population shift towards a state as a sufficient condition for that state's optimal tax. We leverage the structure of the stationary distribution sensitivity to obtain an interpretable formula for this marginal cost and, hence, optimal taxes. Third, we deliver an application to optimal US occupational taxation. This exercise recovers distributional sensitivities and, hence, tax distortion far larger in the long than the short run. Nonetheless, optimal long run occupational taxes are close to those implied by the current US tax system under a benchmark welfare criterion. Low income occupations with high turnover, however, receive more favorable tax treatments.

We begin by laying out a dynamic discrete choice framework for optimal tax work and, in this context, deriving a Diamond-Mirrlees-Saez optimal tax formula for the long run. This formula has a familiar structure: It equates marginal redistributive benefits to marginal excess burdens of taxation. The key departure is that the latter now depend on long run behavioral responses to tax reform. Quantifying such responses has long been recognized as a key challenge for tax design.[1] In our framework, it requires evaluation of sensitivities of the stationary work state distribution to consumption variation.

Towards quantification of these sensitivities, we first derive a propagation equation that treats short-run distributional responses to consumption variation as multi-dimensional impulses and converts them into stationary distribution responses. If the relevant short run responses can be measured in the data, then the equation permits evaluation of stationary distribution sensitivities even if these are not explicitly observed. Absent direct evidence on equilibrium short-run response across (all) behavioral margins relevant to a given tax design problem, in particular, if the goal is to evaluate responses at a counterfactual equilibrium, a structural model of short-run response is needed. For a class of such models, we relate an equilibrium's short run sensitivities to its Markov transition, stationary distribution

[1] For example, Saez et al. (2012) note that: "The long-term response [of earnings to tax change] is of most interest for policymaking, although also .... more difficult to identify empirically."

and a (small) set of preference parameters. Counterfactual equilibrium transitions and distributions can be generated from those observed and the same set of preference parameters via the dynamic hat algebra used in quantitative trade models (Caliendo et al. (2019)). Thus, given these parameters, an observed equilibrium and our formulas, counterfactual stationary distribution sensitivities and, hence, long run marginal excess burdens are available for any alternate consumption allocation, including the optimal one.[2] We refer to the required preference parameters as *structurally sufficient* for dynamic optimal tax analysis.

Diamond-Mirrlees-Saez optimal tax expressions are intuitive, interpretable and informative about tradeoffs at the optimum. However, they provide only implicit descriptions of taxes. This renders them less informative about which taxes are large and which are small. To address this we derive a new optimal tax equation that "inverts" the marginal excess burden component of the classic formulas. The resulting equation identifies the marginal cost of inducing a long run proportional population shift towards a state as a sufficient statistic for that state's optimal tax (up to a government spending shifter). Such marginal costs are given by covariances between the short run flow payoff adjustments needed to induce long run population shifts and the societal costs of delivering those payoff adjustments.

We say that an economy exhibits *proportional attraction and dispersion* if when a payoff increment at a state occurs it attracts workers in proportion to population from other states and if when workers disperse from a state, because of changed circumstances or in pursuit of new opportunities, they do so in proportion to population at destination states. Such behavior is a feature of benchmark models like the repeated logit or variants of the repeated logit in which stickiness of choice is introduced via Calvo-like random re-optimization opportunities. Proportional attraction and dispersion implies that the marginal cost of inducing a long run proportional population shift towards a state is simply the adjusted reciprocal of marginal utility (of consumption) at the target state relative to mean. If, in addition, the social criterion is utilitarian and utility from consumption is log, such marginal costs are affine in consumption. These in turn imply an optimal affine income tax.

More generally, payoff reallocations must be *tuned* to induce a proportional long run population reallocation and suppress non-proportionalities in attraction and dispersion. This, in turn, has implications for the marginal cost of inducing such a reallocation and for optimal taxes. In particular, to induce a proportional long run population reallocation towards a target state, payoffs must be relatively reduced in

[2]Well known procedures for empirically recovering these structurally sufficient parameters exist in the applied dynamic discrete choice literature.

downstream states (i.e. in states that disproportionately receive dispersing workers from the targeted state) and relatively increased in upstream ones (from which the targeted state disproportionately attracts). If a state is a stepping stone, channeling workers from low to high income opportunities, this tuning implies a shift of payoff from low to high marginal utility states, which releases resources. The marginal cost of a long run proportional population shift to a stepping stone is correspondingly reduced and it receives a more favorable optimal tax treatment. The inverted optimal tax formula also indicates what can go wrong if a policymaker relies on short run sensitivities to evaluate tax designs. When states are persistent, short run sensitivities imply larger marginal costs of inducing population shifts than long run suggesting much greater scope for redistribution and greater optimal variation in taxes across states.

In the context of a structural model of occupational choice, we deploy the machinery described above to explore the optimal design of policies that reach below incomes and tax occupations.[3] Our benchmark model implies a small number of preference parameters that are structurally sufficient for tax evaluation. We estimate the key marginal utility parameter using U.S. occupational transition data and combine this and other calibrated parameters with our formulas to calculate long and short run distributional sensitivities at the prevailing empirical allocation. We then use the estimated parameters and our formulas to calculate and interpret optimal tax functions. Three main results emerge.

1. Long run distributional sensitivities at the prevailing allocation are an order of magnitude greater than short run. This implies much greater long run behavioral distortion from taxation than would be suggested by short-run sensitivities. We estimate that in the long run an extra dollar extracted from high earning lawyers yields \$0.52 of revenue after taking into account long run substitution of the legal profession for lower tax alternatives. Conversely, an extra dollar taken from low earning maintenance workers generates \$1.71 of revenue as workers migrate to higher earning and higher tax occupations. These values compare with short-run revenue (annual) impacts of \$0.95 and \$1.06 respectively.

2. Under a welfare criterion that identifies the marginal social welfare weight with reciprocals of consumption as in Saez (2002), the optimal tax structure is not too distant from actual. Both are approximately affine in occupational income,

[3] There is a history of occupation taxes in the U.S. at the state and local level, see Appleton (2012).

with similarly sized deductions and slopes. Thus, apparently large long run occupational choice distortions rationalize an approximation to current policy.

3. Low earning/low retention occupations receive more generous optimal tax treatments. Food services is such an occupation. This treatment is rationalized by dampened long run distributional sensitivities and marginal excess burdens. Viewed through the lens of our inverted optimal tax formula, long run population reallocation costs for such occupations are relatively reduced.

We evaluate the robustness of our results to modifications of our benchmark quantitative model. The first permits workers to make an initial education choice. Thereafter, workers are segmented into high (college) and low (high school) educational groups. The second (elaborated in the appendix) allows for unobserved heterogeneity and partitions workers into unobserved "mobile" and "immobile" types. In each case, we re-estimate the model, recover type-specific distributional sensitivities and evaluate optimal taxes. Other robustness exercises decompose management into finer occupational categories, vary structurally sufficient model parameters around estimated values, and vary long run elasticities of substitution between occupations. Each exercise generates additional insight, but preserves the main results listed above.[4]

The paper proceeds as follows. Section 2 lays out a benchmark dynamic discrete choice environment and policy problem. Section 3 provides a formula for the stationary distribution sensitivity that factors it into short run substitution and long run propagation components. We describe empirical strategies for quantitatively operationalizing the formula. Section 4 introduces the inverse optimal tax equation and relates tax design to the marginal cost of inducing population movement. Section 5 gives extensions. Section 6 develops our application to a salient occupational choice model. Section 7 concludes.

Literature A large literature considers optimal taxation in settings in which household choices respond smoothly and immediately to tax variation. Contributions of Diamond and Mirrlees (1971), Mirrlees (1971), and Saez (2001) are seminal. Rothschild and Scheuer (2013) initiate a line of research in which taxes are designed for workers making discrete occupational and continuous effort choices. See

[4]For example, when augmenting with education, we find that the tax schedule retains its affine shape, but that high education/low income occupation combinations receive relatively generous tax treatments. These combinations have low long run own population elasticities underpinned by low retention of highly educated workers. This mitigates the tax distortion incurred in assisting these populations.

also Ales et al. (2015). In these papers workers have no inherent preferences over occupations: They select the occupation that maximizes their income and make small income adjustments in response to small tax changes. Work on static discrete choice optimal tax design originates with Saez (2002, 2004). Recent developments include Ales and Sleet (2022), Fajgelbaum and Gaubert (2020), Laroque and Pavoni (2017) and Lockwood et al. (2017). Our paper builds on these by considering long run optimal tax design in explicitly dynamic settings. Like the static discrete choice literature, the key behavioral input is a distributional sensitivity to consumption variation. Now, however, it is a long run stationary distribution sensitivity.

Saez et al. (2012) observe that slow behavioral adjustment on career and human capital margins may elevate long term earnings responses to tax variation. They emphasize, however, the difficulties in empirically identifying such responses. We show how to construct long term behavioral responses from short in stationary settings and how to leverage dynamic discrete choice models to quantify long term responses at counterfactual policies. Keane (2011) emphasizes costly human capital accumulation and the elevated long-run responses to tax variation that this implies. Guvenen et al. (2014) evaluate implications of such long run responses for equilibrium earnings distributions. Several papers in the macro-public finance tradition consider implications of human capital formation for tax design. Examples include Heathcote et al. (2017), Heathcote et al. (2020), and Krueger and Ludwig (2016). These papers focus on continuous, beginning of career human capital choices that have long-lasting effects on worker productivity. In each the human capital margin dampens income tax progressivity prescriptions. Closer to us in formulation is Coen-Pirani (2021). He provides a tractable model that integrates discrete location choice into a worker's productivity process. He characterizes the complete transition of an optimal parametric income tax function.

In stressing granular tax designs that reach beneath incomes, we depart from work in public finance that collapses adjustment along all behavioral margins into a single reduced form elasticity of taxable income. This work focuses on income tax design and evaluation for which the elasticity of taxable income, appropriately measured, serves as a sufficient statistic, see Gruber and Saez (2002) or Saez et al. (2012). Our approach requires selecting a subset of margins on which to focus and developing a deeper understanding of short and long-run responses on those margins. It permits analysis of more granular tax designs. Inevitably, however, it omits some adjustment margins relevant for full income tax analysis. We regard it as a complement to papers that directly measure elasticities of taxable income.

Our dynamic discrete choice model is closest to that introduced into trade literature by Artuç et al. (2010). Variations on this have been used to examine the impact of trade and other shocks on worker choice of sector, industry or location. Examples include Artuç et al. (2010), Artuç and McLaren (2015), Caliendo et al. (2019) and Kleinman et al. (2023). We utilize (a variation on) the estimation approach of Artuç et al. (2010) to obtain values for structural parameters and the dynamic hat algebra methodology of Caliendo et al. (2019) to construct counterfactual equilibria. Kleinman et al. (2023) evaluate first order responses of dynamic discrete environments to disturbances. They focus on transition paths to productivity shocks. In contrast, we analyze long run responses to tax variation and use the resulting distributional sensitivities as inputs into optimal tax equations. Our distributional sensitivity formulas are redolent of sensitivity formulas obtained in the macroeconomic production network literature, e.g., Baqaee and Farhi (2019). These describe the response of sectoral Domar weights (sales shares) to microeconomic TFP or markup shocks. Domar weight vectors and input-output matrices in network models are analogous to stationary population distributions and Markov transitions in our set up.

2 Benchmark Environment and Policy Problem

This section lays out a simple baseline environment for steady state dynamic discrete choice tax analysis.

Notational conventions We denote vectors and matrices with bold face letters and write $f(\mathbf{x})$ to denote the element-wise application of a function f to a vector or matrix. A superscript $^\top$ denotes a transpose; $\mathbf{I}$ is an identity matrix; $\mathbf{D_x}$ a diagonal matrix with vector $\mathbf{x}$ on its leading diagonal; $\mathbf{\Pi_x}$ a matrix with each column equal to (the same) vector $\mathbf{x}$. If $\mathbf{P}$ is the stationary distribution of a Markov chain, we call $\mathbf{\Pi_P}$ a stationary distribution matrix. If $\mathbf{Q}$ is the transition of a Markov chain, then $\mathbf{Q}(j,i)$ is the probability of moving *to* j *from* i. Given a matrix $\mathbf{X}$ each column of which is a vector of outcomes and a Markov transition $\mathbf{Q}$, we denote by $\mathbb{E}_\mathbf{Q}[\mathbf{X}] = \mathbf{Q}^\top\mathbf{X}$ and $\hat{\mathbb{E}}_\mathbf{Q}[\mathbf{X}] = (\mathbf{I} - \mathbf{Q}^\top)\mathbf{X}$, respectively, the matrix of conditional expectations and deviations from conditional expectations under $\mathbf{Q}$.

Work states and demographics Let $\mathcal{I} = \{1, \ldots, I\}$ be a fixed discrete work-state space. Depending on the application, these states may represent work intensities, locations, occupations, skills or combinations of the preceding. Our later quantitative analysis will focus on occupation choices. A perpetual youth structure is assumed. In each period a mass δ of new born entrants augments a mass $1-\delta$ of survivors from the preceding period.

Labor supply of survivors Surviving workers select a work state at the beginning of each period to maximize their lifetime payoff inclusive of moving costs and payoff shocks. Let $\mathbf{c} \in \mathbb{R}^I_+$ be a time invariant stationary allocation describing consumption available to workers in each state. Lifetime payoffs are assembled from $u : \mathbb{R}_+ \to \mathbb{R}$, an increasing, strictly concave and differentiable function mapping current consumption to current utility, β, a discount factor inclusive of survival probability, $\boldsymbol{\kappa} : \mathcal{I} \times \mathcal{I} \to \mathbb{R}$, an adjustment cost function, and $\{\boldsymbol{\varepsilon}_t\}_{t=0}^{\infty}$, $\boldsymbol{\varepsilon}_t \in \mathbb{R}^I$, a process for payoff shocks. Such shocks are drawn by workers in successive periods of their lives independently of past draws and the draws of others. Expected maximal shock draws are assumed finite: $E[\max_{\mathcal{I}} \boldsymbol{\varepsilon}(j)] \in \mathbb{R}$. Given a consumption allocation $\mathbf{c}$, initial state i_0 and process for states adapted to shock draws, $i^\infty = \{i_t\}_{t=1}^{\infty}$, a surviving worker obtains lifetime payoff (exclusive of initial shock and adjustment cost):

$$U(i_0, i^\infty; \mathbf{c}) = u(\mathbf{c}(i_0)) + E\left[\sum_{t=1}^{\infty} \beta^t \{u(\mathbf{c}(i_t)) - \boldsymbol{\kappa}(i_t, i_{t-1}) + \boldsymbol{\varepsilon}_t(i_t)\}\right]. \tag{1}$$

Let $\mathbf{V}(\mathbf{c})(i)$ be the optimal payoff of a surviving worker who is initially at i and selects states in subsequent periods to solve: $\mathbf{V}(\mathbf{c})(i) := \sup_{i^\infty} U(i, i^\infty; \mathbf{c})$. It is readily shown that, given $\mathbf{c}$, $\mathbf{V}(\mathbf{c})$ is the unique bounded solution to the recursion:

$$\mathbf{V}(\mathbf{c}) = u(\mathbf{c}) + \beta \boldsymbol{\mathcal{K}}[\mathbf{V}(\mathbf{c})], \tag{2}$$

where $\boldsymbol{\mathcal{K}} : \mathbb{R}^I \to \mathbb{R}^I$ is a payoff aggregator satisfying:

$$\forall i \in \mathcal{I}, \qquad \boldsymbol{\mathcal{K}}[\mathbf{v}](i) := E\left[\max_{j \in \mathcal{I}} \{\mathbf{v}(j) - \boldsymbol{\kappa}(j, i) + \boldsymbol{\varepsilon}(j)\}\right]. \tag{3}$$

The map $\boldsymbol{\mathcal{K}} : \mathbb{R}^I \to \mathbb{R}^I$ is monotone, convex, commutes with the addition of constant vectors[5] and, under a regularity condition on the distribution of shocks, is smooth. Optimizing transition behavior of workers is described by a Markov matrix $\boldsymbol{\mathcal{Q}}(\mathbf{V}(\mathbf{c}))$, where the map $\boldsymbol{\mathcal{Q}} : \mathbb{R}^I \to \mathcal{M}$, with $\mathcal{M}$ the space of Markov matrices, is related to $\boldsymbol{\mathcal{K}}$ via:[6]

$$\boldsymbol{\mathcal{Q}}(\mathbf{v}) = \left(\frac{\partial \boldsymbol{\mathcal{K}}[\mathbf{v}]}{\partial \mathbf{v}}\right)^\top. \tag{4}$$

[5]That is $\boldsymbol{\mathcal{K}}[\mathbf{v} + r\mathbf{1}] = \boldsymbol{\mathcal{K}}[\mathbf{v}] + r\mathbf{1}$, where r is a scalar and $\mathbf{1} \in \mathbb{R}^I$ is a unit vector.

[6]This is the content of the Williams-Daley-Zachary theorem, see, e.g., Fosgerau et al. (2013).

It follows from (4) and the properties of $\mathcal{K}$ that (i) $\mathcal{Q}(\mathbf{v})(j,i)$ is non-decreasing in $\mathbf{v}(j)$ and non-increasing in $\mathbf{v}(k)$, $k \neq j$, and that (ii) $\mathcal{Q}(\mathbf{v})$ is constant with respect to uniform increases in payoff, i.e $\mathcal{Q}(\mathbf{v}) = \mathcal{Q}(\mathbf{v} + r\mathbf{1})$ for any scalar r and unit vector $\mathbf{1}$. Further, twice differentiability of $\mathcal{K}$ implies that $\mathcal{Q}$ is differentiable with $\frac{\partial(\mathcal{Q}(\mathbf{v}))}{\partial \mathbf{v}} = \frac{\partial^2(\mathcal{K}[\mathbf{v}]^\top)}{\partial \mathbf{v}^2}$.

Example 1. In a dynamic logit model with $\boldsymbol{\varepsilon}$ a vector of component-wise independent Gumbel shocks, the payoff aggregator (up to a constant) and Markov maps are:

$$\mathcal{K}[\mathbf{v}](i) = \log \sum_{j \in \mathcal{I}} \boldsymbol{\omega}(j,i) \exp^{\mathbf{v}(j)} \quad \text{and} \quad \mathcal{Q}(\mathbf{v})(j,i) = \frac{\boldsymbol{\omega}(j,i) \exp^{\mathbf{v}(j)}}{\sum_{j' \in \mathcal{I}} \boldsymbol{\omega}(j',i) \exp^{\mathbf{v}(j')}}, \tag{5}$$

with $\boldsymbol{\omega}(j,i) := \exp^{-\boldsymbol{\kappa}(j,i)}$.

Labor supply of entrants Let $\mathbf{P_0}$ denote a distribution of entrants over work states. This distribution may be treated parametrically or endogenized by introducing an additional entry state 0, extending $\boldsymbol{\kappa}$ onto $\mathcal{I} \times \mathcal{I}_0$, $\mathcal{I}_0 = \mathcal{I} \cup \{0\}$, and treating entrants as selecting work states to solve:

$$\mathcal{K}_0[\mathbf{V}(\mathbf{c})] := E\left[\max_{j \in \mathcal{I}} \{\mathbf{V}(\mathbf{c})(j) - \boldsymbol{\kappa}(j,0) + \boldsymbol{\varepsilon}(j)\}\right]. \tag{6}$$

Analogous to the case of survivors, (B.17) implies $\mathbf{P_0}(\mathbf{c}) = \mathcal{P}_0(\mathbf{V}(\mathbf{c}))$, where $\mathcal{P}_0(\mathbf{v}) := \frac{\partial \mathcal{K}_0[\mathbf{v}]}{\partial \mathbf{v}}^\top$. In particular, if a logit structure is assumed, then:

$$\mathcal{P}_0(\mathbf{v})(j) = \frac{\boldsymbol{\omega}(j,0) \exp^{\mathbf{v}(j)}}{\sum_{j' \in \mathcal{I}} \boldsymbol{\omega}(j',0) \exp^{\mathbf{v}(j')}}, \quad \text{with} \quad \boldsymbol{\omega}(j,0) := \exp^{-\boldsymbol{\kappa}(j,0)}. \tag{7}$$

Stationary Distribution/Value function pairing Together a tuple $(\mathcal{Q}, \mathcal{P}_0, \delta, \mathbf{V}, \mathbf{c})$ defines a Markov transition $\mathbf{Q}(\mathbf{c})$ that incorporates demographics and behavior:

$$\mathbf{Q}(\mathbf{c}) = (1-\delta)\mathcal{Q}(\mathbf{V}(\mathbf{c})) + \delta \boldsymbol{\Pi}_{\mathcal{P}_0(\mathbf{V}(\mathbf{c}))}. \tag{8}$$

We assume that the induced $\mathbf{Q}(\mathbf{c})$ is ergodic and has unique stationary distribution $\mathbf{P}(\mathbf{c})$. The stationary labor supply block of the model is then summarized by:

$$\mathbf{V}(\mathbf{c}) = u(\mathbf{c}) + \beta \mathcal{K}[\mathbf{V}(\mathbf{c})] \quad \text{and} \quad \mathbf{P}(\mathbf{c}) = \mathbf{Q}(\mathbf{c})\mathbf{P}(\mathbf{c}). \tag{9}$$

In the sequel to streamline the presentation, arguments $\mathbf{c}$ and $\mathbf{v}$ of functions are omitted when the dependence is understood.

Long run and short run Our model implies immediate reactions of lifetime payoffs $\frac{\partial \mathbf{V}}{\partial \mathbf{c}}$ and transition probabilities $\frac{\partial \mathbf{Q}}{\partial \mathbf{c}}$ to permanent consumption perturbations. The stationary distribution P responds more slowly as the modified flows $\frac{\partial \mathbf{Q}}{\partial \mathbf{c}}$ propagate and population stocks adjust. We denote the one period or *short run sensitivity* of $\mathbf{P}(j)$ to a permanent consumption perturbation $\mathbf{c}(i)$ by:

$$\mathbf{\Phi}(j,i) := \sum_{k\in\mathcal{I}} \frac{\partial \mathbf{Q}(j,k)}{\partial \mathbf{c}(i)} \mathbf{P}(k) \tag{10}$$

and collect these sensitivities into a matrix $\mathbf{\Phi}$.[7] The Jacobian $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ gives the *long run sensitivity* of P to a permanent consumption perturbation. We denote corresponding short and long run semi-elasticities by, respectively, $\mathbf{\Psi} = \mathbf{D_P}^{-1}\mathbf{\Phi}$ and $\frac{1}{\mathbf{P}}\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$.

Remark 1. The interpretation of $\mathbf{\Phi}$ as a short run distributional sensitivity is subject to three caveats. First, "short run" is defined in the theoretical model as response over a single period. In quantitative work, a stance must be taken on the length of a period. In such work, we identify a model period with a year. Second, $\mathbf{\Phi}$ gives the short run (single model period) response to a *permanent* consumption perturbation. If a policymaker engineers a long run consumption perturbation slowly via a sequence of tax and equilibrium wage adjustments distributed over periods, then the initial population response to this reform will deviate from $\mathbf{\Phi}$. Third, $\mathbf{\Phi}$ describes a short run (single model period) distributional response on the underlying state space. Suppose that a policymaker only partly observes a worker's state. Let $\overline{\mathrm{Q}}$ denote the matrix of transition probabilities across subsets of the state space that the policymaker can distinguish. Although Q responds immediately to a permanent consumption perturbation and the interpretation of $\mathbf{\Phi}$ remains intact, the aggregated transition $\overline{\mathrm{Q}}$ responds more slowly as distributions on distinguishable subsets evolve. We return to this observation and its implications in Section 5.

Interpretations The specification given in (9) captures the essential elements that differentiate long from short run analyses: frictions that delay migration to higher average payoff states and repeated shocks that create churn in the population.

Remark 2 (Interpreting Costs of Adjustment)**.** Equation (9) omits an explicit amenity value for each state. It is readily shown, however, that amenity values can be absorbed into costs of adjustment to deliver the framework that we use. The costs $\boldsymbol{\kappa}$ may encode ladders and stepping stones with, for example, $\boldsymbol{\kappa}(i_n, i_1) >$

[7] In matrix form, $\mathbf{\Phi} = \left[\mathbf{P}^\top \otimes \mathbf{I}\right] \frac{\partial(\mathrm{vec}\,\mathbf{Q})}{\partial \mathbf{c}}$, with $\otimes$ the Kronecker product, vec $\mathbf{Q}$ matrix $\mathbf{Q}$ reorganized as a vector by stacking its columns and $\frac{\partial(\mathrm{vec}\,\mathbf{Q})}{\partial \mathbf{c}}$ the Jacobian of vec $\mathbf{Q}$ with respect to $\mathbf{c}$.

$\sum_{m=1}^{n-1} \boldsymbol{\kappa}(i_{m+1}, i_m)$, implying that it is less costly for a worker to move to i_n from i_1 in a series of steps. Thus, human capital ladders can be incorporated into the model.

Remark 3 (Interpreting Payoff Shocks)**.** Our model attributes pairings $(\mathcal{K}, \mathcal{Q})$ to discrete choice in the face of payoff shocks. Such shocks may be interpreted as disturbances to preferences, to adjustment costs or, under appropriate assumptions on u and taxes, to wages. Alternative rationales are available. A dynamic logit pairing of $(\mathcal{K}, \mathcal{Q})$ may, for example, be motivated by appeals to costly search (Wu (2020)) or bounded rationality (Mattsson and Weibull (2002)). A more agnostic interpretation treats $\mathcal{Q}$ (and $\mathcal{P}_0$) as modeling devices designed to generate stochastic choice behavior and behavioral elasticities that better match the data.

Remark 4 (Random Re-optimization)**.** Random re-optimization opportunities may be incorporated into the model.[8] If ρ is the re-optimization probability, then (2) holds, with $\mathcal{K}[\mathbf{V}(\mathbf{c})]$ redefined as:

$$\mathcal{K}[\mathbf{V}(\mathbf{c})](i) = (1-\rho)\mathbf{V}(\mathbf{c})(i) + \rho E\left[\max_{j \in \mathcal{I}}\{\mathbf{V}(\mathbf{c})(j) - \boldsymbol{\kappa}(j,i) + \boldsymbol{\varepsilon}(j)\}\right] \tag{11}$$

and (8) becomes:

$$\mathbf{Q}(\mathbf{c})(j,i) = (1-\delta)\{(1-\rho)\mathbf{I}(j,i) + \rho\mathcal{Q}(\mathbf{V}(\mathbf{c}))(j,i))\} + \delta\mathcal{P}_0(\mathbf{V}(\mathbf{c}))(j). \tag{12}$$

Remark 5 (Exogenous Transitions; Persistent Types)**.** A worker's state could encompass its productivity, skill or health. It is then natural to think of some transitions or retentions as exogenous rather than chosen.[9] To explicitly incorporate these, split the state space as $\mathcal{I} = \mathcal{X} \times \mathcal{A}$, with $\mathcal{X}$ a set of chosen and $\mathcal{A}$ a set of exogenously determined state components. Assume that the exogenous components or "types" are determined before choice within a period and that they evolve according to a Markov chain with matrix ρ and entry distribution ρ_0. The worker's problem is as in (2), but with $\mathcal{K}$ now modified as:

$$\mathcal{K}[\mathbf{V}(\mathbf{c})](x,\alpha) = \sum_{\alpha' \in \mathcal{A}} E\left[\max_{x' \in \mathcal{X}}\{\mathbf{V}(\mathbf{c})(x',\alpha') - \boldsymbol{\kappa}(x',\alpha',x) + \boldsymbol{\varepsilon}(x')\}\right]\rho(\alpha'|\alpha), \tag{13}$$

where $|$ notation is used to indicate conditioning. The Markov matrix over states

[8] These may be incorporated into our framework by permitting $\varepsilon_t(j) = -\infty$ shock draws, allowing the distribution for $\{\varepsilon_t\}$ to depend on a worker's current state and with some probability to assign $-\infty$ payoff to all but the current state. In this remark, we describe a more explicit treatment.

[9] As in the random re-optimization case, exogenous transitions can be incorporated via penalizing ε_t shock draws. Again we describe a more explicit treatment.

becomes:

$$\mathbf{Q}(\mathbf{c})(x', \alpha'|x, \alpha) = (1-\delta)\boldsymbol{\mathcal{Q}}(\mathbf{V}(\mathbf{c}))(x'|\alpha', x)\rho(\alpha', \alpha) + \delta\boldsymbol{\mathcal{P}}_0(\mathbf{V}(\mathbf{c}))(x', \alpha')\rho_0(\alpha'), \tag{14}$$

with $\frac{\partial \boldsymbol{\mathcal{K}}[\mathbf{v}](x,\alpha)}{\partial \mathbf{v}(x',\alpha')} = \boldsymbol{\mathcal{Q}}(\mathbf{v})(x'|\alpha', x)\rho(\alpha'|\alpha)$ and $\boldsymbol{\mathcal{Q}}(\mathbf{v})(x'|\alpha', x)$ the probability that a worker selects x' given chosen state x and updated type α'.

Special cases Logit models are widely used in applied work. We use three logit special cases of our framework for illustrative purposes. Each case supposes that payoff shocks are Gumbel and behavior (conditionally) logit.

1. (*Repeated logit*). This case imposes $\boldsymbol{\kappa}(j, i) = \boldsymbol{\kappa}(j)$. Thus, adjustment costs are independent of a worker's current state, choice is repeated rather than persistent and long and short run substitution responses coincide. In addition, the logit assumption induces *proportional attraction and dispersion* behavior: when a payoff increment at a state occurs it attracts workers in proportion to population from other states, $\frac{1}{\mathbf{P}(j)}\frac{\partial \mathbf{P}(j)}{\partial \mathbf{v}(i)} = -\mathbf{P}(i)$, for $j \neq i$, and when workers disperse from a state in response to new shock draws they do so in proportion to population at destination states.

2. (*Calvo-logit*). This case imposes exogenous retention except at randomly occurring re-optimization dates. At these, $\boldsymbol{\kappa}(j, i) = \boldsymbol{\kappa}(j)$ and behavior is logit. The associated transition $\mathbf{Q} = (1-\psi)\mathbf{I} + \psi\mathbf{\Pi}_{\mathbf{P}}$ is obtained from (12) by setting $\psi = (1-\delta)\rho + \delta$. The latter is interpreted as a re-optimization probability that combines the probabilities of replacement by an optimizing entrant and re-optimization by a survivor. As discussed below, choice is sticky and short run substitution suppressed relative to long, $|\mathbf{\Phi}| \leq |\frac{\partial \mathbf{P}}{\partial \mathbf{c}}|$, but the proportional attraction and dispersion property is preserved.

3. (*Dynamic logit*). Our third case closes down retirement and entry, $\delta = 0$, but allows costs of adjustment to depend on a worker's current state. In this case $\mathbf{Q} = \boldsymbol{\mathcal{Q}}(\mathbf{V})$, with $\mathbf{Q}$ given as in (5). Churn of workers over states is generated by payoff shocks. It divorces short from long run substitution patterns and permits state-contingent heterogeneity in both.

Production, resource-feasibility and equilibrium A function $F : \mathbb{R}^I_+ \to \mathbb{R}$ describes the economy's technology for converting allocations of workers over states $\mathbf{p} \in \mathbb{R}^I_+$ into final goods. The function is assumed to be increasing, to have constant returns to scale, to have a continuous derivative $\frac{\partial F}{\partial \mathbf{p}} = \left(\frac{\partial F}{\partial \mathbf{p}(1)}, \ldots, \frac{\partial F}{\partial \mathbf{p}(I)}\right)$ and to satisfy an Inada condition at zero.[10] Let $G \in \mathbb{R}_+$ denote an exogenous level of government spending. A consumption allocation $\mathbf{c}$ is *resource-feasible* at steady state if:

$$F(\mathbf{P}(\mathbf{c})) - \mathbf{c} \cdot \mathbf{P}(\mathbf{c}) - G \geq 0. \tag{15}$$

A resource-feasible allocation $\mathbf{c}$ can be implemented as part of a *stationary competitive equilibrium* (SCE) in which profit maximizing firms pay workers their marginal products, $\mathbf{w} = \frac{\partial F(\mathbf{P}(\mathbf{c}))}{\partial \mathbf{p}}^\top$, and taxes are equated to the wedges between private consumptions and marginal products, $\mathbf{T} = \mathbf{w} - \mathbf{c}$. Conversely, an SCE $\mathbf{c}$ is resource-feasible. In short, $\mathbf{c}$ is an SCE allocation if and only if it satisfies (15).

Policy choice We initially assume that the policymaker can observe fully and condition taxes upon a worker's current state. This implies that it can implement any resource-feasible (and, hence, SCE) consumption allocation via appropriate choice of tax policy. We thus adopt the standard approach of formulating the policymaker's problem as a choice of resource-feasible consumption allocation $\mathbf{c}$ and then back out supporting taxes $\mathbf{T} = \frac{\partial F(\mathbf{P}(\mathbf{c}))}{\partial \mathbf{p}}^\top - \mathbf{c}$. In the context of our benchmark model, this implies that only transitory payoff shocks ε and re-optimization opportunities are privately observed by the worker. The extension of our results to settings with privately observed type-states that persist over time is described in Section 5.[11]

Let $\mathrm{M} : \mathbb{R}^I_+ \to \mathbb{R}$ be a smooth, increasing social objective defined over stationary consumption allocations. For example, the social objective may be a Pareto-weighted sum of lifetime utilities: $\mathrm{M}(\mathbf{c}) = \boldsymbol{\lambda}^\top \mathbf{V}(\mathbf{c})$. The policymaker's problem is then:

$$\max_{\mathbf{c}} \{\mathbf{M}(\mathbf{c}) \quad | \quad F(\mathbf{P}(\mathbf{c})) - \mathbf{c} \cdot \mathbf{P}(\mathbf{c}) - G \geq 0\}. \tag{16}$$

This is a familiar discrete choice tax design problem with the key caveat that the distribution of workers over states is the stationary distribution satisfying (9).[12]

[10] In principle F can be elaborated to describe the production of intermediate goods and the transportation of goods across space. Such rich production structures are important components of trade and spatial models where labor demand implications of productivity or foreign import supply shocks are focuses of analysis. Our analysis concerns implications of tax design for labor supply and so we emphasize the (reduced) map from long run worker allocations to final goods.

[11] In our benchmark model, the policy space is, thus restricted, in the sense that tax functions are time invariant, consistent with our focus on long run designs, and independent of histories of individual states. On the other hand, it is richer than is assumed in applied models that restrict attention to taxes as parametric functions of earnings.

[12] See the seminal contributions of Saez (2002, 2004) and Ales and Sleet (2022), Laroque and Pavoni (2017), and Fajgelbaum and Gaubert (2020) for recent applications. All are static.

Optimal tax equations The first order condition from (16) supplies a discrete choice Diamond-Mirrlees-Saez optimal tax equation:

$$\underbrace{\mathbf{D}_{\mathbf{P}}{}^{-1}\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^{\top}}_{\text{MSW}} = \underbrace{\mathbf{1} - \mathbf{D}_{\mathbf{P}}^{-1}\frac{\partial \mathbf{P}}{\partial \mathbf{c}}^{\top}\mathbf{T}}_{1+\text{MEB}}, \tag{17}$$

where, on the left side, Υ is the Lagrange multiplier on the resource constraint. The complete left side term is interpreted as a vector of optimal marginal social welfare (MSW) weights. Its i-th element gives the social value of a permanent consumption increment at state i deflated by Υ and expressed in resource units. The right side gives the cost of delivering a consumption increment at each state. It nets from the direct unit resource cost the extra tax revenues $-\mathbf{D}_{\mathbf{P}}{}^{-1}\frac{\partial \mathbf{P}}{\partial \mathbf{c}}^{\top}\mathbf{T}$ generated by workers as they move in response to consumption increments. The latter, with elements $\text{MEB}(i) = -\frac{1}{\mathbf{P}(i)}\sum_{j\in\mathcal{I}}\mathbf{T}(j)\frac{\partial \mathbf{P}(j)}{\partial \mathbf{c}(i)}$, is interpreted as a vector of long run marginal excess burdens of taxation. Together with the resource constraint, equation (17) may be used to characterize or calculate the optimal taxes and equilibrium associated with a particular welfare criteria. Alternatively, it may be used to recover the marginal social welfare weights that rationalize a particular observed tax function $\mathbf{T}$. The MEB vector can be used independently to evaluate the long run budgetary cost of a small permanent tax reform and the projects that it funds.

Long run distribution sensitivities The preceding discussion identifies stationary distribution sensitivities $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ as key inputs into long-run marginal excess burden calculations and, hence, long-run optimal tax equations. Quantitative activation of the optimal tax equation (17) requires information on $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$. However, long run behavioral responses to tax reform are difficult to identify directly. The high dimensional nature of $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ introduces additional complication. Further, if (17) is used to characterize optimal taxes at a fixed welfare criteria, then $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ must also be calculated at a counterfactual equilibrium.

3 The structure of long run substitution

This section develops an expression for the stationary distribution Jacobian $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$. The expression indicates how information in transition data may be leveraged to obtain estimates of long run substitution responses at prevailing and counterfactual equilibria. It highlights the central role of post-substitution transition behavior in relating short to long substitution responses.

3.1 Propagation

Let $\mathbf{c}$ denote a stationary consumption allocation and $\mathbf{Q}$ an induced Markov matrix describing worker transition behavior. The matrix $\mathbf{Q}$ may be derived from the behavioral model of Section 2. However, that is not necessary for the first result of this section. Let $\mathbf{P}$ be a stationary distribution of workers over states satisfying the (implicit) equation: $(\mathbf{I}-\mathbf{Q})\mathbf{P}=0$. Singularity of $\mathbf{I}-\mathbf{Q}$ precludes direct application of the implicit function theorem to this last equation requiring a different approach to recovery of $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$. Define the generalized (group) inverse of matrix $\mathbf{A}$ to be the unique matrix $\mathbf{A}^{\#}$ satisfying $\mathbf{A}^{\#}\mathbf{A}\mathbf{A}^{\#}=\mathbf{A}^{\#}$, $\mathbf{A}\mathbf{A}^{\#}=\mathbf{A}^{\#}\mathbf{A}$, and $\mathbf{A}\mathbf{A}^{\#}\mathbf{A}=\mathbf{A}$.[13]

Proposition 1 (Propagation)**.** *If $\mathbf{Q}$ is differentiable with respect to $\mathbf{c}$ and $\mathbf{P}$ is a probability distribution that solves $(\mathbf{I}-\mathbf{Q})\mathbf{P}=0$, then the Jacobian $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ satisfies:*

$$\frac{\partial \mathbf{P}}{\partial \mathbf{c}}=(\mathbf{I}-\mathbf{Q})^{\#}\boldsymbol{\Phi}, \tag{18}$$

with $\boldsymbol{\Phi}$ the short run sensitivity matrix. If $\mathbf{Q}$ defines an ergodic chain, then $(\mathbf{I}-\mathbf{Q})^{\#}=\sum_{m=0}^{\infty}\{\mathbf{Q}^m-\boldsymbol{\Pi}_{\mathbf{P}}\}$, $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}=\sum_{m=0}^{\infty}\mathbf{Q}^m\boldsymbol{\Phi}$ or, in terms of semi-elasticities,

$$\frac{1}{\mathbf{P}}\frac{\partial \mathbf{P}}{\partial \mathbf{c}}=\boldsymbol{\Psi}+\mathbf{D}_{\mathbf{P}}{}^{-1}\sum_{m=1}^{\infty}\mathrm{Cov}\left(\mathbf{Q}^m,\boldsymbol{\Psi}\right), \tag{19}$$

with $\boldsymbol{\Psi}=\mathbf{D}_{\mathbf{P}}{}^{-1}\boldsymbol{\Phi}$ the short run semi-elasticity matrix.

Proof. See Appendix A. □

Formula (18) implies that the group inverse $(\mathbf{I}-\mathbf{Q})^{\#}$ acts as a *propagation factor* that converts the short run sensitivity $\boldsymbol{\Phi}$ into the long run sensitivity $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$.

Interpreting Proposition 1. In the ergodic case, the propagation factor in Proposition 1 has the form $(\mathbf{I}-\mathbf{Q})^{\#}=\sum_{m=0}^{\infty}\{\mathbf{Q}^m-\boldsymbol{\Pi}_{\mathbf{P}}\}$ permitting straightforward computation and interpretation of long given short run substitution responses. Consider a small permanent consumption increment at a state i. This relatively raises lifetime payoffs at i and, possibly, at other states from which it is cheap to access i. In each subsequent period, flows to these payoff enhanced states will be elevated, generating successive waves of substitution, $\boldsymbol{\Phi}(\cdot,i)$. Having substituted in response to a consumption perturbation, workers then behave to a first order according to $\mathbf{Q}$. Thus, $\mathbf{Q}^m\boldsymbol{\Phi}(\cdot,i)$ gives the first order impact on $\mathbf{P}$ of workers who substituted m

[13] If $\mathbf{A}$ is invertible, then the group inverse is the inverse. In the applied math literature, $(\mathbf{I}-\mathbf{Q})^{\#}$, with $\mathbf{Q}$ a Markov matrix, is referred to as the deviation matrix or the ergodic potential of $\mathbf{Q}$.

periods ago. From (18), the long run response to the consumption perturbation cumulates these terms: $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}(\cdot, i) = \sum_{m=0}^{\infty} \mathbf{Q}^m \mathbf{\Phi}(\cdot, i)$. Intuitively, if extra consumption at i shifts workers between states j and k that have very similar post-substitution behavior (i.e. such that columns $\mathbf{Q}^m(\cdot, j)$ and $\mathbf{Q}^m(\cdot, k)$ are similar), then each wave of short run substitution dissipates and short and long run substitution patterns are similar. Conversely, if the extra consumption diverts workers to states with very different post-substitution transition behavior, then short run and long run substitution will be very different.[14] Expression (19) quantifies these effects via the sum of the cross-covariance matrices $\text{Cov}(\mathbf{Q}^m, \mathbf{\Psi})$. The following examples illustrate.

Example 2 (*No persistence*)**.** If adjustment costs are zero and there is no persistence in the chain, then $\mathbf{Q} = \mathbf{\Pi_P}$ and (18) implies $\frac{\partial \mathbf{P}}{\partial \mathbf{c}} = \mathbf{\Phi} + \sum_{m=1}^{\infty} \mathbf{\Pi_P}^m \mathbf{\Phi} = \mathbf{\Phi}$. In this case, substituting workers disperse according to $\mathbf{P}$ in the period after substitution. No persistence thus implies identity of long and short-run responses to consumption perturbations and an absence of propagation. □

Example 3 (*Slow diffusion with two states*)**.** In the two state case with transition elements $\mathbf{Q}(1,1) = p$ and $\mathbf{Q}(2,2) = q$, evaluation of (18) implies: $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}(j, i) = \frac{1}{2-(p+q)} \mathbf{\Phi}(j, i)$. Define $p + q$ to be the persistence of $\mathbf{Q}$ and call $\mathbf{Q}$ persistent if $p + q > 1$. It follows that the greater the persistence of $\mathbf{Q}$, the greater the long run amplification of short run responses. Intuitively, if the chain is persistent, a consumption increment at i leads to an accumulation of workers at i over time. □

Example 4 (*Calvo-logit*)**.** In the Calvo-logit model $\mathbf{Q} = (1-\psi)\mathbf{I} + \psi \mathbf{\Pi_P}$. Thus, $(\mathbf{I} - \mathbf{Q})^{\#} = \frac{1}{\psi}(\mathbf{I} - \mathbf{\Pi_P})$ and $\frac{\partial \mathbf{P}}{\partial \mathbf{c}} = (\mathbf{I} - \mathbf{Q})^{\#} \mathbf{\Phi} = \frac{1}{\psi} \mathbf{\Phi}$. In this case, post-substitution, workers draw shocks and re-optimize with probability ψ. Hence, they accumulate in states to which they have substituted. Further, when workers re-optimize they disperse according to $\mathbf{P}$. Together these effects imply that long-run sensitivities are uniformly increased by a factor of $1/\psi$ relative to short run. □

Environments with many states, persistence, and heterogeneous patterns of diffusion admit more complex patterns of propagation.

Example 5 (Stepping stones and dead ends)**.** Consider the scenario illustrated in Figure 1, where blocks indicate work states and arrows positive flows of workers. Entrant workers replace retirees and are born into a state labeled "School" (s). From there they may choose to go to a "Stepping Stone" (ss) or a "Dead End" (de) job. Suppose that these alternatives deliver identical low levels of pay and similar

[14] In particular, if $\mathbf{Q}$ is persistent with large values for $\mathbf{Q}(i, i)$ and $\mathbf{Q}(j, j)$, then each wave of workers that substitutes from j to i accumulates in i rather than j implying enhanced long run substitution.

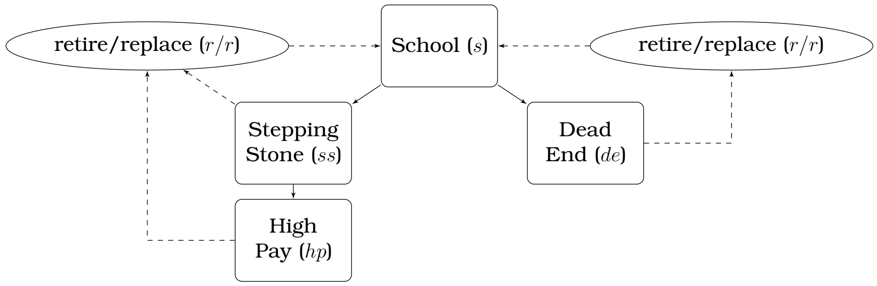
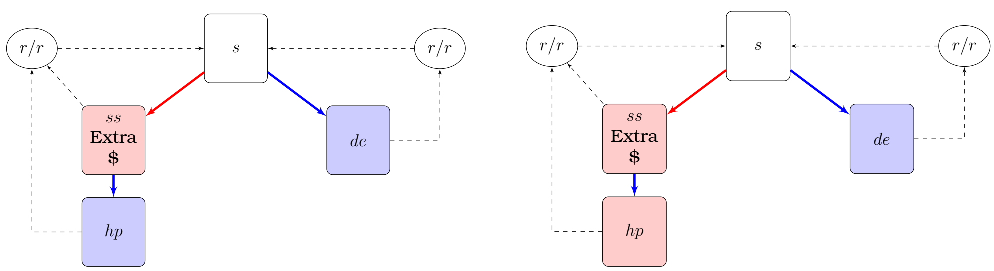


Figure 1: Job Network. Blocks are states; Arrows are positive flows. Exogenous flows to retirement and replacement with new entrants are shown as dashed arrows.

average lifetime payoffs. The ss job, however, has prospects and provides a positive probability of moving to a "High Pay" (hp) job. The de job, in contrast, offers a less stressful, low effort life and a higher amenity value, but no opportunity for advancement. Assume that an exogenous fraction of workers from each job state retire in each period. Although ss and de deliver identical incomes (and would receive the same treatment under an income tax), the different transitions from each imply different long run substitution patterns over jobs and incomes.

Figure 2 illustrates short and potential long run effects of permanently increasing consumption at ss. Such an increase immediately raises $\mathbf{Q}(ss, s)$, while reducing



(a) Short run effects. (b) Long run effects.

Figure 2: Short and long run effects of extra consumption at the "Stepping Stone". Blue arrows show reduced transitions; red arrows increased transitions. Blue boxes show reduced worker populations. Red boxes show increased worker populations.

$\mathbf{Q}(de, s)$ and $\mathbf{Q}(hp, ss)$ (shown by red and blue arrows, respectively, in Figure 2).[15] Initially, worker populations increase at ss and decline at the other two job states (shown by red and blue boxes in Figure 2a). In particular, the initial decline in

[15]We continue to assume that direct flows between de and hp and s and hp are not possible.

$\mathbf{Q}(hp, ss)$ causes a population reduction at hp of:

$$\mathbf{\Phi}(hp, ss) = -\frac{\partial \mathbf{Q}(ss, ss)}{\partial \mathbf{c}(ss)}\mathbf{P}(ss) < 0.$$

In later periods the build up of workers at ss relative to de combined with the positive transition rate from ss to hp and zero transition from de to hp, offsets the effect of the reduction in $\mathbf{Q}(hp, ss)$ on the numbers arriving at hp. Precisely, let $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}(hp, ss) = \sum_{m=0}^{\infty} \mathbf{\Gamma}_m$, with $\mathbf{\Gamma}_m := \sum_{k \in \mathcal{I}} \mathbf{Q}^m(hp, k)\mathbf{\Phi}(k, ss)$ the impact on the population at hp of substitution m periods ago. The terms $\mathbf{\Gamma}_m$ are increasing in m, since $\mathbf{\Gamma}_0 = \mathbf{\Phi}(hp, ss) < 0 < \mathbf{\Phi}(ss, ss)$ and:

$$\mathbf{\Gamma}_m = \mathbf{Q}(hp, hp)\mathbf{\Gamma}_{m-1} + \mathbf{Q}(hp, ss)\mathbf{Q}(ss, ss)^{m-1}\mathbf{\Phi}(ss, ss), \tag{20}$$

with the second right hand side term in (20) giving the impact on the size of the m-th substitution cohort at hp of those who substituted m periods ago to ss and only now move to hp, The terms $\mathbf{\Gamma}_m$ eventually become positive, arresting the decline in the population at hp and, potentially, as they cumulate, implying $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}(hp, ss) > 0$. In this case, long run populations are increased at ss and hp, but depleted at de (shown by red and blue boxes in Figure 2b). □

Connecting to data In the absence of direct evidence on long run substitution, the formulas in Proposition 1 imply that long run sensitivities $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ at an observed equilibrium can be constructed from short using observed transitions $\mathbf{Q}$. In particular, given data on $\mathbf{Q}$, the propagation matrix $(\mathbf{I} - \mathbf{Q})^{\#}$ can be constructed and applied to estimates of short run sensitivities $\mathbf{\Phi}$. In the absence of direct or complete evidence on $\mathbf{\Phi}$ a structural model may be adopted to relate short run substitution to observed transition behavior. We turn to this next.

3.2 Integrating Short Run Substitution

We proceed in three steps. First, we derive a general characterization of short run substitution for models of the form considered in Section 2 and combine it with Proposition 1 to provide a more complete description of long run substitution. Then in a second step, we impose additional structure that permits evaluation using data. We conclude by deriving implications for the marginal excess burden.

Lemma 1 below provides a first characterization of short run substitution. It is independent of the particular preference shock structure assumed in the benchmark model and implies that, after possible normalization of utilities, a consumption increment at i generates greater short run substitution *from* $j \neq i$ if it reduces the lifetime payoff at j *relative to that expected under a Markov chain* S defined below. The lemma utilizes the following symmetry-like property for Markov chains.

Definition 1. *Let* R *be the transition of an ergodic Markov chain with stationary distribution* $\mathbf{P}$. *The chain is said to be* reversible *if* $\mathbf{R} = \mathbf{D_P}\mathbf{R}^\top\mathbf{D_P}^{-1}$.

Lemma 1 (Short run substitution). *Let* $\mathbf{S} = \mathbf{I} - \mathbf{D_P}\Xi\mathbf{D_P}^{-1}$, *with* $\Xi(j,i) = \frac{1}{\mathbf{P}(j)}\sum_{k\in\mathcal{I}}\frac{\partial\mathbf{Q}(j,k)}{\partial\mathbf{v}(i)}\mathbf{P}(k)$. *After possible normalization of utilities,* S *is a reversible Markov chain with stationary distribution* $\mathbf{P}$ *and the short run semi-elasticity* $\boldsymbol{\Psi}$ *satisfies:*

$$\boldsymbol{\Psi} = \hat{\mathbb{E}}_{\mathbf{S}}\left[\frac{\partial\mathbf{V}}{\partial\mathbf{c}}\right] := \frac{\partial\mathbf{V}}{\partial\mathbf{c}} - \mathbb{E}_{\mathbf{S}}\left[\frac{\partial\mathbf{V}}{\partial\mathbf{c}}\right] = (\mathbf{I} - \mathbf{S}^\top)\frac{\partial\mathbf{V}}{\partial\mathbf{c}}. \tag{21}$$

Proof. See Appendix A. □

We call S the *(short run) substitution matrix*; its $j \neq i$-th off-diagonal element gives the one period out-flow from j in response to a payoff increment at i (normalized by the population at i). In our setting reversibility of S is analogous to Slutsky symmetry.[16] Lemma 1 can be combined with Proposition 1 to give the following characterization of long run substitution.

Proposition 2 (Long run substitution). *The (stationary distribution) semi-elasticity matrix* $\frac{\mathbf{1}}{\mathbf{P}}\frac{\partial\mathbf{P}}{\partial\mathbf{c}}$ *satisfies:*

$$\frac{\mathbf{1}}{\mathbf{P}}\frac{\partial\mathbf{P}}{\partial\mathbf{c}} = \hat{\mathbb{E}}_{\mathbf{S}}\left[\frac{\partial\mathbf{V}}{\partial\mathbf{c}}\right] + \mathbf{D_P}^{-1}\sum_{m=1}^{\infty}\mathrm{Cov}\left(\mathbf{Q}^m, \hat{\mathbb{E}}_{\mathbf{S}}\left[\frac{\partial\mathbf{V}}{\partial\mathbf{c}}\right]\right). \tag{22}$$

Proof. See Appendix A. □

Expression (22) decomposes the long run response of the distribution $\mathbf{P}$ to a permanent consumption increment into a current short run response (the first term) and an accumulation of past responses (the second summation term). The latter implies that long run substitution between two states $j \neq i$ is enhanced relative to short run (and more workers are drawn from j by a consumption increment at i) if the covariance terms $\mathrm{Cov}\left(\mathbf{Q}^m(j,\cdot), \hat{\mathbb{E}}_{\mathbf{S}}\left[\frac{\partial\mathbf{V}}{\partial\mathbf{c}}\right](\cdot,i)\right)$ are negative. This occurs if the consumption increment at i raises (resp. reduces) lifetime payoffs relative to

[16] That $\mathbf{S}$ has positive off-diagonal elements, columns summing to one and satisfies $\mathbf{S} = \mathbf{D_P}\mathbf{S}^\top\mathbf{D_P}^{-1}$ is independent of the particular cardinal representation of utilities. But the scale of $\mathbf{I} - \mathbf{S}$ (and, hence, the sign of the diagonal elements of $\mathbf{S}$) requires their normalization.

those expected under S in states with low (resp. high) probabilities of subsequently transitioning to j. In particular, if i or states upstream to i that are payoff enhanced by the consumption increment have low probabilities of subsequent transition to j, then the covariance in (22) will be smaller (more negative) and long run substitution between i and j will be elevated.

Remark 6. Equation (22) is reminiscent of expressions found in the macro-network literature, e.g. Baqaee and Farhi (2019). Those expressions describe how Domar weight (sales share) vectors are perturbed by shocks. In our setting, Domar weight vectors are replaced by stationary distributions of workers, input-output matrices by Markov transition matrices, forward-looking prices by marginal lifetime value functions and productivity shocks by consumption perturbations. □

Imposing additional structure; Connecting to data We show in the appendix that the benchmark model implies a map $\mathcal{H}$ such that $\mathbf{S} = \mathcal{H}(\mathbf{Q})$, where $\mathcal{H}$ is determined by the structure of preference shocks and re-optimization opportunities. Examples follow.

Example 6 (Calvo-Logit)**.** Evaluation of $\mathbf{S}$ in the Calvo-logit case gives: $\mathbf{S} = \mathbf{Q} = (1-\psi)\mathbf{I} + \psi\mathbf{\Pi_P}$. Thus, off-diagonal elements, $j \neq i$, satisfy: $\mathbf{S}(j,i) = \psi\mathbf{P}(j)$. This case exhibits *proportional attraction*: the outflow from j in response to an increment in $\mathbf{v}(i)$ is proportional to $\mathbf{P}(j)$ (scaled by the re-optimization probability ψ). The repeated logit corresponds to the limiting case with $\psi = 1$ and $\mathbf{S}(j,i) = \mathbf{P}(j)$. □

Example 7 (Dynamic Logit)**.** For the dynamic logit without perpetual youth: $\mathbf{S} = \mathbf{Q}\mathbf{D_P}\mathbf{Q}^\top\mathbf{D_P}^{-1}$, with off-diagonal ($j \neq i$) elements of $\mathbf{S}$ satisfying:

$$\mathbf{S}(j,i) = \mathbf{P}(j) + \mathbf{P}(j)\mathrm{Cov}\left(\frac{\mathbf{Q}(j,\cdot)}{\mathbf{P}(j)}, \frac{\mathbf{Q}(i,\cdot)}{\mathbf{P}(i)}\right). \tag{23}$$

In this case, proportional attraction is disrupted by the covariance term. It implies that short-run substitutability between j and i is elevated when $\mathbf{Q}(j,\cdot)$ and $\mathbf{Q}(i,\cdot)$ covary positively. Intuitively, in this case j and i attract workers from similar states and, so, are closer short-run substitutes with util increments at one siphoning off workers who would have transitioned to or remained at the other. In contrast, if $\mathbf{Q}$ is very persistent, then j mainly "attracts" workers from j and similarly for i. In this case, the covariance is negative and short run substitution is suppressed.[17] □

[17] Let $\hat{\mathbf{Q}} := \mathbf{D_P}\mathbf{Q}^\top\mathbf{D_P}^{-1}$. Then $\hat{\mathbf{Q}}$ is the "time-reversed" version of $\mathbf{Q}$ and $\mathbf{S} = \mathbf{Q}\hat{\mathbf{Q}}$ defines a Markov chain called the *multiplicative reversibilization* of $\mathbf{Q}$. An economic logic underpins the identity of $\mathbf{S}$

The workers' envelope conditions imply:

$$\frac{\partial \mathbf{V}}{\partial \mathbf{c}} = (\mathbf{I} - \beta \boldsymbol{\mathcal{Q}})^{-1} \mathbf{D}_{\partial \mathbf{u}}. \tag{24}$$

If $\boldsymbol{\mathcal{H}}$ and β and equilibrium values for $\mathbf{Q}$, $\boldsymbol{\mathcal{Q}}$ and $\mathbf{D}_{\partial \mathbf{u}}$ are known, then together (21), $\mathbf{S} = \boldsymbol{\mathcal{H}}(\mathbf{Q})$ and (24) permit recovery of the corresponding equilibrium short run semi-elasticity $\boldsymbol{\Psi}$ or, equivalently, sensitivity $\boldsymbol{\Phi} = \mathbf{D_P}^{-1}\boldsymbol{\Psi}$. It is common in the applied literature to impose a Gumbel restriction on preference shocks, which pins down $\boldsymbol{\mathcal{H}}$ as in (23), and to calibrate β. In addition, for this case, well known procedures exist for estimating the parameters of $\mathbf{D}_{\partial \mathbf{u}}$. Then given (non-parametric) estimates of $\mathbf{Q}$, $\boldsymbol{\mathcal{Q}}$ and $\mathbf{c}$, equilibrium values for $\boldsymbol{\Phi}$ or $\boldsymbol{\Psi}$ may be constructed. The long run sensitivity $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ is obtained for this equilibrium using (18). Explicit estimates of the potentially high dimensional cost of adjustment parameter $\boldsymbol{\kappa}$ are not required. Although such costs affect behavior, their impact on long run substitution is fully encoded in the estimated $\mathbf{Q}$ and $\boldsymbol{\Phi}$.

Construction of $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ for counterfactual equilibria proceeds along similar lines, but requires corresponding counterfactual values for $\mathbf{Q}$. The dynamic hat algebra approach of Caliendo et al. (2019) shows how such values for $\mathbf{Q}$ may be constructed from those of observed equilibria under a dynamic logit assumption (and a log assumption on flow utility from consumption). No additional parameter estimates are needed beyond those used to evaluate long run sensitivities at an observed equilibrium. The discount factor and marginal utility parameters are structurally sufficient for counterfactual evaluation. We elaborate this procedure in the appendix and implement it in Section 6.

Marginal Excess Burdens Long run marginal excess burden formulas are readily recovered from (22).

Proposition 3. *Long run marginal excess burdens satisfy:*

$$-\mathbf{D_P}^{-1} \frac{\partial \mathbf{P}}{\partial \mathbf{c}}^{\top} \mathbf{T} = -\mathbf{D_P}^{-1} \sum_{m=0}^{\infty} \mathrm{Cov}\left(\hat{\mathbb{E}}_{\mathbf{S}} \left[\frac{\partial \mathbf{V}}{\partial \mathbf{c}} \right], (\mathbf{Q}^m)^{\top} \mathbf{T} \right). \tag{25}$$

The covariance term in (25) cumulates the impact on tax revenues of waves of substituting workers who, post substitution, diffuse according to $\mathbf{Q}$ in successive

with $\mathbf{Q}\hat{\mathbf{Q}}$. Off-diagonal $j \neq k$ elements of $\mathbf{S}$ and corresponding substitutability are large if inflows to j and k originate from common sources with high probability. In such cases, positive correlation between $\hat{\mathbf{Q}}(j, \cdot)$ and $\mathbf{Q}(k, \cdot)$ occurs: if workers at i came from j with high probability ($\hat{\mathbf{Q}}(j, i)$ is large), and workers leave i for k with high probability ($\mathbf{Q}(k, i)$ is large), then i is a common source for j and k.

periods. In static models only the $m = 0$ covariance is present. The Calvo-Logit supplies a simple benchmark case in which the covariance terms at successive horizons m are proportionate.

Example 8 (Calvo-logit)**.** In the Calvo-logit model $\frac{\partial \mathbf{P}}{\partial \mathbf{c}} = (\mathbf{I} - \mathbf{Q})^{\#}\mathbf{\Phi} = \frac{1}{\psi}\mathbf{\Phi}$ and $\mathbf{S} = (1-\psi)\mathbf{I} + \psi\mathbf{\Pi}_{\mathbf{P}}$. Substituting these and the envelope condition into (22) gives:

$$\frac{1}{\mathbf{P}}\frac{\partial \mathbf{P}}{\partial \mathbf{c}}(j,i) = \frac{1}{\psi} \times \underbrace{\frac{\psi}{\nu}\frac{\partial u(\mathbf{c}(i))}{\partial c}(\mathbf{I}(j,i) - \mathbf{P}(j))}_{\mathbf{\Phi}(j,i)}, \tag{26}$$

with $\nu := 1 - \beta(1-\psi)$. In this case, an extra unit of consumption at i draws workers proportionately to population from states j with, respectively, short and long run intensities $\frac{\psi}{\nu}\frac{\partial u(\mathbf{c}(i))}{\partial c}$ and $\frac{1}{\nu}\frac{\partial u(\mathbf{c}(i))}{\partial c}$. Evaluating the long run MEB (25) using (26) gives:

$$-\mathbf{D_P}^{-1}\frac{\partial \mathbf{P}}{\partial \mathbf{c}}^{\top}\mathbf{T} = -\frac{1}{\nu}\mathbf{D}_{\partial \mathbf{u}}(\mathbf{T} - G\mathbf{1}). \tag{27}$$

The simple form in (27) relies on the proportional attraction and dispersion properties of the Calvo-logit model: in the long run, a consumption increment at i draws workers in proportion to population towards i with intensity $\frac{1}{\nu}\frac{\partial u(\mathbf{c}(i))}{\partial c}$ and with each proportional unit shift generating revenues $\mathbf{T}(i) - G$. $\square$

More generally, state-contingent costs of adjustment imply that a consumption increment at i attracts most strongly from states that have low costs of moving to i and disperses most strongly towards states that have low costs of moving from i (inclusive of i itself). Long run substitution patterns are modified accordingly. Equation (22) describes exactly how, with (25) giving implications for long run marginal excess burdens.

4 Inverse optimal tax equations

Stationary distribution sensitivities $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ facilitate quantitative evaluation of long run marginal excess burdens. They may be inserted into (17) and used to characterize optimal tax distortions. We pursue this in Section 6. As we now show, their structure also facilitates derivation of "inverted" optimal tax equations that provide more explicit characterizations of taxes.

The goal of this section is the conversion:

$$\mathbf{D_P}^{-1}\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^{\top} = \mathbf{1} - \mathbf{D}_{\mathbf{P}}^{-1}\frac{\partial \mathbf{P}}{\partial \mathbf{c}}^{\top}\mathbf{T} \implies \mathbf{T} = \mathbf{H} + G\mathbf{1} \tag{28}$$

for an interpretable and quantifiable matrix $\mathbf{H}$. Such conversion is straightforward with two states. Then, (17) may be rearranged, combined with the policymaker budget constraint and the expressions for long run sensitivities $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ from Example 3 to obtain:

$$\mathbf{T}(i) = \frac{1-\mathbf{P}(i)}{\mathbf{\Psi}(i,i)/\sum_{j\in\mathcal{I}}(1-\mathbf{Q}(j,j))}\left(1-\frac{1}{\mathbf{P}(i)}\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}(i)\right)+G. \tag{29}$$

Intuitively, higher taxes are associated with low marginal social welfare weights, higher policymaker spending and reduced own semi-elasticity values $\frac{1}{\mathbf{P}(i)}\frac{\partial \mathbf{P}(i)}{\partial \mathbf{c}(i)} = \frac{\mathbf{\Psi}(i,i)}{\sum_{j\in\mathcal{I}}(1-\mathbf{Q}(j,j))}$; the latter a simple inverse (semi-)elasticity result. In (29), $\sum_{j\in\mathcal{I}}(1-\mathbf{Q}(j,j))$ is the inverted counterpart of the propagation matrix from our earlier sensitivity formulas. For persistent chains, it acts as a compression factor implying lower taxes than would be suggested by inspection of short run semi-elasticities $\mathbf{\Psi}(i,i)$.

More generally, utilizing our previous expressions for long run distributional sensitivities, the following inverted optimal tax formula emerges.

Proposition 4 (Inverted Optimal Tax Formula)**.** *Optimal taxes satisfy:*

$$\mathbf{T} - G\mathbf{1} = \mathrm{Cov}(\mathbf{\Omega}, \mathbf{C}), \tag{30}$$

where $\mathbf{\Omega} = \mathbf{AB}$ *and:*

$$\mathbf{A} := \underbrace{(\mathbf{I}-\mathbf{Q}^\top)}_{\substack{\text{Reverse}\\ \text{Propagation}}} \qquad \mathbf{B} := \underbrace{(\mathbf{I}-\mathbf{S}^\top)^{\#}\mathbf{D_P}^{-1}(\mathbf{I}-\beta\mathcal{Q})}_{\substack{\text{Reverse Short Run}\\ \text{Substitution Matrix}}} \qquad \mathbf{C} := \underbrace{\mathbf{D}_{\partial\mathbf{u}}{}^{-1}\left(\mathbf{1}-\mathbf{D_P}^{-1}\frac{1}{\Upsilon}\frac{\partial\mathbf{M}}{\partial\mathbf{c}}^\top\right)}_{\substack{\text{Redistribution}\\ \text{Vector}}}. \tag{31}$$

Proof. See Appendix A. □

Interpreting the inverted formula Our earlier optimal tax equation (17) was formulated in terms of the marginal benefits and costs of a *consumption reallocation*. In contrast, (30) is expressed in terms of the marginal benefits and costs of a *population reallocation*. To see this first define the unit population reallocation matrix $\mathbf{\Delta} := \mathbf{I} - \mathbf{\Pi_P}$ and interpret its i-th column: $\mathbf{\Delta}(\cdot,i) = \begin{pmatrix} -\mathbf{P}(1) & \ldots & 1-\mathbf{P}(i) & \ldots & -\mathbf{P}(I)\end{pmatrix}^\top$ as a proportional reallocation of workers from each j to i. The extra revenues accruing to the policymaker from reallocation $\mathbf{\Delta}(\cdot,i)$ are:

$$\mathbf{T}^\top\mathbf{\Delta}(\cdot,i) = \mathbf{T}(i) - \sum_{j\in\mathcal{I}}\mathbf{T}(j)\mathbf{P}(j) = \mathbf{T}(i) - G.$$

Thus, the i-th element $\mathbf{T}(i) - G$ on the left side of (30) is the *fiscal benefit of a long run unit population reallocation to* i. Expression (30) equates such benefits to the social costs of long run unit population reallocations implying that the latter are sufficient statistics for optimal taxes (up to a government spending shifter). These social costs are obtained as covariances between the flow payoff variations required to induce long run population reallocations (rows of $\boldsymbol{\Omega}$) and the societal costs of delivering such variations ($\mathbf{C}$).[18] The flow payoff variation matrix $\boldsymbol{\Omega}$ further factors into matrices $\mathbf{A}$ and $\mathbf{B}$, with the following interpretations:

1. The *reverse propagation matrix* $\mathbf{A}$ describes the *short run* population reallocations needed to induce *long run* unit population reallocations in the face of post-substitution diffusion.

2. The *reverse short run substitution matrix* $\mathbf{B}$ gives the flow payoff reallocations needed to induce *short run* unit population reallocations holding expected lifetime utility fixed.

This factorization is a direct consequence of the stationary distribution sensitivity formulas of the preceding section. The vector $\mathbf{C}$ includes both the direct resource cost of delivering flow payoff variation (given by reciprocals of marginal utilities of consumption) and redistributive attitudes towards such variation. To illuminate (30) consider first the Calvo-logit benchmark.

Calvo-logit inverted formula In the Calvo-logit case $\mathbf{Q} = \mathcal{Q} = \mathbf{S} = (1-\psi)\mathbf{I} + \psi\boldsymbol{\Pi}_{\mathbf{P}}$. Substituting these into (30) and simplifying gives:

$$\boldsymbol{\Omega} = \nu({\mathbf{D}_{\mathbf{P}}}^{-1} - \mathbf{E}), \tag{32}$$

where $\nu := (1 - \beta(1-\psi))$ and $\mathbf{E}$ is a matrix of ones. Again the simple form (32) stems from the proportionality of attraction and dispersion implied by the Calvo-logit model. Removal of a unit of payoff from workers in each state and allocation of $1/\mathbf{P}(i)$ to those in i induces a proportional long run population shift to i. The constant ν in (32) adjusts for frequency of decision-making and scales to ensure a unit long run

[18] Adão et al. (2023) derive an interesting inverted optimal tax equation in the context of optimal tariff design. In particular, they use an inversion step to decompose optimal tariffs additively into efficiency, redistributive and other components. Although the settings, static continuous demand for tradable goods vs. dynamic flows of workers across discrete states, and tax formulas are different, a related decomposition is available in our setting. Writing (30) as $\mathbf{T} = \boldsymbol{\Omega}\mathbf{D}_{\mathbf{P}}\mathbf{C}$, with $G = 0$ for simplicity, it follows that $\mathbf{T} = \boldsymbol{\Omega}\mathbf{D}_{\mathbf{P}}{\mathbf{D}_{\partial\mathbf{u}}}^{-1} - \boldsymbol{\Omega}{\mathbf{D}_{\partial\mathbf{u}}}^{-1}\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^{\top}$, where the first term gives the resource cost of inducing a long run population reallocation holding expected lifetime utility constant and the second gives an additional redistributive cost utilizing the societal payoff function M.

population shift.[19] Substitution of (32) into (30) yields:

$$\mathbf{T} - G\mathbf{1} = \nu \hat{\mathbb{E}}_{\mathbf{P}}[\mathbf{C}], \tag{33}$$

where $\hat{\mathbb{E}}_{\mathbf{P}}[\mathbf{x}] = \mathbf{x} - \mathbb{E}_{\mathbf{P}}[\mathbf{x}]\mathbf{1}$ gives deviations from expected values at $\mathbf{P}$. Thus, costs of inducing long run population reallocations and, hence, optimal taxes, vary across states in proportion to the costs of delivering flow payoff variation (net of mean) described by $\mathbf{C}$.

Setting $\mathrm{M}(\mathbf{c}) = \mathbf{P}^\top \mathbf{V}(\mathbf{c})$, using (30) to express $\mathbf{C}$ and, hence, (33) as an implicit function of $\mathbf{c}$ and inverting permits yields:

$$\mathbf{T} = \mathbf{w} - \mathcal{C}\left(\mathbf{w} + \nu E\left[\frac{1}{\partial u(\mathbf{c})/\partial c}\right] - G\mathbf{1}\right), \tag{34}$$

with $\mathcal{C} : \mathbb{R}^I_+ \to \mathbb{R}^I_+$ defined implicitly by $\mathcal{C}(\mathbf{r})(i) + \frac{1-\beta(1-\psi)}{\partial u(\mathcal{C}(\mathbf{r})(i))/\partial c} = \mathbf{r}(i)$ for $\mathbf{r} \in \mathbb{R}^I_+$. Hence, in this case an income tax is optimal with shape controlled by the curvature of $\frac{1}{\partial u/\partial c}$ via $\mathcal{C}$. In particular, if $\frac{1}{\partial u/\partial c}$ is convex (resp. concave), then optimal taxes are convex in income and progressive (resp. concave and regressive). If $u(c) = a\log c$, then $\frac{1}{\partial u/\partial c}$ is linear in c and optimal taxes are affine in income with:

$$\mathbf{T} = \frac{1}{1+b}\hat{\mathbb{E}}_{\mathbf{P}}[\mathbf{w}] + G\mathbf{1} \quad \text{and} \quad b = \frac{a}{\nu} = \frac{a}{1-\beta(1-\psi)}. \tag{35}$$

Larger values of a and smaller values of ψ translate into smaller marginal income tax rates.

Payoff tuning More generally, a tax designer assessing the cost of a long run unit population reallocation to a state (and trading it off against the tax revenue benefits) must consider the downstream propagation and upstream substitution implications of assigning payoff to the state. The designer must "tune" payoffs accordingly.

Consider, for example, a long run unit population reallocation towards the "Stepping Stone" (ss) job in Example 5. This requires a 1% reduction in the population of workers at all jobs and a reallocation of these workers to ss. The removal of a unit of payoff from workers in each state and allocation of $1/\mathbf{P}(ss)$ to those in ss, as in the Calvo-logit case, will, over time, raise the number of workers at ss. However, it will not draw workers proportionately from across the job network, as required by a unit population reallocation. It will draw disproportionately from "Dead End" (de),

[19] Large values of ψ (and ν) imply frequent re-optimization, short spells in states and larger required variation in flow payoffs to induce long run unit population reallocation.

because this is a close "upstream" substitute. It will draw less than proportionately from "High Pay" (hp) because this is "downstream" from ss and a build up of workers in the latter will propagate to hp. To generate a long run unit population shift, the policymaker must *tune* the payoff adjustment. The matrix $\boldsymbol{\Omega}$ for this case describes exactly how this should be done. Intuitively it requires relatively raising payoff at de to deter excessive depletion, while relatively reducing it at hp to deter onward propagation and excessive build up. If hp workers have greater consumption and lower marginal utilities of consumption than de workers, this tuning will release resources. In turn, this will reduce the resource cost of inducing a long run unit population shift towards ss. This creates a force for lower optimal taxes at ss compared to similarly earning de.[20]

Approximate inverted optimal tax formula Given a dynamic discrete choice equilibrium with transition $\mathbf{Q}$, transition for survivors $\boldsymbol{\mathcal{Q}}$, substitution matrix $\mathbf{S}$, and stationary distribution $\mathbf{P}$, define *best fit Calvo-logit behavior* as a $\psi \in [0,1]$ and transition $\tilde{\mathbf{Q}} := (1-\psi)\mathbf{I} + \psi\boldsymbol{\Pi}_{\mathbf{P}}$ such that ψ minimizes $\max\{\|\boldsymbol{\Delta}_{\mathbf{Q}}\|, \|\boldsymbol{\Delta}_{\mathbf{S}}\|, \|\boldsymbol{\Delta}_{\boldsymbol{\mathcal{Q}}}\|\}$, where:

$$\boldsymbol{\Delta}_{\mathbf{Q}} := \frac{1}{\psi}(\mathbf{Q} - \tilde{\mathbf{Q}}) \qquad \boldsymbol{\Delta}_{\boldsymbol{\mathcal{Q}}} := \frac{\beta}{\nu}\mathbf{D}_{\mathbf{P}}(\boldsymbol{\mathcal{Q}} - \tilde{\mathbf{Q}})^{\top}\mathbf{D}_{\mathbf{P}}{}^{-1} \qquad \boldsymbol{\Delta}_{\mathbf{S}} := \frac{1}{\psi}(\mathbf{S} - \tilde{\mathbf{Q}}), \tag{36}$$

$\nu = 1 - \beta(1-\psi)$ and $\|\cdot\|$ is a matrix norm. If the equilibrium is generated by a Calvo-logit model, then the best fit behavior is exact and $\boldsymbol{\Delta}_{\mathbf{Q}} = \boldsymbol{\Delta}_{\boldsymbol{\mathcal{Q}}} = \boldsymbol{\Delta}_{\mathbf{S}} = 0$. Otherwise the matrices $\boldsymbol{\Delta}_{\mathbf{Q}}$, $\boldsymbol{\Delta}_{\boldsymbol{\mathcal{Q}}}$, and $\boldsymbol{\Delta}_{\mathbf{S}}$ describe deviations in $\mathbf{Q}$, $\boldsymbol{\mathcal{Q}}$ and $\mathbf{S}$ from the values implied by the best fit Calvo-logit. They capture non-proportionalities in transition and substitution behavior.

As detailed previously, Calvo-logit models impart a simple structure to $\boldsymbol{\Omega}$ and, hence, to optimal taxes. For situations in which behavior is close to Calvo-logit, an interpretable approximation to $\boldsymbol{\Omega}$ is available that makes explicit how downstream propagation and upstream substitution shape taxes. In Appendix A, we derive a polynomial expansion of $\boldsymbol{\Omega}$ around its best fit Calvo-logit value in terms of $\boldsymbol{\Delta}_{\mathbf{Q}}$, $\boldsymbol{\Delta}_{\boldsymbol{\mathcal{Q}}}$, and $\boldsymbol{\Delta}_{\mathbf{S}}$ and show that for $\max\{\|\boldsymbol{\Delta}_{\mathbf{Q}}\|, \|\boldsymbol{\Delta}_{\mathbf{S}}\|, \|\boldsymbol{\Delta}_{\boldsymbol{\mathcal{Q}}}\|\}$ small enough:

$$\boldsymbol{\Omega} \approx \nu\left(\mathbf{D}_{\mathbf{P}}{}^{-1} - \mathbf{E}\right) + \nu\left(\boldsymbol{\Delta}_{\mathbf{S}}^{\top} - \boldsymbol{\Delta}_{\mathbf{Q}}^{\top} - \boldsymbol{\Delta}_{\boldsymbol{\mathcal{Q}}}^{\top}\right)\mathbf{D}_{\mathbf{P}}{}^{-1}. \tag{37}$$

The first right hand component of (37), $\nu\left(\mathbf{D}_{\mathbf{P}}{}^{-1} - \mathbf{E}\right)$, coincides with that in (32) and gives the flow payoff adjustments needed to target long run population reallocations

[20] This may be countered by redistributive considerations. If the policymaker has particular concern for those workers in hp and little concern for those in de, it may perceive this to be tuning to be costly, discouraging it from a lower tax on ss.

on particular states in the best fit Calvo-logit model. As before it implies that to induce population movement towards i, flow payoff must be increased at i by $\nu/\mathbf{P}(i)$ and reduced at each $j \neq i$ by $-\nu$. The remaining terms in (37) give first order approximations to the payoff tuning necessitated by deviations from best fit Calvo-logit. These terms are readily interpretable:

1. $\mathbf{\Delta}_{\mathbf{S}}^{\top}(i,j) = \mathbf{\Delta}_{\mathbf{S}}(j,i) > 0$, $j \neq i$, indicates that j is a relatively close short run substitute to i and that extra lifetime payoff at i disproportionately attracts workers from j. To ensure that j is not excessively depleted and a long run unit population reallocation to i occurs some extra lifetime and, hence, flow payoff is needed at j relative to the Calvo-logit benchmark. The converse is true if $\mathbf{\Delta}_{\mathbf{S}}(j,i) < 0$.

2. $\mathbf{\Delta}_{\mathbf{Q}}^{\top}(i,j) = \mathbf{\Delta}_{\mathbf{Q}}(j,i) > 0$ indicates that outflows to j from i are relatively large. To deter a build up of workers in j and ensure that a long run unit population reallocation to i occurs, lifetime and, hence, flow payoff in j must be depressed relative to the benchmark.

3. $\mathbf{\Delta}_{\boldsymbol{\mathcal{Q}}}^{\top}(i,j) = \mathbf{\Delta}_{\boldsymbol{\mathcal{Q}}}(j,i) > 0$ indicates that surviving workers have a higher probability of flowing to j from i relative to the benchmark making (via the envelope condition) long run payoffs more sensitive to payoff at j. Less extra flow payoff at j is needed to induce an additional unit of long run payoff at i.

The best fit Calvo-logit model implies optimal taxes of the form (33). Approximate deviations in optimal taxes from these are generated by the societal "pricing" of the payoff tuning terms in (37) using C. Results are most transparent under a slight variation of the assumptions used in the Calvo-logit setting: $u = a \log$ and $\frac{\partial \mathbf{M}}{\partial \mathbf{c}} = \mathbf{D}_{\partial \mathbf{u}} \mathbf{P}$.[21] Combining (37) with (30) delivers the approximation:

$$\mathbf{T} \approx \frac{1}{1+b} \hat{\mathbb{E}}_{\mathbf{P}}[\mathbf{w}] + \left(\frac{1}{1+b} \right)^2 \left\{ \mathbf{\Delta}_{\mathbf{S}}^{\top} - \mathbf{\Delta}_{\mathbf{Q}}^{\top} - \mathbf{\Delta}_{\boldsymbol{\mathcal{Q}}}^{\top} \right\} \hat{\mathbb{E}}_{\mathbf{P}}[\mathbf{w}] + G\mathbf{1}, \tag{38}$$

with $b = a/\nu$. The linearity of taxes in incomes found in (35) is disrupted by variations in the rows of $\mathbf{\Delta}_{\mathbf{S}}^{\top} - \mathbf{\Delta}_{\mathbf{Q}}^{\top} - \mathbf{\Delta}_{\boldsymbol{\mathcal{Q}}}^{\top}$.

Consider a "stepping stone" state i that attracts workers from low income states following a payoff increment ($\mathbf{\Delta}_{\mathbf{S}}(j,i) > 0$ for j such that $\hat{\mathbb{E}}_{\mathbf{P}}[\mathbf{w}](j) < 0$) and propagates them onto high income states ($\mathbf{\Delta}_{\mathbf{Q}}(k,i) > 0$ for k such that $\hat{\mathbb{E}}_{\mathbf{P}}[\mathbf{w}](k) > 0$). Equation (38) indicates that taxes will then be depressed at i relative to the Calvo

[21] This is equivalent to assuming $\frac{\partial \mathbf{M}}{\partial \mathbf{c}} = \boldsymbol{\lambda}^{\top} \frac{\partial \mathbf{V}}{\partial \mathbf{c}}$, with $\boldsymbol{\lambda} = \frac{1}{1-\beta}(\mathbf{I} - \beta \boldsymbol{\mathcal{Q}})\mathbf{P}$ and so coincides with utilitarianism (with respect to lifetime payoffs) if $\mathbf{Q} = \boldsymbol{\mathcal{Q}}$ and there is no entry and replacement.

benchmark since the corresponding elements of $\mathbf{\Delta}_{\mathbf{S}}^{\top} - \mathbf{\Delta}_{\mathbf{Q}}^{\top} - \mathbf{\Delta}_{\mathcal{Q}}^{\top}$ correlate negatively with those of $\hat{\mathbb{E}}_{\mathbf{P}}[\mathbf{w}]$. Intuitively, to ensure a unit population shift to i, a relative increase in payoff at upstream states j and reduction at downstream states k must occur. Since workers at upstream states have low consumptions and high marginal utilities and workers at downstream states the reverse, this payoff adjustment releases resources and, hence, lowers the cost of inducing a long run population shift to i. At the optimum the fiscal benefits of such a shift and, hence, taxes at i must be correspondingly lower.

Mistaking short for long run responses In static models, the optimal tax equation (30) reduces to: $\mathbf{T} = \text{Cov}(\mathbf{B}, \mathbf{C}) - G\mathbf{1}$, with $\mathbf{B} = (\mathbf{I} - \mathbf{S}^{\top})^{\#}\mathbf{D}_{\mathbf{P}}{}^{-1}$. Dynamic considerations introduce the reverse propagation matrix $\mathbf{A}$ that converts short into long run population reallocation costs. A policymaker that mistook short for long run sensitivities and implemented optimal taxes satisfying $\mathbf{T} = \text{Cov}(\mathbf{B}, \mathbf{C}) + G\mathbf{1}$ might regard themselves as having balanced the costs and benefits of population shifts and as having attained an optimum. However, such an assessment omits the long term implications of post-substitution propagation. These implications are contained within the reverse propagation matrix $\mathbf{A}$. If states are highly persistent, $\mathbf{A}$ is close to zero indicating that small short run population adjustments are sufficient to induce unit long run shifts. The overall social marginal cost of inducing a long run population shift is correspondingly reduced. At an optimum, equality of marginal costs and benefits implies that optimal tax variation around the mean G is compressed (and much smaller than would be inferred from consideration of $\text{Cov}(\mathbf{B}, \mathbf{C})$ alone).

Connecting to data Quantitative evaluation of $\mathbf{\Omega} = \mathbf{AB}$ at empirical or counterfactual equilibria proceeds analogously to that for $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$. The reverse propagation matrix $\mathbf{A}$ is immediately recovered from transition data at an empirical equilibrium. A dynamic logit model (with or without perpetual youth) relates $\mathbf{S}$ to $\mathbf{Q}$. It thus permits evaluation of $\mathbf{B}$ using transition data. Vector $\mathbf{C}$ is obtained after specification of a social criterion and estimation of parameters describing marginal utilities. Assembling the elements then gives the cost vector $\text{Cov}(\mathbf{\Omega}, \mathbf{C}) = \mathbf{\Omega D_P C}$ at the equilibrium. Costs $\text{Cov}(\mathbf{\Omega}, \mathbf{C})$ are obtained at a counterfactual consumption allocation $\mathbf{c}$ after recovery of the counterfactual $\mathbf{Q}$ and $\mathbf{P}$ via dynamic hat algebra.

5 Persistent Private Heterogeneity

Various elaborations of our model are possible. Here we focus on one: Persistent private heterogeneity.[22] As described in Remark 5, our model of worker choice readily incorporates persistent worker states (or types) that evolve exogenously. Following that remark, partition a worker's state into chosen x and type α components: $i = (x, \alpha) \in \mathcal{I} = \mathcal{X} \times \mathcal{A}$, where $\mathcal{X}$ and $\mathcal{A}$ have cardinalities n_X and n_A, respectively. Let ρ denote the Markov transition over exogenous states and ρ_0 the initial distribution of entrants over these states.

We now suppose that a policymaker cannot condition a worker's consumption on its type: $\mathbf{c}(x, \alpha) = \mathbf{c}(x)$. Underlying this are two assumptions. First, that a worker's type α affects its costs of attaining or remaining in a choice state x, but not the pre-tax income it receives from x. Second, that the policymaker cannot condition taxes directly on α for informational (or, perhaps, other) reasons. In this sense there is persistent private heterogeneity. The formula for stationary distribution responses (22) continues to apply, with $\mathbf{P}$ and $\mathbf{Q}$ defined on $\mathcal{I} = \mathcal{X} \times \mathcal{A}$ as in Remark 5. The interpretation of the components of this formula as representing short run responses and their propagation remains valid. However, the informational restriction on policy design implies that the optimal tax equation is modified as:

$$\underbrace{\mathbf{D}_{\overline{\mathbf{P}}}^{-1} \frac{1}{\Upsilon} \frac{\partial \mathbf{M}}{\partial \mathbf{c}}^{\top}}_{\text{MSW}} = \underbrace{\mathbf{1} - \mathbf{D}_{\overline{\mathbf{P}}}^{-1} \frac{\partial \mathbf{P}}{\partial \mathbf{c}}^{\top} \mathbf{N}^{\top} \mathbf{T}}_{1+\text{MEB}}, \tag{39}$$

where $\mathbf{N} = (\mathbf{I} \ \ldots \ \mathbf{I})$, comprised of n_A identity matrices each of dimension n_X, is an aggregator matrix that converts distributions over the complete state space $\mathcal{I}$ into marginals over the chosen state space $\mathcal{X}$ and $\overline{\mathbf{P}} = \mathbf{N}\mathbf{P}$ is the marginal of $\mathbf{P}$ over $\mathcal{X}$. $\mathbf{N}^{\top}$ functions as a distributor matrix that distributes vectors on the chosen state space into ones on the complete state space. Quantitative implementation of the formulas in Proposition 2 and (39) for this case requires disentangling the Markov chain (ρ, ρ_0) and the components $\mathcal{Q}$ and $\mathcal{P}_0$ from data on steady state flows between states over different time horizons. In Appendix D.8 we do so for a simple case.[23]

The optimal tax equation can be recast in terms of the "aggregated Markov chain" over chosen states which the policymaker observes and conditions taxes on. Let

[22] In Appendix E, we consider stochastic ageing, transitions, and externalities.

[23] If α-types are observable to the econometrician, then ρ and ρ_0 and the chosen components of $\mathbf{Q}$ may be estimated non-parametrically from data. For types that are unobservable, we follow the simple case of Artuç et al. (2010). Kasahara and Shimotsu (2009) considers the case of unobservable permanent types with degenerate Markov transition $\rho = \mathbf{I}$.

$\overline{\mathbf{Q}}$ be the matrix of transition probabilities between chosen states at the stationary distribution:

$$\overline{\mathbf{Q}}(x', x) = \sum_{\alpha,\alpha' \in \mathcal{A}} \mathbf{Q}((x', \alpha'), (x, \alpha)) \frac{\mathbf{P}(x, \alpha)}{\overline{\mathbf{P}}(x)}.$$

It is immediate that $\overline{\mathbf{Q}}$ defines a Markov chain with stationary distribution $\overline{\mathbf{P}}$.[24] The argument of Proposition 1 applies and, for the ergodic case,

$$\frac{\partial \overline{\mathbf{P}}}{\partial \mathbf{c}} = (\mathbf{I} - \overline{\mathbf{Q}})^{\#} \overline{\boldsymbol{\Phi}} = \sum_{m=0}^{\infty} (\overline{\mathbf{Q}})^m \overline{\boldsymbol{\Phi}}, \tag{40}$$

where $\overline{\boldsymbol{\Phi}}(x', y) = \sum_{x \in \mathcal{X}} \frac{\partial \overline{\mathbf{Q}}(x', x)}{\partial \mathbf{c}(y)} \overline{\mathbf{P}}(x)$. The optimal tax equation can then be recast in terms of the aggregated chain as:

$$\mathbf{D}_{\overline{\mathbf{P}}}{}^{-1} \frac{1}{\Upsilon} \frac{\partial \mathbf{M}^{\top}}{\partial \mathbf{c}} = \mathbf{1} - \mathbf{D}_{\overline{\mathbf{P}}}{}^{-1} \overline{\boldsymbol{\Phi}}^{\top} (\mathbf{I} - \overline{\mathbf{Q}}^{\top})^{\#} \mathbf{T}. \tag{41}$$

In this setting, $\overline{\boldsymbol{\Phi}}$ describes the response to consumption perturbations of steady state flows between observable states. However, it no longer describes an immediate response since such flows depend on the mix of types at each choice state, which takes time to adjust. In Appendix B.4, we use $\overline{\boldsymbol{\Phi}}$ to define a substitution matrix $\overline{\mathbf{S}}$ for this setting and, hence, obtain an inverted optimal tax equation that expresses optimal taxes in terms of marginal costs of inducing population adjustment across observable states.

6 Optimal tax design with dynamic occupational choice

This section describes a quantitative application of our framework to an occupational choice setting. The occupational choice margin is a natural candidate for our approach. Varied authors have found evidence that a major component of human capital is occupation specific and have identified occupational variation as an important determinant of steady state earnings variation.[25] Further, training and relocation costs make adjusting occupations expensive and occupational choice inherently dynamic. Such costs can delay occupational adjustment, divorcing long from short-run responses to tax change.

[24] However, the chain defined by $\overline{\mathbf{Q}}$ does *not* describe the evolution of workers over observable states. See Appendix B.4 for discussion.

[25] See, inter alia, Cortes and Gallipoli (2018), Kambourov and Manovskii (2009). The model that we quantify is related to those used in the trade literature to evaluate sectoral and occupational labor supply and wage responses to trade shocks.

6.1 Data and estimation

Model We estimate an occupational choice model similar to the benchmark model described in Section 2. The state space $\mathcal{I}$ is identified with a set of I occupations. A perpetual youth structure accommodates the occupational churn associated with retirement and replacement and implies greater mobility earlier in life.[26] A worker's effective discount factor is given by $\beta = b(1-\delta)$, with $1-\delta$ the survival probability. Workers' per period preferences net of Gumbel shocks are set to $u(\mathbf{c}(i)) = a \log \mathbf{c}(i)$. The $\boldsymbol{\kappa}(j,i)$ cost values, $j \neq i$, are interpreted as combining occupational amenity differentials and the effort costs of retraining for and adjusting to a new occupation. We assume that each new generation of workers is distributed exogenously across occupations according to $\mathbf{P_0}$ and, hence, treat $\mathbf{P_0}$ as a parameter rather than an equilibrium object.[27] A Cobb-Douglas production function over occupations of the form $F(\mathbf{p}) = A \prod_{i=1}^{I} \mathbf{p}(i)^{\phi(i)}$, $\sum_{i=1}^{I} \boldsymbol{\phi}(i) = 1$, is assumed.[28] We initially consider a consolidated model in which workers are distinguished by an exogenously given birth state occupation and realized Gumbel shocks. We then consider versions in which these differences are augmented by permanent, observable educational types generated by entrants in an initial choice. In Appendix D.8, we consider persistent hidden mobility types.

Method The parameters $\{a, \beta, \delta\}$ are structurally sufficient for optimal tax analysis. We set β and δ to values from the literature. Our estimation procedure for a exploits the inversion and finite dependence ideas of Hotz and Miller (1993) and is similar to Artuç et al. (2010)'s development and implementation of those ideas.[29] We obtain a stationary $\mathcal{Q}$ via a smoothed non-parametric estimator applied to transition data for non-entrants and set $\mathbf{P_0}$ to the distribution of twenty-five year olds over occupations. From these an estimated transition $\mathbf{Q}$ is obtained. Details of these steps are provided in Appendix D. Estimates of the structurally sufficient parameters $\{a, \beta, \delta\}$ are combined with an estimated steady state $\mathbf{Q}$, $\mathbf{c}$ and $\mathbf{P_0}$ to recover counterfactual $\mathbf{Q}$ and $\mathbf{P}$ via dynamic hat algebra. See Appendix C for details. Long-run stationary distribution sensitivities can then be obtained using our formulas. Cobb Douglas

[26]Our model implies that mobility declines with age as workers move from lower paying entry occupations to higher paying ones. However, we abstract from occupational learning, which augments early in life mobility.

[27]We relax this assumption in Subsection 6.4.

[28]We consider implications of alternative degrees of substitutability between occupations in the appendix.

[29]Some restrictions on adjustment costs are needed for identification. Relative to Artuç et al. (2010) we adopt a more flexible time invariant cost structure that relates costs of adjustment to multi-dimensional required skill differentials across occupations.

production function parameters ϕ are selected to be consistent with occupational income shares. GDP is normalized to one and the TFP parameter A set accordingly.

Data Our primary data source for estimation of the empirical Q and a is the March Supplement of the Current Population Survey (ASEC-CPS) for the years from 2003 to 2020. We restrict our sample to full-time wage-earners aged 25 to 65. Further details of the sample selection are deferred to Appendix D.1. We aggregate 2010 Census occupation codes into 2-digit SOC occupations, minus Farming, fishing, and forestry occupations.[30] Along with their occupation, responders provide information on their pre-tax wage income. We use TAXSIM of NBER to estimate federal and state income taxes and calculate after-tax incomes. A valuable aspect of ASEC-CPS is the inclusion of information on occupational transitions: Responders report their current occupation and the occupation they held last-year. This permits non-parametric estimation of the survivor Markov transition matrix $\mathcal{Q}$.

Calibration and Estimation Results Following Heathcote et al. (2017), we set $b = .96$ and $\delta = .029$ implying an effective discount rate of $\beta = .93$. The key additional parameter requiring estimation is a. Our procedure (elaborated in Appendix D) uses a smoothed non-parametric estimator of the survivor transition matrix $\mathcal{Q}$ and a parametric specification of the cost of adjustment structure. Estimates of a show some sensitivity to choice of smoothing procedure and cost structure. We give results for several configurations in Table 1. We obtain a baseline value, $a = 0.142$,

Cost Function (κ)	Interpolation Method ($\mathcal{Q}$)	Estimates of (a)
Type-I	Imputation	0.207(0.074)
Type-II	Imputation	0.180(0.066)
Type-I	PPML	0.088(0.038)
Type-II	PPML	0.092(0.045)

Table 1: Estimation of a parameter.

Notes: The left column shows the type of cost function. Types are explicitly defined in Appendix D. The middle column indicates the smoothing procedure used. Imputation replaces zero transition flows in the data with small positive constants and adjusts other elements down to ensure columns of the transition matrix sum to one. PPML indicates that the transition matrix is smoothed using the Poisson pseudo-maximum likelihood method. The right column shows the estimates of a.

by averaging these estimates. This value is within the 95% confidence interval of all

[30] The two digit level decomposition represents a compromise between reliable estimation of flows between occupations, which argues for greater aggregation, and isolation of differently paying options for workers, which motivates a finer decomposition. Management is a large occupation at the three digit level, which encompasses an array of differently paying roles. We present additional results in Appendix D in which we further disaggregate the management occupation.

estimations reported in Table 1. In Appendix D, we undertake sensitivity analysis around this baseline value. G is set to \$19,804 (2019 dollars).

6.2 Elasticities, Marginal Excess Burdens at the data allocation

Short and long run distributional elasticities Given the selected values of β and δ and the estimates of a, $\mathcal{Q}$, P_0 and, hence, $\mathbf{Q}$, we compute the model-implied short and long-run elasticity matrices $\mathbf{\Psi}\mathbf{D_c}$ and $\frac{\mathbf{c}}{\mathbf{P}}\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ for the empirical allocation using (22). We summarize these results below.[31] We find stationary long-run own elasticities that are an order of magnitude greater in absolute value than short-run elasticities. A 1% permanent increase in consumption at an occupation implies an approximate 0.1% to 0.16% change in the fraction of workers at that occupation over a year in most cases. Cross occupation responses are smaller still at between -0.00% and -0.03%. However, in the long-run, a 1% consumption increment induces own fraction changes of between 1.12% and 1.59%, while cross occupation responses vary between 0.01% and -0.23%. All cross elasticities in the short and most in the long-run are negative implying that a consumption increment in one reduces population in most others regardless of time horizon and that they are substitutes.[32]

Short and long run marginal excess burdens Figure 3 reports the marginal excess burdens $-\mathbf{D_P}^{-1}\mathbf{\Phi}^{\top}\mathbf{T}$ (blue) and $-\mathbf{D_P}^{-1}\frac{\partial \mathbf{P}}{\partial \mathbf{c}}^{\top}\mathbf{T}$ (red) at short and long horizons and at the prevailing empirical allocation. In the figure, occupations are ordered (right to left) by income. We retain this ordering in all successive figures.[33] The marginal

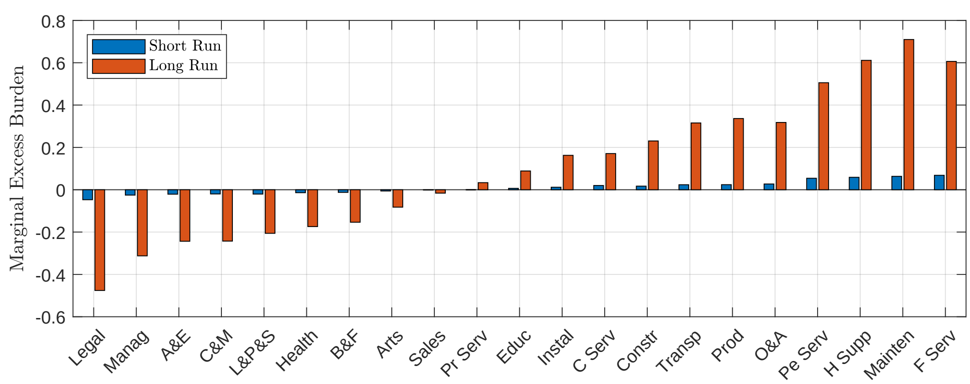


Figure 3: Short and Long Run Marginal Excess Burdens in the Data

[31]Full results are displayed in Appendix D.
[32]Health and personal services are exceptions. They are weak long run complements.
[33]Occupational labels in this and later figures are abbreviated. Full occupational labels are reported in Table D.1 in the appendix.

excess burden values in Figure 3 give the additional short and long-run per capita revenues induced by behavioral adjustment when the policymaker delivers a dollar of consumption to workers at each occupation. In the short-run the marginal excess burdens are close to zero indicating small behavioral revenue consequences. An extra dollar of consumption delivered to a worker in the highest paid and highest taxed legal occupation costs $0.95 in the short-run because some workers migrate to legal from lower earning and less taxed occupations or are retained by legal generating a small offsetting revenue gain of $0.05. Conversely, an extra dollar delivered to a worker in the low paid and low taxed maintenance occupation costs $1.06. The additional cost comes from a small short-run population increase in this low tax occupation. Confronted with this evidence, a policymaker concerned with redistributing from higher to lower earning workers might judge the incentive costs to be small and be encouraged to undertake a strongly redistributive reform across occupations.[34] However, the long-run marginal excess burdens are significantly larger. The long run cost of delivering resources to those in the legal occupation is $0.52 on the dollar, while the long run cost of delivering to those in maintenance is $1.71 on the dollar. A dollar taken from those working in the legal occupation and delivered to those in maintenance depresses revenues by $1.19 in the long-run.

The lowest paid occupation, food services, stands out. Since taxes are monotone in income and own elasticities are positive, a force is introduced for marginal excess burdens that fall with the income-ranking of an occupation. This suggests that the marginal excess burden should be highest in the food service occupation since a dollar delivered to those in this occupation draws workers towards the lowest income and tax payment. The short run marginal excess burden in food services is highest, but this is not the case in the long run, when the marginal excess burden is lower than in the higher earning maintenance and health service occupations. The relatively low long run marginal excess burden at the food services occupation is driven by relatively low long run own and (in absolute value) cross semi-elasticities $\frac{1}{\mathbf{P}}\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$. Our long run substitution formula (19) attributes this to relatively low retention by food services that dampens long run accumulation of workers in this occupation.

[34] This evidence only relates to the occupational choice margin. The policymaker may be concerned with distortion to the hours margin or other margins along which short run adjustment is possible. However, the empirical public finance literature has emphasized low compensated elasticities of labor supply and attributed larger short run responses to timing and evasion. It argues that the latter are best confronted by broadening the tax base and removing avoidance opportunities rather than lowering tax rates. See Saez et al. (2012).

6.3 Optimal tax results

Welfare criteria Saez (2002) advocates for direct specifications of marginal social welfare weights that encode transparently the policymaker's taste for local redistribution at alternate consumption allocations. We first follow this approach and set optimal marginal social welfare weights equal to $\frac{1}{\Upsilon}\frac{\partial u(\mathbf{c})}{\partial c} = \frac{a}{\Upsilon}\frac{1}{\mathbf{c}}$. This choice aligns with Saez (2002)'s specification in a static occupational choice setting. It also implies a direct inverted relationship between consumption allocations and marginal excess burdens at the optimum since then from (17):

$$\frac{\mathbf{c}}{E[\mathbf{c}]} = \frac{1}{1 + \text{MEB}}. \tag{42}$$

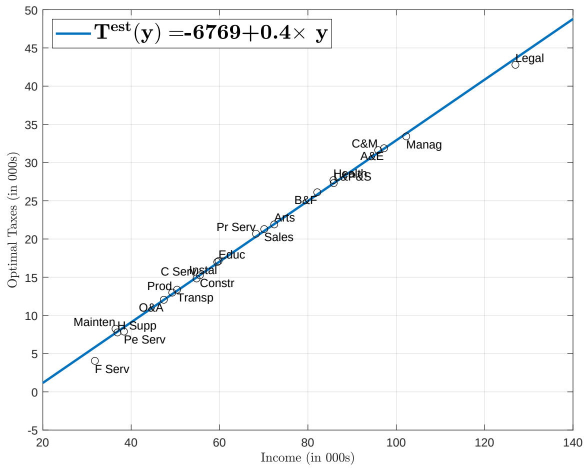
In the macroeconomics literature, criteria such as $\mathrm{M}(\mathbf{c}) = \mathbf{P_0}^\top \mathbf{V}(\mathbf{c})$, with $\frac{\partial \mathbf{M}}{\partial \mathbf{c}} = \mathbf{P_0}^\top \frac{\partial \mathbf{V}}{\partial \mathbf{c}}$, or $\mathrm{M}(\mathbf{c}) = \mathbf{P}(\mathbf{c})^\top \mathbf{V}(\mathbf{c})$, with $\frac{\partial \mathbf{M}}{\partial \mathbf{c}} = \mathbf{P}^\top \frac{\partial \mathbf{V}}{\partial \mathbf{c}} + \mathbf{V}^\top \frac{\partial \mathbf{P}}{\partial \mathbf{c}}$, are frequently employed. These alternatives tilt the social weighting towards, respectively, lower consumptions encountered earlier in life or higher consumptions encountered later in life.[35] We consider alternative welfare criteria later.

Optimal taxes under the benchmark welfare criterion Under our benchmark welfare specification given the parameter values described above and utilizing the argument in Appendix C to calculate counterfactual transitions and state distributions, equation (17) is solved to yield long-run stationary optimal consumption, wage, and tax allocations.

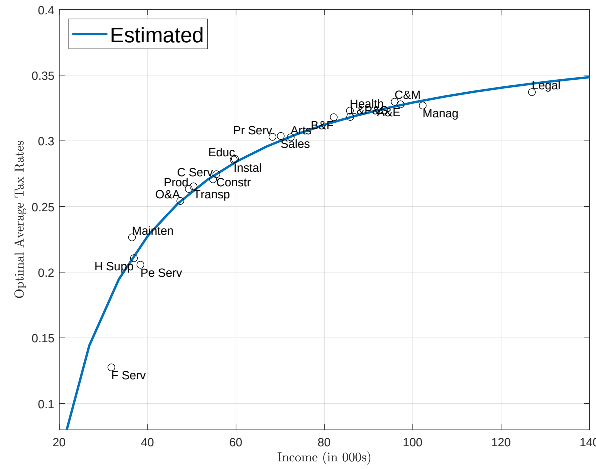
Figure 4 illustrates optimal and actual taxes and average income tax rates by occupation. Actual values are averaged within occupation and describe the effective occupational tax schedule in the U.S.. The optimal tax code is well approximated by an affine income tax (with the partial exception of food services, which is discussed below). This approximated code features an intercept of -$6,769 and a slope of 0.4 and, hence, is equivalent to a deduction of $17,059 and a flat marginal tax rate of 0.4.[36] It is close to (but slightly more redistributive) than the actual effective occupational income tax code which is well approximated by an affine function with intercept -$6,594 (deduction: $16,739) and marginal tax rate 0.39. Thus, the large long run distribution sensitivities and marginal excess burdens obtained at the data allocation (and also found at the optimum, see Appendix D for tables of optimal

[35] Our benchmark criterion can be formulated as a weighting over lifetime utilities $\boldsymbol{\lambda}^\top \mathbf{V}$, with $\boldsymbol{\lambda} = \left\{\left(1 - \frac{\beta}{1-\beta}\frac{\delta}{1-\delta}\right)\mathbf{P} + \left(\frac{\beta}{1-\beta}\frac{\delta}{1-\delta}\right)\mathbf{P_0}\right\}$ fixed in the optimization.

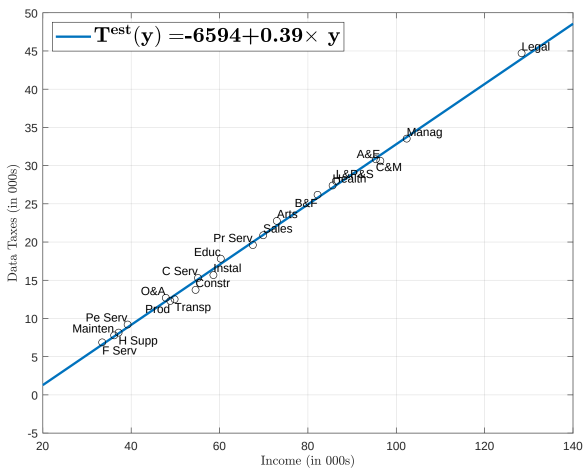
[36] Here and subsequently all dollar amounts are in 2019 US dollars.



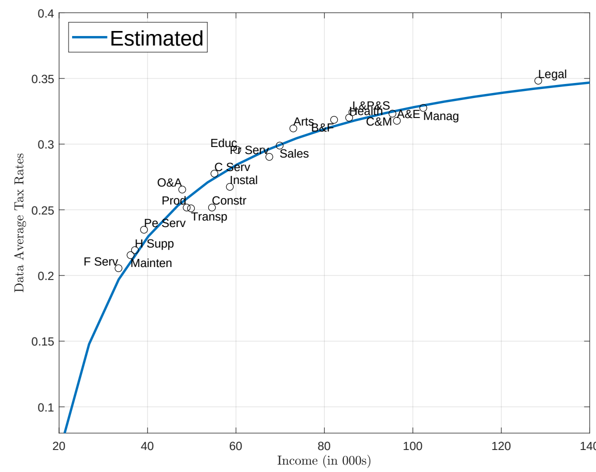
(a) Taxes: Optimal



(b) Average Income Tax Rates: Optimal



(c) Taxes: Data



(d) Average Income Tax Rates: Data

Figure 4: Taxes and Average Income Tax Rates by Occupation at the Optimum and in the Data. Benchmark welfare criterion. Incomes and tax payments in 2019 USD.

values) rationalize an optimal occupational tax function not very distant from actual under the selected welfare criterion. As noted, the low short run elasticities and marginal excess burdens would, in contrast, suggest much greater potential for reform of actual taxes at the criterion.

The food services occupation again emerges as a moderate outlier. The average tax rate paid by workers in this occupation is about 6% below that implied by the approximately optimal affine code and about 8% below that paid in the data. This amounts to a reduction in tax of about \$1,790 or an additional tax deduction of \$4,511 for food services workers. Again, this is underpinned by the relatively greater churn of workers through food services. This underpins lower own and cross elasticities and marginal excess burden values for this occupation relative to other low paying ones and rationalizes correspondingly lower taxes.[37]

Optimal taxes through the lens of the inverted optimal tax formula Table 2 shows the costs of inducing long and short run unit population movements towards selected occupations in thousands of 2019 dollars at the optimum.

Occupation	Long Run Cost	Short Run Cost
Legal	22.75	180.68
Management	13.39	129.52
Healthcare practitioner and technical	7.67	75.31
Business and financial	6.07	58.73
Education, training, and library	−2.93	−32.05
Installation, maintenance, and repair	−3.05	−34.89
Construction and extraction	−4.78	−52.88
Transportation and material moving	−6.67	−71
Office and administrative support	−7.99	−74.6
Personal care and service	−12.15	−90.7
Healthcare support	−12.27	−101.54
Building and grounds cleaning and maintenance	−11.78	−104.53
Food preparation and serving	−15.98	−114.37

Table 2: Long and short run costs of inducing population movement (selected occupations).

From (30) the long run cost is exactly the optimal tax variation $\mathbf{T} - G\mathbf{1} = \mathrm{Cov}(\mathbf{\Omega}, \mathbf{C})$ around the mean. Table 2 indicates that in most cases short run costs are an order of magnitude larger than their long run counterparts in absolute value. Intuitively, low short run population elasticities necessitate large consumption reallocations to induce population movement towards a target occupation. These, in turn, imply large redistributive costs or benefits depending on whether the reallocation is directed towards a high or low paid occupation. In contrast much larger long run elasticities of population adjustment require much smaller consumption reallocations to induce long run population movements towards target occupations.

6.4 Optimal Taxation by Educational Group

A concern with the benchmark perpetual youth model and the previous results is that all exogenous heterogeneity is attributed to initial occupation and transitory

[37]For example, the optimal long run own semi-elasticity in food services (2.77) falls sharply relative to higher paid maintenance (3.57).

shocks. Different persistent exogenous worker types may exhibit different mobility patterns and may merit different tax treatments (if they can be identified by the policymaker). A natural starting point is to use education to proxy persistent worker skill types.[38] This is problematic, however, as education is in part a choice and policy contingent on education may distort that choice. Consequently, we extend our benchmark model to allow entrant workers an education and initial occupation choice contingent on a shock draw. In subsequent periods, surviving workers select only occupations, but with payoffs and a cost of adjustment that depend on their education. Thus, in these periods, education functions as a "type". We assume that the policymaker can implement education and occupational contingent taxes and consumption allocations and investigate the extent to which different mobility patterns of education types (and different entrant choices) shape policy.

Extended model with an educational choice Formally, the worker state space is extended to include two components, $i = (o, s) \in \mathcal{O} \times \mathcal{S} = \mathcal{I}$, with $o \in \mathcal{O} = \{1, \ldots, O\}$ an occupation and $s \in \mathcal{S} = \{1, \ldots, S\}$ an education type. Entrants select a pair (o, s) and have expected maximized value:

$$\boldsymbol{\mathcal{K}}_0[\mathbf{V}(\mathbf{c})] := E\left[\max_{s\in\mathcal{S}, o\in\mathcal{O}}\{\mathbf{V}(\mathbf{c}|s)(o) - \boldsymbol{\kappa}_0(o, s) + \boldsymbol{\varepsilon}(o, s)\}\right], \tag{43}$$

with $\mathbf{c} \in \mathbb{R}^I_+ = \mathbb{R}^{OS}_+$, $\boldsymbol{\varepsilon}(o, s)$ a Gumbel preference shock and $\boldsymbol{\kappa}_0$ an entrant cost function. The entrant choice distribution is:

$$\boldsymbol{\mathcal{P}}_0(\mathbf{V}(\mathbf{c}))(o, s) = \frac{\boldsymbol{\omega}_0(o, s)\exp^{\mathbf{V}(\mathbf{c}|s)(o)}}{\sum_{o'\in\mathcal{O}, s'\in\mathcal{S}}\boldsymbol{\omega}_0(o', s')\exp^{\mathbf{V}(\mathbf{c}|s')(o')}}, \quad \text{with} \quad \boldsymbol{\omega}_0(o, s) := \exp^{-\boldsymbol{\kappa}_0(o,s)}. \tag{44}$$

Survivors can update their occupations, but not their educations. The latter, however, influence payoffs and occupational costs of adjustment. Survivor payoffs evolve as: $\mathbf{V}(\mathbf{c}|s) = u(\mathbf{c}, s) + \beta\boldsymbol{\mathcal{K}}[\mathbf{V}(\mathbf{c}|s), s]$, with $u(\mathbf{c}, s) = \{u(\mathbf{c}(o, s), s)\}_{o\in\mathcal{O}}$ and for each $s \in \mathcal{S}$:

$$\forall o \in \mathcal{O}, \qquad \boldsymbol{\mathcal{K}}[\mathbf{v}(s), s](o) := E\left[\max_{o'\in\mathcal{O}}\{\mathbf{v}(o', s) - \boldsymbol{\kappa}(o', o, s) + \boldsymbol{\varepsilon}(o')\}\right]. \tag{45}$$

[38] In Appendices B.4.3 and D.8, we pursue a different approach. We consider a special case of the persistent type cases considered in Section 5. Specifically, we augment the baseline model with latent mobility types. A worker is either mobile or immobile. If mobile, it is able to re-optimize its work state; if immobile it cannot. A Markov chain describes the evolution of types over choice states. This model is consistent with occupational mobility rates that are decreasing in tenure as documented in Seo and Oh (2023) for sectors. Contingent on the utility parameter a, this model has similar long run optimal tax implications to our benchmark.

With Gumbel shocks, the corresponding education-specific Markov matrix is:

$$\mathcal{Q}[\mathbf{V}(\mathbf{c}|s), s](o', o) = \frac{\boldsymbol{\omega}(o', o, s) \exp^{\mathbf{V}(\mathbf{c}|s)(o')}}{\sum_{r \in \mathcal{O}} \boldsymbol{\omega}(r, o, s) \exp^{\mathbf{V}(\mathbf{c}|s)(r)}},$$

where $\boldsymbol{\omega}(o', o, s) := \exp^{-\boldsymbol{\kappa}(o', o, s)}$.

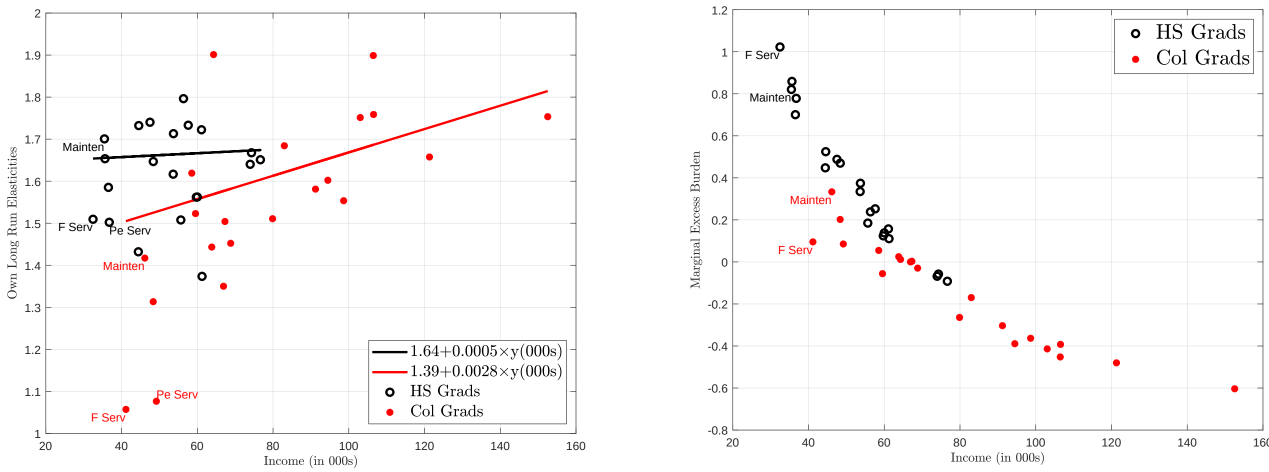
The Cobb-Douglas technology is modified to accommodate different educational types: $F(\mathbf{p}) = A \prod_{o \in \mathcal{O}} \left(\sum_{s \in \mathcal{S}} \boldsymbol{\psi}(o, s) \mathbf{p}(o, s) \right)^{\phi(o)}$, where $\mathbf{p}(o, s)$ and $\boldsymbol{\psi}(o, s)$ are, respectively, measures and productivities of workers over education and occupation. Implied equilibrium pre-tax earnings are then: $\mathbf{w}(o, s) = \boldsymbol{\psi}(o, s) \boldsymbol{\phi}(o) F(\mathbf{P}) / \sum_{s' \in \mathcal{S}} \mathbf{P}(o, s')$. The policymaker is assumed to observe and condition taxes on (o, s). In our empirical implementation of this framework, we assume two education choices: high school and college. Estimation and calculation of optimal taxes proceeds similarly to the benchmark case. See Appendix B.5.

Distributional elasticities and marginal excess burdens We begin by describing evaluations of our long run elasticity and marginal excess burden formulas at the empirical allocation. As in the benchmark case, long run elasticities are an order of magnitude larger than short run. However, there are marked differences in these elasticities across educational groups (see Figure 5a, which shows own long run elasticities for different occupation/educational combinations plotted against the average incomes of these combinations). In particular, at low paid occupations, college educated workers have lower own long run elasticities than high school workers (e.g. 1.057 (C) vs. 1.509 (HS) on food services, 1.076 (C) vs. 1.502 (HS) on personal services). This is underscored by low attraction of college educated entrants to low paid occupations and low retention at these occupations of college educated survivors.[39] Both forces suppress long run elasticities, the latter reducing long run relative to short run elasticities for college educated at low paid occupations.

On average, college educated workers have larger absolute pay and, hence, tax differentials across occupations than do the high school educated. Nonetheless, the low elasticities associated with low income occupations translate into relatively low long run marginal excess burdens for college workers in these occupations at the empirical educational/occupational allocation. See Figure 5b.

Optimal tax results Figure 6 shows optimal taxes under our benchmark welfare criterion. Overall, the optimal tax code is less redistributive than for the general

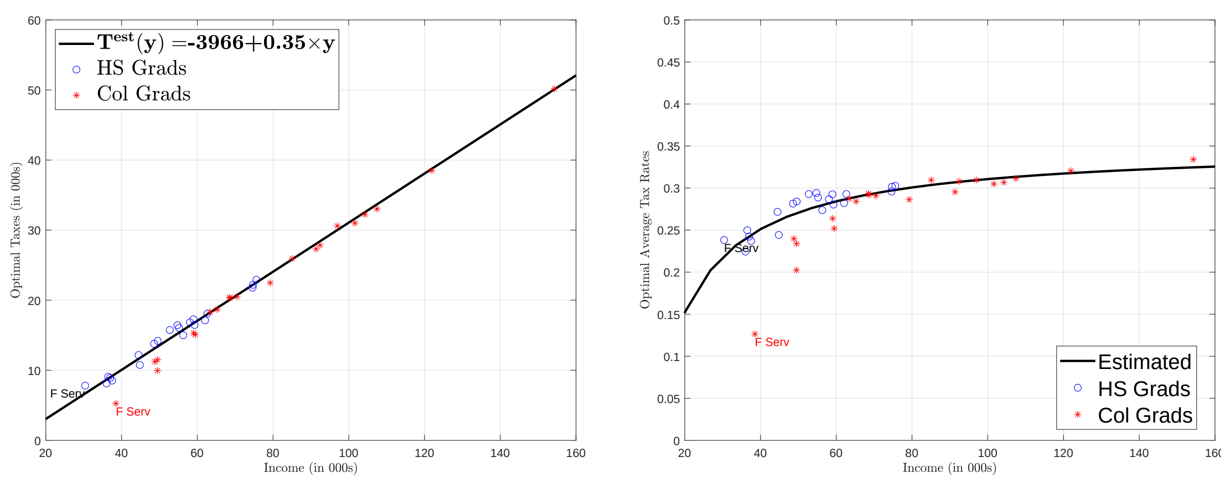
[39]For example, retention of college educated workers in food services is 0.8 versus 0.88 for high school educated workers.



(a) Long run own elasticities. Black ∘ are elasticities for high school educated workers; red • for college educated workers. Population-weighted best fit lines for high schoolers (–) and college educated (–).

(b) Marginal Excess Burdens. Black ∘ show elasticities for high school educated workers; red • for college educated workers.

Figure 5: Long run own elasticities and marginal excess burdens at the empirical allocation.



(a) Taxes: Optimal

(b) Average Income Tax Rates: Optimal

Figure 6: Taxes and Average Income Tax Rates by Occupation at the Optimum. Benchmark welfare criterion.

population, with the approximating optimal affine tax function featuring a reduced deduction of $11,321 and a lower marginal tax rate of 0.35. Although the tax code retains a broadly affine structure when taxes are plotted against income at different occupation/education type combinations, college educated workers in low paid occupations are outliers and pay low optimal taxes. Such tax treatments reflect low own elasticities for these populations both at the empirical allocation as described above and at the optimum.

7 Conclusion

Policymakers selecting tax designs for the long run must evaluate long run substitution patterns over income-generating activities. In equilibrium workers travel between work states as they pursue opportunities, are dislodged by shocks, or retire and are replaced. If tax variation shifts worker flows amongst states with different onward diffusion characteristics, then long run responses to tax variation differ from short run. In particular, long run responses exceed short run responses if retention by destination states is large and adjusted flows take time to accumulate in payoff enhanced states. Variation in short run substitution and post substitution diffusion motivates differential optimal tax treatment of states with similar incomes, but different accessibility and prospects. This paper develops interpretable formulas that capture these dimensions of behavior and that describe the long run response of distributions of workers over income-generating states to consumption variation. It integrates these formulas into optimal tax equations. In addition, it provides new "inverted" formulas that identify the long run social cost of targeting a proportional reallocation of workers onto a state as a sufficient statistic for that state's optimal tax. In benchmark logit and Calvo specifications with a utilitarian objective, such costs reduce to scaled reciprocals of marginal utilities of consumption net of mean. In more complex settings, the consumption reallocations needed to induce a proportional population reallocation must be "tuned" to suppress non-uniformities in substitution and diffusion.

The formulas developed in this paper are readily connected to data via established techniques in structural dynamic discrete choice estimation and the dynamic hat algebra approach of trade. The latter leverages information contained in observed transitions across work states and requires identification and estimation of only a small number of "structurally sufficient" preference parameters. We implement these procedures to explore the optimal taxation of occupations. We find that long-run occupational choice elasticities are an order of magnitude greater than their short-run counterparts and rationalize an optimal policy similar to the de facto affine-in-income occupational tax schedule prevailing in the U.S.. This affine form is augmented with relatively lower taxes for low income/high churn occupations.

We view our methods and approach as a complement to the rich empirical literature on the elasticity of taxable income, or to work in macro-public finance that has focused on the intensive hours margin, latent general or pre-career human capital formation. Our extension section and appendix indicates further dimensions for enriching and developing our analysis to accommodate unobserved persistent heterogeneity, incomplete tax systems, and transitions. We leave further development in these directions to future work.

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A Appendix: Main Proofs

Proof of Proposition 1 Let $\partial \mathrm{Q}$ denote a perturbation of Q such that $\mathrm{Q}+\partial \mathrm{Q}$ remains a transition matrix and let $\partial \mathrm{P}$ denote the corresponding perturbation of the stationary distribution P. From Golub and Meyer (1986), Theorem 3.2, $\partial \mathbf{P} = (\mathbf{I}-\mathbf{Q})^{\#}\partial \mathbf{F}$, where $\partial \mathbf{F} = \partial \mathbf{Q}\mathbf{P}$ denotes the vector of flow responses associated with the perturbation. Equation (18) follows immediately. By Lamond and Puterman (1989), p. 123, if Q is ergodic and, hence, aperiodic, $(\mathbf{I}-\mathbf{Q})^{\#} = \sum_{m=0}^{\infty}(\mathbf{Q}^m - \mathbf{\Pi}_{\mathbf{P}})$. Equation (19) then follows from (18) since $\mathbf{\Pi}_{\mathbf{P}}\mathbf{\Phi} = 0$. □

Proof of Lemma 1 By definition:

$$(\mathbf{I}-\mathbf{S})\mathbf{D_P} = \left[\mathbf{P}^\top \otimes \mathbf{I}\right] \frac{\partial(\text{vec } \mathbf{Q})}{\partial \mathbf{v}} = \delta \frac{\partial \boldsymbol{\mathcal{P}}_0}{\partial \mathbf{v}} + (1-\delta)\left[\mathbf{P}^\top \otimes \mathbf{I}\right] \frac{\partial(\text{vec } \boldsymbol{\mathcal{Q}})}{\partial \mathbf{v}},$$

and the i-th column of $(\mathbf{I}-\mathbf{S})\mathbf{D_P}$ gives the adjustment in population flows induced by an additional util in the i-th state. Since the overall population is constant, the total adjustment is zero. Hence, summing the elements in the i-th column gives $0 = \mathbf{P}(i) - \sum_{j\in\mathcal{I}} \mathbf{S}(j,i)\mathbf{P}(i)$ or $1 = \sum_{j\in\mathcal{I}} \mathbf{S}(j,i)$. Recall from Section 2 that $\boldsymbol{\mathcal{Q}}(j,k)$ and $\boldsymbol{\mathcal{P}}_0(j)$ are decreasing in $\mathbf{v}(i)$, $i \neq j$. Thus, an extra unit of payoff at i reduces the flow of workers to $j \neq i$, for each $j \in \mathcal{I}$, $\mathbf{S}(j,i) \geq 0$. It is possible that $\mathbf{S}(i,i) < 0$ for some i. However, $(\mathbf{I}-\mathbf{S})\mathbf{D_P}^{-1}\frac{\partial \mathbf{V}}{\partial \mathbf{c}} = t(\mathbf{I}-\mathbf{S})\mathbf{D_P}^{-1}\frac{1}{t}\frac{\partial \mathbf{V}}{\partial \mathbf{c}} = (\mathbf{I}-\{(1-t)\mathbf{I}+t\mathbf{S}\})\mathbf{D_P}^{-1}\frac{1}{t}\frac{\partial \mathbf{V}}{\partial \mathbf{c}}$. Thus, a rescaling of utilities (inclusive of costs and shocks) by $1/t \in [1,\infty)$, that leaves behavior unchanged, modifies $\mathbf{S}$ as: $\tilde{\mathbf{S}} = (1-t)\mathbf{I} + t\mathbf{S}$. Hence, it is always possible to renormalize utilities so that $\mathbf{S}$ is positive and Markov. Further, since, from Section 2, adding an extra util at all states leaves behavior unchanged, it follows that each row sum must also be zero: $\mathbf{0} = (\mathbf{I}-\mathbf{S})\mathbf{D_P}\mathbf{1}$. Hence, $\mathbf{P} = \mathbf{SP}$ and $\mathbf{P}$ is a stationary distribution of $\mathbf{S}$. For reversibility, recall that $\boldsymbol{\mathcal{Q}} = \frac{\partial \boldsymbol{\mathcal{K}}}{\partial \mathbf{v}}^\top$ and $\boldsymbol{\mathcal{P}}_0 = \frac{\partial \boldsymbol{\mathcal{K}}_0}{\partial \mathbf{v}}^\top$ and so $(\mathbf{I}-\mathbf{S})\mathbf{D_P} = (1-\delta)\frac{\partial^2(\boldsymbol{\mathcal{K}}^\top \mathbf{P})}{\partial \mathbf{v}^2} + \delta\frac{\partial^2(\boldsymbol{\mathcal{K}}_0^\top)}{\partial \mathbf{v}^2}$ and is the Hessian of $(1-\delta)\boldsymbol{\mathcal{K}}^\top\mathbf{P} + \delta\boldsymbol{\mathcal{K}}_0$. Thus, $(\mathbf{I}-\mathbf{S})\mathbf{D_P}$ and, hence, $\mathbf{SD_P}$ is symmetric. Then $\mathbf{SD_P} = \mathbf{D_P}^\top\mathbf{S}^\top$ and $\mathbf{S}$ is reversible. Finally, $\boldsymbol{\Psi} = \boldsymbol{\Xi}\frac{\partial \mathbf{V}}{\partial \mathbf{c}} = \mathbf{D_P}^{-1}(\mathbf{I}-\mathbf{S})\mathbf{D_P}\frac{\partial \mathbf{V}}{\partial \mathbf{c}} = (\mathbf{I}-(\overline{\mathbf{S}})^\top)\frac{\partial \mathbf{V}}{\partial \mathbf{c}} = (\mathbf{I}-\mathbf{S}^\top)\frac{\partial \mathbf{V}}{\partial \mathbf{c}}$, where the last equality uses the reversibility of $\mathbf{S}$. □

Lemma A.1 (Short run substitution). *There exists a function $\boldsymbol{\mathcal{H}} = \{\boldsymbol{\mathcal{H}}_k\}$ such that:*

$$\forall j,k, \qquad \mathbf{S}(j,k) = \mathbf{I}(j,k) - \frac{1}{\mathbf{P}(k)} \sum_{i\in\mathcal{I}} \boldsymbol{\mathcal{H}}_k(\mathbf{Q})(j,i)\mathbf{P}(i),$$

where $\boldsymbol{\mathcal{H}}$ depends only on the distribution over payoff shocks.

Proof. By definition: $\mathbf{S}(j,k) = \mathbf{I}(j,k) - \frac{1}{\mathbf{P}(k)}\sum_{i\in\mathcal{I}} \frac{\partial \mathbf{Q}(\mathbf{v})(j,i)}{\partial \mathbf{v}(k)}\mathbf{P}(i)$. For expositional simplicity, set aside perpetual youth and assume that payoff shocks are identically distributed at each state. Then: $\mathbf{Q}(\mathbf{v})(j,i) = \boldsymbol{\mathcal{Q}}(\mathbf{v})(j,i) = \tilde{\boldsymbol{\mathcal{Q}}}(\{\mathbf{v}(k) - \boldsymbol{\kappa}(k,i) - \mathbf{v}(I) + \boldsymbol{\kappa}(I,i)\}_{k=1}^{I-1})(j)$, where $\tilde{\boldsymbol{\mathcal{Q}}} : \mathbb{R}^{I-1} \to \mathcal{D}$, with $\mathcal{D}$ the set of probability distributions on $\mathcal{I}$. Hence, $\mathbf{S}(j,k) = \mathbf{I}(j,k) - \frac{1}{\mathbf{P}(k)}\sum_{i\in\mathcal{I}} \frac{\partial \tilde{\boldsymbol{\mathcal{Q}}}(\{\mathbf{v}(k)-\boldsymbol{\kappa}(k,i)-\mathbf{v}(I)+\boldsymbol{\kappa}(I,i)\}_{k=1}^{I-1})(j)}{\partial \mathbf{v}(k)}\mathbf{P}(i)$. By an argument in Hotz and Miller (1993), $\tilde{\boldsymbol{\mathcal{Q}}}$ is invertible. The result follows. □

Proof of Proposition 2 Substituting $(\mathbf{I}-\mathbf{Q})^\# = \sum_{m=0}^\infty\{\mathbf{Q}^m - \boldsymbol{\Pi}_\mathbf{P}\}$ into (18) and using the reversibility of $\mathbf{S}$ gives: $\frac{\partial \mathbf{P}}{\partial \mathbf{c}} = \sum_{m=0}^\infty\{\mathbf{Q}^m - \boldsymbol{\Pi}_\mathbf{P}\}(\mathbf{I}-\mathbf{S})\mathbf{D_P}\frac{\partial \mathbf{V}}{\partial \mathbf{c}} = \sum_{m=0}^\infty\{\mathbf{Q}^m - \boldsymbol{\Pi}_\mathbf{P}\}\mathbf{D_P}(\mathbf{I} - \mathbf{D_P}^{-1}\mathbf{SD_P})\frac{\partial \mathbf{V}}{\partial \mathbf{c}} = \sum_{m=0}^\infty\{\mathbf{Q}^m - \boldsymbol{\Pi}_\mathbf{P}\}\mathbf{D_P}(\mathbf{I}-\mathbf{S}^\top)\frac{\partial \mathbf{V}}{\partial \mathbf{c}}$. Separating out the first time and using the definition of $\hat{\mathbb{E}}_\mathbf{S}$ then yields: $\frac{\partial \mathbf{P}}{\partial \mathbf{c}} = \hat{\mathbb{E}}_\mathbf{S}\left[\frac{\partial \mathbf{V}}{\partial \mathbf{c}}\right] + \sum_{m=1}^\infty\{\mathbf{Q}^m - \boldsymbol{\Pi}_\mathbf{P}\}\mathbf{D_P}(\mathbf{I}-\mathbf{S}^\top)\frac{\partial \mathbf{V}}{\partial \mathbf{c}}$. Observing that the (j,k)-th element of: $\{\mathbf{Q}^m - \boldsymbol{\Pi}_\mathbf{P}\}\mathbf{D_P}(\mathbf{I}-\mathbf{S}^\top)\frac{\partial \mathbf{V}}{\partial \mathbf{c}}$ is an expectation (over i under $\mathbf{P}$) of $\{\mathbf{Q}^m(j,i) - \mathbf{P}(j)\}\left\{\frac{\partial \mathbf{V}(i)}{\partial \mathbf{c}(k)} - \mathbb{E}_\mathbf{S}\left[\frac{\partial \mathbf{V}(l)}{\partial \mathbf{c}(k)}\middle| i\right]\right\}$ and that since both terms in the product have a zero expectation under $\mathbf{P}$, this element equals the (j,k)-th term of $\text{Cov}\left(\mathbf{Q}^m, \hat{\mathbb{E}}_\mathbf{S}\left[\frac{\partial \mathbf{V}}{\partial \mathbf{c}}\right]\right)$. The right hand side covariance in (22) follows. □

Proof of Proposition 4 Begin with (17), $\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^\top = \mathbf{P} - \left(\frac{\partial \mathbf{P}}{\partial \mathbf{c}}\right)^\top \mathbf{T}$. Replacing $\mathbf{P}$ with $\mathbf{D}_{\mathbf{P_Q}}\mathbf{1}$ and substituting for $\left(\frac{\partial \mathbf{P}}{\partial \mathbf{c}}\right)^\top = \left(\frac{\partial \mathbf{V}}{\partial \mathbf{c}}\right)^\top \mathbf{D_P}(\mathbf{I}-\mathbf{S}^\top)(\mathbf{I}-\mathbf{Q}^\top)^\#$ gives: $\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^\top = \mathbf{D_P}\mathbf{1} - \left(\frac{\partial \mathbf{V}}{\partial \mathbf{c}}\right)^\top \mathbf{D_P}(\mathbf{I}-\mathbf{S}^\top)(\mathbf{I}-\mathbf{Q}^\top)^\#\mathbf{T}$. Reorganizing: $(\mathbf{I}-\mathbf{S}^\top)(\mathbf{I}-\mathbf{Q}^\top)^\#\mathbf{T} = \boldsymbol{\theta}$, where $\boldsymbol{\theta} := \mathbf{D_P}^{-1}\left(\frac{\partial \mathbf{V}}{\partial \mathbf{c}}^\top\right)^{-1}\mathbf{D_P}\mathbf{1} - \mathbf{D_P}^{-1}\left(\frac{\partial \mathbf{V}}{\partial \mathbf{c}}^\top\right)^{-1}\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^\top$. Since $\mathbf{S}$ is a Markov matrix, the group inverse $(\mathbf{I}-\mathbf{S}^\top)^\#$ of $\mathbf{I}-\mathbf{S}^\top$ exists and: $(\mathbf{I}-\mathbf{Q}^\top)^\#\mathbf{T} = (\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + \mathbf{n_S}$, where $\mathbf{n_S}$ is an element of the null space of $\mathbf{I}-\mathbf{S}^\top$. Since $\mathbf{S}$ is a Markov matrix $\mathbf{n_S} = g_S\mathbf{1}$ for some constant g_S. Using $(\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)(\mathbf{I}-\mathbf{Q}^\top)^\# = (\mathbf{I}-\mathbf{Q}^\top)^\#$, $(\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)(\mathbf{I}-\mathbf{S}^\top)^\# = (\mathbf{I}-\mathbf{S}^\top)^\#$ and $(\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)\mathbf{1} = 0$, we obtain: $(\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)(\mathbf{I}-\mathbf{Q}^\top)^\#\mathbf{T} = (\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)(\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + (\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)g_S\mathbf{1} \implies (\mathbf{I}-\mathbf{Q}^\top)^\#\mathbf{T} = (\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta}$. Next noting that $(\mathbf{I}-\mathbf{Q}^\top)$ is the group inverse of $(\mathbf{I}-\mathbf{Q}^\top)^\#$, we have: $\mathbf{T} = (\mathbf{I}-\mathbf{Q}^\top)(\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + \mathbf{n_Q}$, for some $\mathbf{n_Q}$ in the null space of $(\mathbf{I}-\mathbf{Q}^\top)^\#$. Recalling that $(\mathbf{I}-\mathbf{Q}^\top)^\# = \sum_{n=0}^{\infty}((\mathbf{Q}^\top)^n - \boldsymbol{\Pi}_\mathbf{P}^\top)$, we have that $\mathbf{n_Q} = g_Q\mathbf{1}$ for some constant g_Q, so that: $\mathbf{T} = (\mathbf{I}-\mathbf{Q}^\top)(\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + g_Q\mathbf{1}$. Next observe that: $\mathbf{P}^\top\mathbf{T} = \mathbf{P}^\top(\mathbf{I}-\mathbf{Q}^\top)(\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + g_Q\mathbf{P}^\top\mathbf{1} = g_Q$, where we use the fact that: $\mathbf{P}^\top(\mathbf{I}-\mathbf{Q}^\top) = 0$. Thus, $G = \mathbf{P}^\top\mathbf{T} = g_Q$. Hence, we have: $\mathbf{T} = (\mathbf{I}-\mathbf{Q}^\top)(\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + G\mathbf{1}$, which completes the proof. □

Proof of Proposition 4 Begin with (17), $\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^\top = \mathbf{P} - \left(\frac{\partial \mathbf{P}}{\partial \mathbf{c}}\right)^\top \mathbf{T}$. Replacing $\mathbf{P}$ with $\mathbf{D}_{\mathbf{P_Q}}\mathbf{1}$ and substituting for $\left(\frac{\partial \mathbf{P}}{\partial \mathbf{c}}\right)^\top = \left(\frac{\partial \mathbf{V}}{\partial \mathbf{c}}\right)^\top \mathbf{D_P}(\mathbf{I}-\mathbf{S}^\top)(\mathbf{I}-\mathbf{Q}^\top)^\#$ gives: $\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^\top = \mathbf{D_P}\mathbf{1} - \left(\frac{\partial \mathbf{V}}{\partial \mathbf{c}}\right)^\top \mathbf{D_P}(\mathbf{I}-\mathbf{S}^\top)(\mathbf{I}-\mathbf{Q}^\top)^\#\mathbf{T}$. Reorganizing: $(\mathbf{I}-\mathbf{S}^\top)(\mathbf{I}-\mathbf{Q}^\top)^\#\mathbf{T} = \boldsymbol{\theta}$, where $\boldsymbol{\theta} := \mathbf{D_P}^{-1}\left(\frac{\partial \mathbf{V}}{\partial \mathbf{c}}^\top\right)^{-1}\mathbf{D_P}\mathbf{1} - \mathbf{D_P}^{-1}\left(\frac{\partial \mathbf{V}}{\partial \mathbf{c}}^\top\right)^{-1}\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^\top$. Since $\mathbf{S}$ is a Markov matrix, the group inverse $(\mathbf{I}-\mathbf{S}^\top)^\#$ of $\mathbf{I}-\mathbf{S}^\top$ exists and: $(\mathbf{I}-\mathbf{Q}^\top)^\#\mathbf{T} = (\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + \mathbf{n_S}$, where $\mathbf{n_S}$ is an element of the null space of $\mathbf{I}-\mathbf{S}^\top$. Since $\mathbf{S}$ is a Markov matrix $\mathbf{n_S} = g_S\mathbf{1}$ for some constant g_S. Using $(\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)(\mathbf{I}-\mathbf{Q}^\top)^\# = (\mathbf{I}-\mathbf{Q}^\top)^\#$, $(\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)(\mathbf{I}-\mathbf{S}^\top)^\# = (\mathbf{I}-\mathbf{S}^\top)^\#$ and $(\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)\mathbf{1} = 0$, we obtain: $(\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)(\mathbf{I}-\mathbf{Q}^\top)^\#\mathbf{T} = (\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)(\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + (\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top)g_S\mathbf{1} \implies (\mathbf{I}-\mathbf{Q}^\top)^\#\mathbf{T} = (\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta}$. Next noting that $(\mathbf{I}-\mathbf{Q}^\top)$ is the group inverse of $(\mathbf{I}-\mathbf{Q}^\top)^\#$, we have: $\mathbf{T} = (\mathbf{I}-\mathbf{Q}^\top)(\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + \mathbf{n_Q}$, for some $\mathbf{n_Q}$ in the null space of $(\mathbf{I}-\mathbf{Q}^\top)^\#$. Recalling that $(\mathbf{I}-\mathbf{Q}^\top)^\# = \sum_{n=0}^{\infty}((\mathbf{Q}^\top)^n - \boldsymbol{\Pi}_\mathbf{P}^\top)$, we have that $\mathbf{n_Q} = g_Q\mathbf{1}$ for some constant g_Q, so that: $\mathbf{T} = (\mathbf{I}-\mathbf{Q}^\top)(\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + g_Q\mathbf{1}$. Next observe that: $\mathbf{P}^\top\mathbf{T} = \mathbf{P}^\top(\mathbf{I}-\mathbf{Q}^\top)(\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + g_Q\mathbf{P}^\top\mathbf{1} = g_Q$, where we use the fact that: $\mathbf{P}^\top(\mathbf{I}-\mathbf{Q}^\top) = 0$. Thus, $G = \mathbf{P}^\top\mathbf{T} = g_Q$. Hence, we have: $\mathbf{T} = (\mathbf{I}-\mathbf{Q}^\top)(\mathbf{I}-\mathbf{S}^\top)^\#\boldsymbol{\theta} + G\mathbf{1}$, which completes the proof. □

Lemma A.2 below provides an expansion for $\boldsymbol{\Omega}$ in terms of behavioral deviation from Calvo-logit matrices. It utilizes the expansion to obtain an approximation to $\boldsymbol{\Omega}$.

Lemma A.2. *Suppose a dynamic discrete choice equilibrium with transition* $\mathbf{Q}$*, transition for survivors* $\mathcal{Q}(\mathbf{V})$*, substitution matrix* $\mathbf{S}$*, and stationary distribution* $\mathbf{P}$*. Let* ψ *and* $\tilde{\mathbf{Q}}$ *define best fit Calvo-logit behavior to this equilibrium and let* $\boldsymbol{\Delta}_\mathbf{Q}$*,* $\boldsymbol{\Delta}_\mathbf{S}$*, and* $\boldsymbol{\Delta}_\mathcal{Q}$ *be as in* (36)*. Then:*

$$\boldsymbol{\Omega} = \nu(\mathbf{I}-\boldsymbol{\Delta}_\mathbf{Q}^\top)\Big(\mathbf{I}-\boldsymbol{\Pi}_\mathbf{P}^\top + \psi\sum_{n=0}^{\infty}(1-\psi)^n\sum_{m=1}^{\infty}\binom{m+n}{m}\psi^m(\boldsymbol{\Delta}_\mathbf{S}^\top)^m\Big)(\mathbf{I}-\boldsymbol{\Delta}_\mathcal{Q}^\top)\mathbf{D_P}^{-1}, \quad \text{(A.1)}$$

where $\binom{m+n}{m} = \frac{n+m!}{m!n!}$ *is a binomial coefficient. Let* $\|\cdot\|$ *denote a (sub-multiplicative)* *matrix norm. If* $\max\{\|\mathbf{\Delta_Q}\|, \|\mathbf{\Delta_S}\|, \|\mathbf{\Delta_{\mathcal{Q}}}\|\} \leq \epsilon$ *for some* $\epsilon \in (0,1)$*, then:*

$$\|\mathbf{\Omega D_P} - \nu\left(\mathbf{I} - \mathbf{\Pi_P^\top} + \mathbf{\Delta_S^\top} - \mathbf{\Delta_Q^\top} - \mathbf{\Delta_{\mathcal{Q}}^\top}\right)\| \leq 4\nu\frac{\epsilon^2}{1-\epsilon}. \tag{A.2}$$

Proof. Recall: $\mathbf{\Omega D_P} = (\mathbf{I} - \mathbf{Q}^\top)(\mathbf{I} - \mathbf{S}^\top)^{\#}\mathbf{D_P}^{-1}(\mathbf{I} - \beta\boldsymbol{\mathcal{Q}})\mathbf{D_P}$. Proceed term-by-term. First: $\mathbf{I} - \mathbf{Q}^\top = \mathbf{I} - \tilde{\mathbf{Q}}^\top - \psi\mathbf{\Delta_Q^\top} = \psi(\mathbf{I} - \mathbf{\Pi_P^\top}) - \psi\mathbf{\Delta_Q^\top}$. Second: $\mathbf{D_P}^{-1}(\mathbf{I} - \beta\boldsymbol{\mathcal{Q}})\mathbf{D_P} = \nu\mathbf{I} - \beta\psi\mathbf{D_P}^{-1}\mathbf{\Pi_P D_P} - \nu\mathbf{\Delta_{\mathcal{Q}}^\top} = \nu\mathbf{I} - \beta\psi\mathbf{\Pi_P^\top} - \nu\mathbf{\Delta_{\mathcal{Q}}^\top}$. Now turn to $\mathbf{S}$ and recall: $\mathbf{S} = \tilde{\mathbf{Q}} + \psi\mathbf{\Delta_S} = (1-\psi)\mathbf{I} + \psi\mathbf{\Pi_P} + \psi\mathbf{\Delta_S}$. Since $\mathbf{\Pi_P}$ is a stationary distribution for $\mathbf{S}$ and $\tilde{\mathbf{Q}}$, $\mathbf{\Delta_S\Pi_P} = 0$. In addition, since $\mathbf{\Delta_S}$ is a matrix whose columns are differences of (scaled) probability distributions, $\mathbf{1}^\top\mathbf{\Delta_S} = 0$. Consequently, since the rows of $\mathbf{\Pi_P}$ are constant, $\mathbf{\Pi_P\Delta_S} = 0$. Substituting for $\mathbf{S}$ in $\mathbf{S}^n - \mathbf{\Pi_P}$, expanding and using these relations gives: $\sum_{n=0}^\infty(\mathbf{S}^n - \mathbf{\Pi_P}) = \sum_{n=0}^\infty(1-\psi)^n(\mathbf{I} - \mathbf{\Pi_P}) + \sum_{n=0}^\infty\sum_{m=1}^n\binom{n}{m}\psi^m(1-\psi)^{n-m}\mathbf{\Delta_S^m} = \frac{1}{\psi}(\mathbf{I} - \mathbf{\Pi_P}) + \sum_{n=0}^\infty(1-\psi)^n\sum_{m=1}^\infty\binom{m+n}{m}\psi^m\mathbf{\Delta_S^m}$, where $\binom{m+n}{m} = \frac{(m+n)!}{m!n!}$ is a binomial coefficient and the second right hand sums are well defined given $\|\mathbf{\Delta_S}\| < \epsilon < 1$. As for $\mathbf{\Delta_S}$, $\mathbf{\Delta_Q\Pi_P} = 0$ and $\mathbf{\Pi_P\Delta_Q} = 0$. Using this and assembling elements:

$$\mathbf{\Omega} = \nu\psi(\mathbf{I} - \mathbf{\Delta_Q^\top})\left(\frac{1}{\psi}(\mathbf{I} - \mathbf{\Pi_P^\top}) + \sum_{n=0}^\infty(1-\psi)^n\sum_{m=1}^\infty\begin{pmatrix} m+n \\ m\end{pmatrix}\psi^m\mathbf{\Delta_S^m}\right)\left(\mathbf{I} - \mathbf{\Delta_{\mathcal{Q}}^\top}\right)\mathbf{D_P}^{-1}.$$

Extracting the $\mathbf{\Delta_S}$ term from the right hand side sum gives: $\mathbf{\Omega} = \nu(\mathbf{I} - \mathbf{\Pi_P^\top} - \mathbf{\Delta_Q^\top})(\mathbf{I} - \mathbf{\Pi_P^\top} + \mathbf{\Delta_S^\top} + \psi\sum_{n=0}^\infty(1-\psi)^n\sum_{m=2}^\infty\binom{m+n}{m}\psi^m\mathbf{\Delta_S^m})(\mathbf{I} - \mathbf{\Delta_{\mathcal{Q}}^\top})$. Expanding the right hand side: $\mathbf{\Omega} - \nu(\mathbf{I} - \mathbf{\Pi_P^\top} + \mathbf{\Delta_S^\top} - \mathbf{\Delta_Q^\top} - \mathbf{\Delta_{\mathcal{Q}}^\top}) = \nu(-\mathbf{\Delta_Q^\top\Delta_S^\top} - \mathbf{\Delta_S^\top\Delta_{\mathcal{Q}}^\top} + \mathbf{\Delta_Q^\top\Delta_{\mathcal{Q}}^\top} + \mathbf{\Delta_Q^\top\Delta_S^\top\Delta_{\mathcal{Q}}^\top} + \psi(\mathbf{I} - \mathbf{\Delta_Q^\top})\sum_{n=0}^\infty(1-\psi)^n\sum_{m=2}^\infty\binom{m+n}{m}\psi^m\mathbf{\Delta_S^m})(\mathbf{I} - \mathbf{\Delta_{\mathcal{Q}}^\top})$. Thus, $\|\mathbf{\Omega} - \nu(\mathbf{I} - \mathbf{\Pi_P^\top} + \mathbf{\Delta_S^\top} - \mathbf{\Delta_Q^\top} - \mathbf{\Delta_{\mathcal{Q}}^\top})\| = \nu\|(-\mathbf{\Delta_Q^\top\Delta_S^\top} - \mathbf{\Delta_S^\top\Delta_{\mathcal{Q}}^\top} + \mathbf{\Delta_Q^\top\Delta_{\mathcal{Q}}^\top} + \mathbf{\Delta_Q^\top\Delta_S^\top\Delta_{\mathcal{Q}}^\top} + \psi(\mathbf{I} - \mathbf{\Delta_Q^\top})\sum_{n=0}^\infty(1-\psi)^n\sum_{m=2}^\infty\binom{m+n}{m}\psi^m\mathbf{\Delta_S^m})(\mathbf{I} - \mathbf{\Delta_{\mathcal{Q}}^\top})\|$. Using the sub-multiplicativity of the matrix norm and the bounds in the proposition:

$$\left\|\mathbf{\Omega} - \nu\left(\mathbf{I} - \mathbf{\Pi_P^\top} + \mathbf{\Delta_S^\top} - \mathbf{\Delta_Q^\top} - \mathbf{\Delta_{\mathcal{Q}}^\top}\right)\right\| = \nu\{3\epsilon^2 + \epsilon^3 + \psi(1+\epsilon)^2K\}, \tag{A.3}$$

where: $K := \left\|\sum_{n=0}^\infty(1-\psi)^n\sum_{m=2}^\infty\binom{m+n}{m}\psi^m\mathbf{\Delta_S^m}\right\|$. Evaluation of the latter term using the sub-multiplicativity of norms, the bounds on the deviation matrices and the definition of the negative binomial distribution, $\epsilon < 1$ yields the bound: $0 < K \leq \frac{\epsilon^2}{\psi(1-\epsilon)}$. Substituting this into (A.3) gives the desired result. □

Online Appendices: Not for Publication

B Appendix: Additional Theoretical Results

B.1 Propagation in terms of mean first passage times

Expression (19) can be compactly reformatted in terms of semi-elasticities and expected travel times or *mean first passage times* between states. Let $\mathbf{m_Q}$ denote the matrix of mean first passage times for $\mathbf{Q}$, with $\mathbf{m_Q}(j,i)$, $j \neq i$, the expected travel time to j from i given $\mathbf{Q}$ and $\mathbf{m_Q}(i,i)$ the expected first return time to i given $\mathbf{Q}$. Proposition 1 has the following corollary.

Corollary 1 (Propagation; Semi-elasticities). *Let* $\mathbf{\Psi}$*, with* (j,k)*-th element* $\mathbf{\Psi}(j,k) := \frac{1}{\mathbf{P}(j)} \sum_{i \in \mathcal{I}} \frac{\partial \mathbf{Q}(j,i)}{\partial \mathbf{c}(k)} \mathbf{P}(i)$*, denote the matrix of short run semi-elasticities of* $\mathbf{P}$*. The long and short-run semi-elasticity matrices of* $\mathbf{P}$ *with respect to* $\mathbf{c}$ *are related via:*

$$\frac{1}{\mathbf{P}} \frac{\partial \mathbf{P}}{\partial \mathbf{c}} = \mathbf{\Psi} - \mathrm{Cov}\left(\mathbf{m_Q}, \mathbf{\Psi}\right), \tag{B.1}$$

with elements $\frac{1}{\mathbf{P}(j)} \frac{\partial \mathbf{P}(j)}{\partial \mathbf{c}(k)} = \mathbf{\Psi}(j,k) - \mathrm{Cov}\left(\mathbf{m_Q}(j,\cdot), \mathbf{\Psi}(\cdot,k)\right)$ *and where covariances are with respect to* $\mathbf{P}$*.*

Proof. From Proposition 1, $\frac{\partial \mathbf{P}}{\partial \mathbf{c}} = (\mathbf{I} - \mathbf{Q})^{\#} \mathbf{\Phi}$. From Cho and Meyer (2000), the (j,i)-th, $j \neq i$, off diagonal element of $(\mathbf{I} - \mathbf{Q})^{\#}$ is given by $\mathbf{a}(j,j) - \mathbf{P}(j)\mathbf{m_Q}(j,i)$, where $\mathbf{a}(j,j)$ is the j-th diagonal element of $(\mathbf{I} - \mathbf{Q})^{\#}$. Consequently,

$$\frac{\partial \mathbf{P}(j)}{\partial \mathbf{c}(k)} = \mathbf{a}(j,j) \sum_{i \in \mathcal{I}} \mathbf{\Phi}(i,k) - \mathbf{P}(j) \sum_{i \neq j} \mathbf{m_Q}(j,i) \mathbf{\Phi}(i,k). \tag{B.2}$$

However, $\sum_{i \in \mathcal{I}} \partial \mathbf{\Phi}(i,k) = 0$. Combining this with the fact that the first return time $\mathbf{m_Q}(j,j)$ equals $\frac{1}{\mathbf{P}(j)}$ (see Kemeny and Snell (1976)), and (B.2) gives: $\frac{1}{\mathbf{P}(j)} \frac{\partial \mathbf{P}(j)}{\partial \mathbf{c}(k)} = \mathbf{\Psi}(j,k) - \sum_{i \in \mathcal{I}} \mathbf{m_Q}(j,i) \mathbf{\Phi}(i,k) = \frac{1}{\mathbf{P}(j)} \frac{\partial \mathbf{P_1}(j)}{\partial \mathbf{c}(k)} - \sum_{i \in \mathcal{I}} \mathbf{m_Q}(j,i) \mathbf{\Psi}(i,k) \mathbf{P}(i)$. Finally since: $\sum_{i \in \mathcal{I}} \mathbf{\Psi}(i,k) \mathbf{P}(i) = \sum_{i \in \mathcal{I}} \mathbf{\Phi}(i,k) = 0$, we obtain the desired result (B.1). □

Equation (B.1) implies that long and short run semi-elasticities of $\mathbf{P}$ deviate from one another to the extent that mean first passage times of $\mathbf{Q}$ covary with short-run semi-elasticities. The latter covariance succinctly captures the role of post-substitution behavior in modifying long run relative to short run substitution patterns. Consider a pair of states j and k and suppose that a consumption increment at k depresses worker flows to states with low mean first passage times to j (resp. enhances worker flows to states with high mean first passage times to j). Then the covariance $\mathrm{Cov}\left(\mathbf{m_Q}(j,\cdot), \mathbf{\Psi}(\cdot,k)\right)$ is positive, $\frac{1}{\mathbf{P}} \frac{\partial \mathbf{P}}{\partial \mathbf{c}}(j,k)$ is reduced relative to $\mathbf{\Psi}(j,k)$ and long run substitution is enhanced. In Example 5 in the main text, workers in dead end (de) jobs have a lower probability of transition and, hence, a higher mean first passage time to high pay (hp) jobs than do workers in stepping stone (ss) jobs. An additional dollar on a de job would draw school leavers from

ss to de and, hence, generate a positive value for $\text{Cov}\left(\mathbf{m_Q}(hp,\cdot),\mathbf{\Psi}(\cdot,de)\right)$. Long run substitution between de and hp is enhanced.[40]

B.2 Further interpretation of the inverted optimal tax equation (30)

Consider each component of the $\mathbf{\Omega} = \mathbf{AB}$ matrix in turn. The first is the *reverse propagation matrix* $\mathbf{A} = \mathbf{I} - \mathbf{Q}^\top$. Its i-th row gives the *short run population shift* needed to induce a long run unit population reallocation to i. It inverts the propagation matrix $(\mathbf{I} - \mathbf{Q}^\top)^\#$. To see this explicitly, note that if the short run population shift $\{\mathbf{I}(k,i) - \mathbf{Q}(k,i)\}_{k\in\mathcal{I}}$ is repeatedly applied, then, cumulating the impact and taking account of post-substitution behavior, the long run change in population at each j is:

$$\sum_{n=0}^{\infty}\sum_{k\in\mathcal{I}}\mathbf{Q}(j,k)^n(\mathbf{I}(k,i) - \mathbf{Q}(k,i)) = \mathbf{I}(j,i) - \mathbf{P}(j) = \mathbf{\Delta}(j,i). \tag{B.3}$$

Thus, the repeated short run population shift $\mathbf{A}(i,\cdot) = (-\mathbf{Q}(1,i)\dots 1 - \mathbf{Q}(i,i)\dots - \mathbf{Q}(I,i))$ implements the long run shift $\mathbf{\Delta}(\cdot,i)$. If $\mathbf{Q} = \mathbf{\Pi_P}$, then $\mathbf{A}(i,\cdot) = \mathbf{\Delta}^\top(i,\cdot)$. In this case a long run population reallocation $\mathbf{\Delta}(\cdot,i)$ is achieved via repeated short run shifts of $\mathbf{\Delta}(\cdot,i)$, each shift diffusing in the subsequent period. More generally, short run shifts must be "tuned" to generate a unit long run population reallocation towards i: short run shifts to states k with high onward propagation from i must be damped (relative to $\mathbf{\Delta}(k,i) = \mathbf{I}(k,i) - \mathbf{P}(k)$), while those with low onward propagation must be relatively enhanced. In particular, for strongly persistent chains the elements of $\mathbf{A} = \mathbf{I} - \mathbf{Q}^\top$ must be reduced towards the zero matrix and the required short run shifts are small.

Define $\mathbf{\Delta_c} \in \mathbb{R}^I$ to be a *mean payoff-preserving* consumption reallocation if $\mathbf{P}^\top\frac{\partial\mathbf{V}}{\partial\mathbf{c}}\mathbf{\Delta_c} = 0$. Mean payoff-preserving consumption reallocations leave expected lifetime payoffs at the stationary equilibrium unchanged. The rows of the matrix $\mathbf{B} = (\mathbf{I} - \mathbf{S}^\top)^\#\mathbf{D_P}^{-1}(\mathbf{I} - \beta\mathcal{Q})$ give the mean payoff-preserving flow payoff reallocations needed to induce *short run* population shifts to desired states. $\mathbf{B}$ may be decomposed as:

$$\mathbf{B} = \underbrace{(\mathbf{I} - \mathbf{S}^\top)^\#\mathbf{D_P}^{-1}}_{\mathbf{B}_1}\underbrace{(\mathbf{I} - \beta\mathcal{Q})}_{\mathbf{B}_2}$$

with $\mathbf{B}_1$ giving the mean preserving *lifetime payoff* reallocations needed to induce short run unit population shifts to desired states and $\mathbf{B}_2$ converting these into *flow payoff* reallocations. If $\mathbf{S} = \mathbf{\Pi_P}$, as is the case in the repeated logit, then $\mathbf{B}_1$ reduces to $(\mathbf{I} - \mathbf{\Pi_P})\mathbf{D_P}^{-1}$ with i-th row $(-1,\dots,1/\mathbf{P}(i) - 1,\dots,-1)$. In this case, lowering payoffs by one in each state and augmenting payoff at i by $1/\mathbf{P}(i)$ induces a short run unit population shift to i. More generally and consistent with our previous discussion, payoff reallocations must be tuned to induce a short run shift to a desired state by

[40] Relatedly, consider the case $j = k$. Equation (B.1) implies that a consumption increment at k has a larger long run impact on the population at k if $-\text{Cov}\left(\mathbf{m_Q}(k,\cdot),\mathbf{\Psi}(\cdot,k)\right) > 0$. In particular, if the consumption increment induces workers to substitute from states with high mean first passage times to k, then the covariance will be negative and large in absolute value.

concentrating reductions on alternatives with low substitutability to i. The matrix B_2 gives the consumption adjustments required to generate an extra unit of payoff in each state. $\mathbf{B}_2 = (\mathbf{I} - \beta\boldsymbol{\mathcal{Q}}(\mathbf{V}))$ and, hence, takes account of future propagation of workers when determining flow payoff adjustments.

The matrix: $\mathbf{C} = \mathbf{D}_{\partial\mathbf{u}}{}^{-1}\left(\mathbf{1} - \mathbf{D}_{\mathbf{P}}{}^{-1}\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^{\top}\right)$ describes the resource cost of shifting a unit of flow payoff towards workers in state i net of its redistributive impact. If M is a weighted sum of lifetime utilities, then $\mathbf{C} = \mathbf{D}_{\partial\mathbf{u}}{}^{-1}\left(\mathbf{1} - \frac{1}{\Upsilon}\mathbf{D}_{\mathbf{P}}{}^{-1}\mathbf{D}_{\partial\mathbf{u}}(\mathbf{I} - \beta\boldsymbol{\mathcal{Q}})^{-1}\boldsymbol{\lambda}\right)$, where $\boldsymbol{\lambda}$ is the Pareto weight vector. In particular, if the policymaker's social criterion is utilitarian (with respect to lifetime utilities), $\boldsymbol{\lambda} = \mathbf{P}$, then $\mathrm{Cov}(\boldsymbol{\Omega}, \mathbf{C}) = \mathbf{ABD}_{\mathbf{P}}\mathbf{D}_{\partial\mathbf{u}}{}^{-1}(\mathbf{1} - \mathbf{D}_{\mathbf{P}}{}^{-1}\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^{\top})$ reduces to $\mathbf{ABD}_{\partial\mathbf{u}}{}^{-1}\mathbf{D}_{\mathbf{P}}\mathbf{1} = \mathbf{ABD}_{\partial\mathbf{u}}{}^{-1}\mathbf{P}$. The rows of $\mathbf{ABD}_{\partial\mathbf{u}}{}^{-1}\mathbf{D}_{\mathbf{P}}$ give the mean payoff-preserving consumption reallocations needed to induce long run population movements to target states. Since such reallocations leave mean lifetime payoffs unchanged, they do not affect this utilitarian criterion and, hence, there is no redistributive cost or benefit in this case, just the direct resource cost.

B.3 Equivalence of models with and without explicit amenity values

The next lemma shows that models with separable amenity values at different states may be renormalized to absorb these values into costs of adjustment, purging them from the analysis. We do so for a dynamic logit without perpetual youth, but the argument is generalizable.

Lemma B.1. *Given a dynamic logit model without perpetual youth in which workers have flow utilities net of Gumbel shocks $a \log \mathbf{c}(i) + \mathbf{h}(i)$ and costs of adjustment $\boldsymbol{\kappa}(j,i)$, there is an alternative model with amenity values $\mathrm{h}' = 0$ and costs of adjustment $\boldsymbol{\kappa}'$ satisfying for all i, $\boldsymbol{\kappa}'(i,i) = 0$, that generates the same transition matrices at all consumption allocations as the original environment.*

Proof. Express lifetime payoffs at a given arbitrary consumption allocation $\mathbf{c}$ as:

$$\mathbf{V}(j) - \boldsymbol{\kappa}(j,i) = \tilde{\mathbf{V}}(j,i) := a \log \mathbf{c}(j) + \mathbf{h}(j) - \boldsymbol{\kappa}(j,i) + \beta\overline{\mathbf{V}}(j),$$

with $\overline{\mathbf{V}}(j) = \log \sum \exp^{\tilde{\mathbf{V}}(k,j)}$. The associated transition is $\mathbf{Q}(j,i) = \frac{\exp^{a \log \mathbf{c}(j) + \mathbf{h}(j) - \boldsymbol{\kappa}(j,i) + \beta\overline{\mathbf{V}}(j)}}{\sum_k \exp^{a \log \mathbf{c}(k) + \mathbf{h}(k) - \boldsymbol{\kappa}(k,i) + \beta\overline{\mathbf{V}}(k)}}$. Define: $\boldsymbol{\kappa}'(j,i) := -\frac{1}{1-\beta}\mathbf{h}(j) + \boldsymbol{\kappa}(j,i) + \frac{1}{1-\beta}\mathbf{h}(i) - \frac{1}{1-\beta}\boldsymbol{\kappa}(i,i) + \frac{\beta}{1-\beta}\boldsymbol{\kappa}(j,j)$, where note $\boldsymbol{\kappa}'(j,j) = 0$. In addition, define: $\overline{\mathbf{V}}'(j) := \overline{\mathbf{V}}(j) - \frac{1}{1-\beta}\mathbf{h}(j) + \frac{1}{1-\beta}\boldsymbol{\kappa}(j,j)$ and $\tilde{\mathbf{V}}'(k,j) :=$

$\tilde{\mathbf{V}}(k,j) - \frac{1}{1-\beta}\mathbf{h}(j) + \frac{1}{1-\beta}\boldsymbol{\kappa}(j,j)$. Then:

$$
\begin{aligned}
\mathbf{Q}(j,i) &= \frac{\exp^{a\log \mathbf{c}(j)+\mathbf{h}(j)-\boldsymbol{\kappa}(j,i)+\beta\overline{\mathbf{V}}(j)}}{\sum_{k\in\mathcal{I}}\exp^{a\log \mathbf{c}(k)+\mathbf{h}(k)-\boldsymbol{\kappa}(k,i)+\beta\overline{\mathbf{V}}(k)}} \\
&= \frac{\exp^{a\log \mathbf{c}(j)+\mathbf{h}(j)-\boldsymbol{\kappa}(j,i)+\frac{\beta}{1-\beta}\mathbf{h}(j)-\frac{\beta}{1-\beta}\boldsymbol{\kappa}(j,j)+\beta\overline{\mathbf{V}}'(j)}}{\sum_{k\in\mathcal{I}}\exp^{a\log \mathbf{c}(k)+\mathbf{h}(k)-\boldsymbol{\kappa}(k,i)+\frac{\beta}{1-\beta}\mathbf{h}(k)-\frac{\beta}{1-\beta}\boldsymbol{\kappa}(k,k)+\beta\overline{\mathbf{V}}'(k)}} \\
&= \frac{\exp^{a\log \mathbf{c}(j)-\{-\frac{1}{1-\beta}\mathbf{h}(j)+\boldsymbol{\kappa}(j,i)+\frac{1}{1-\beta}\mathbf{h}(i)-\frac{1}{1-\beta}\boldsymbol{\kappa}(i,i)+\frac{\beta}{1-\beta}\boldsymbol{\kappa}(j,j)\}+\beta\overline{\mathbf{V}}'(j)}}{\sum_{k\in\mathcal{I}}\exp^{a\log \mathbf{c}(k)-\{-\frac{1}{1-\beta}\mathbf{h}(k)+\boldsymbol{\kappa}(k,i)+\frac{1}{1-\beta}\mathbf{h}(i)-\frac{1}{1-\beta}\boldsymbol{\kappa}(i,i)+\frac{\beta}{1-\beta}\boldsymbol{\kappa}(k,k)\}+\beta\overline{\mathbf{V}}'(k)}} \\
&= \frac{\exp^{a\log \mathbf{c}(j)-\boldsymbol{\kappa}'(j,i)+\beta\overline{\mathbf{V}}'(j)}}{\sum_{k\in\mathcal{I}}\exp^{a\log \mathbf{c}(k)-\boldsymbol{\kappa}'(k,i)+\beta\overline{\mathbf{V}}'(k)}}
\end{aligned}
$$

Thus, workers with zero amenity values and costs of adjustment $\boldsymbol{\kappa}'$ and continuation payoffs $\overline{\mathbf{V}}'$ make the same choices at (arbitrary) $\mathbf{c}$ as agents with amenity values $\mathbf{h}$, costs of adjustment $\boldsymbol{\kappa}$ and continuation payoffs $\overline{\mathbf{V}}$. We conclude by showing that $\overline{\mathbf{V}}'$ is the expected continuation payoff function for workers with costs of adjustment $\boldsymbol{\kappa}'$. First note $\tilde{\mathbf{V}}(j,i) = a\log\mathbf{c}(j) + \mathbf{h}(j) - \boldsymbol{\kappa}(j,i) + \beta\overline{\mathbf{V}}(j)$ and, hence,

$$
\begin{aligned}
\tilde{\mathbf{V}}'(j,i) &= \tilde{\mathbf{V}}(j,i) - \frac{1}{1-\beta}\mathbf{h}(i) + \frac{1}{1-\beta}\boldsymbol{\kappa}(i,i) \\
&= a\log\mathbf{c}(j) + \mathbf{h}(j) - \boldsymbol{\kappa}(j,i) - \frac{1}{1-\beta}\mathbf{h}(i) + \frac{1}{1-\beta}\boldsymbol{\kappa}(i,i) + \beta\overline{\mathbf{V}}(j) \\
&= a\log\mathbf{c}(j) + \mathbf{h}(j) - \boldsymbol{\kappa}(j,i) - \frac{1}{1-\beta}\mathbf{h}(i) + \frac{1}{1-\beta}\boldsymbol{\kappa}(i,i) + \frac{\beta}{1-\beta}\mathbf{h}(j) - \frac{\beta}{1-\beta}\boldsymbol{\kappa}(j,j) + \beta\overline{\mathbf{V}}'(j) \\
&= a\log\mathbf{c}(j) - \boldsymbol{\kappa}'(j,i) + \beta\overline{\mathbf{V}}'(j).
\end{aligned}
$$

Also, $\overline{\mathbf{V}}'(j) = \overline{\mathbf{V}}(j) - \frac{1}{1-\beta}\mathbf{h}(j) + \frac{1}{1-\beta}\boldsymbol{\kappa}(j,j) = \log\sum_{k\in\mathcal{I}}\exp^{\tilde{\mathbf{V}}(k,j)} + \log\exp^{-\frac{1}{1-\beta}\mathbf{h}(j)+\frac{1}{1-\beta}\boldsymbol{\kappa}(j,j)} = \log\sum_{k\in\mathcal{I}}\exp^{\tilde{\mathbf{V}}'(k,j)}$. Combining the previous expressions, $\mathbf{V}'$ is the lifetime expected continuation payoff of agents with zero amenity values and costs of adjustment $\boldsymbol{\kappa}'$ as desired. □

B.4 Tax equations for settings with persistent types

This appendix describes sensitivity formulas and optimal tax equations for settings with exogenously evolving persistent types that are not observed by the policymaker. We first recall and elaborate the setting described in Section 5. Next, we describe an inverted optimal tax equation for this setting that makes use of an aggregated Markov chain. Finally, we describe a specialization of results to the “mobile-immobile” types case considered in the main text.

B.4.1 The persistent type model

We adopt notation similar to that of Remark 5 and Section 5 in the main paper. As there, we split the state space as $\mathcal{I} = \mathcal{X} \times \mathcal{A}$, with $\mathcal{X}$ a finite set of chosen and publicly observed states of size X and $\mathcal{A}$ a finite set of exogenously determined and privately observed states or types of size A. We refer to these as (chosen) states and types below. Types are determined before choice within a period and evolve according to a Markov chain with transition ρ and initial distribution ρ_0. To clearly distinguish conditional from joint probabilities, we modify notation from the main text by placing $|$'s to separate conditioning arguments in probabilities. Let $\boldsymbol{\mathcal{Q}}(\mathbf{V}(\mathbf{c}))(x'|x,\alpha')$ denote the probability that a surviving worker of type α' in chosen state x selects x' at a stationary equilibrium with consumption allocation $\mathbf{c}$ and $\boldsymbol{\mathcal{P}}_0(\mathbf{V}(\mathbf{c}))(x|\alpha')$ the probability that an entrant of type α' selects x given consumption allocation $\mathbf{c}$. The complete Markov matrix over types and chosen states may be written compactly in matrix form as:

$$\mathbf{Q} = (1-\delta)\mathbf{D}_{\boldsymbol{\mathcal{Q}}}(\rho \otimes \mathbf{I}) + \delta \mathbf{D}_{\mathbf{\Pi}_{\boldsymbol{\mathcal{P}}_0}}(\mathbf{\Pi}_{\rho_0} \otimes \mathbf{I}), \tag{B.4}$$

where $\mathbf{D}_{\boldsymbol{\mathcal{Q}}}$ organizes the matrices $\boldsymbol{\mathcal{Q}}^{\alpha} := \boldsymbol{\mathcal{Q}}(\mathbf{V}(\mathbf{c}))(\cdot|\cdot,\alpha)$ onto a block diagonal matrix, $\mathbf{I}$ is an identity matrix of dimension X, and $\mathbf{D}_{\mathbf{\Pi}_{\boldsymbol{\mathcal{P}}_0}}$ organizes the matrices $\mathbf{\Pi}_{\boldsymbol{\mathcal{P}}_0^{\alpha}}$, $\boldsymbol{\mathcal{P}}_0^{\alpha} = \boldsymbol{\mathcal{P}}_0(\mathbf{V}(\mathbf{c}))(\cdot|\alpha)$ onto a block-diagonal matrix.[41] Element-wise $\mathbf{Q}$ has the form:

$$\mathbf{Q}((x',\alpha')|(x,\alpha)) = (1-\delta)\boldsymbol{\mathcal{Q}}(x'|x,\alpha')\rho(\alpha'|\alpha) + \delta\boldsymbol{\mathcal{P}}_0(x'|\alpha')\rho_0(\alpha'), \tag{B.5}$$

where dependence on $\mathbf{c}$ and $\mathbf{v}$ is suppressed in the notation. All sensitivity formulas in the main text hold with $\mathbf{Q}$ defined as in (B.4) (or (B.5)). For example, eq. (18) or (22) hold with component matrices $\mathbf{\Phi}$ and $\mathbf{S}$ defined using $\mathbf{Q}$ as in the main text. The resulting sensitivity formulas can be used to construct long run marginal excess burdens and can be inserted into optimal tax equations as described in the main text in (39).

B.4.2 Inverted optimal tax equations and aggregations of Markov chains

Let

$$\overline{\mathbf{P}}(x) = \sum_{\alpha \in \mathcal{A}} \mathbf{P}(x,\alpha), \tag{B.6}$$

denote the marginal stationary distribution over chosen states and:

$$\overline{\mathbf{Q}}(x'|x) = \sum_{\alpha' \in \mathcal{A}} \sum_{\alpha \in \mathcal{A}} \mathbf{Q}((x',\alpha')|(x,\alpha)) \frac{\mathbf{P}(x,\alpha)}{\overline{\mathbf{P}}(x)} \tag{B.7}$$

the probability that a worker moves to x' from x unconditioned on type at the steady state. In general functions of random variables generated by Markov chains do not evolve as Markov chains. In particular, $x_n = f(x_n, \alpha_n)$ does not evolve as a Markov

[41] As in the main text, $\mathbf{\Pi}_{\mathbf{z}}$ denotes a matrix that repeats a vector $\mathbf{z}$ on its columns.

chain (outside of special cases):

$$\begin{aligned}\text{Prob}(x''|x',x) &= \frac{\text{Prob}(x'',x'|x)}{\text{Prob}(x'|x)} \\ &= \sum_{\mathcal{A}\times\mathcal{A}\times\mathcal{A}} \mathbf{Q}(x'',\alpha''|x',\alpha') \left(\frac{\mathbf{Q}(x',\alpha'|x,\alpha)\frac{\mathbf{P}(x,\alpha)}{\overline{\mathbf{P}}(x)}}{\sum_{\mathcal{A}\times\mathcal{A}} \mathbf{Q}(x',\alpha'|x,\alpha)\frac{\mathbf{P}(x,\alpha)}{\overline{\mathbf{P}}(x)}} \right) \\ &\underbrace{\neq}_{\text{in general}} \sum_{\mathcal{A}\times\mathcal{A}} \mathbf{Q}(x'',\alpha''|x',\alpha') \frac{\mathbf{P}(x',\alpha')}{\overline{\mathbf{P}}(x')} = \text{Prob}(x''|x').\end{aligned}$$

Despite this $\overline{\text{Q}}$ is a Markov chain with stationary distribution $\overline{\text{P}}$.

Lemma B.2. *$\overline{\text{P}}$ is a stationary distribution of the Markov chain $\overline{\text{Q}}$.*

Proof. First, observe that for each x', $\overline{\mathbf{P}}(x') = \sum_{\alpha,\alpha'\in\mathcal{A}} \sum_{x\in\mathcal{X}} \mathbf{Q}(x',\alpha'|x,\alpha)\mathbf{P}(x,\alpha) = \sum_{x\in\mathcal{X}} \sum_{\alpha,\alpha'\in\mathcal{A}} \mathbf{Q}(x',\alpha'|x,\alpha)\mathbf{P}(x,\alpha) = \sum_{x\in\mathcal{X}} \overline{\mathbf{Q}}(x'|x)\overline{\mathbf{P}}(x)$. Further, each $\overline{\mathbf{Q}}(x'|x)$ is non-negative and $\sum_{x'\in\mathcal{X}} \overline{\mathbf{Q}}(x'|x) = 1$. □

We refer below to $\overline{\text{Q}}$ as the aggregated (over type) transition matrix, but stress again that the process describing worker transitions over observed states is not Markov. Since $\overline{\text{P}}$ is a stationary distribution of $\overline{\text{Q}}$, $\overline{\mathbf{P}} = (\mathbf{I} - \overline{\mathbf{Q}})\overline{\mathbf{P}}$ and we continue to have by the argument in the paper:

$$\frac{\partial \overline{\mathbf{P}}}{\partial \mathbf{c}} = (\mathbf{I} - \overline{\mathbf{Q}})^{\#}\overline{\mathbf{\Phi}}, \tag{B.8}$$

with:

$$\overline{\mathbf{\Phi}}(x'|x) = \sum_{x''\in\mathcal{X}} \frac{\partial \overline{\mathbf{Q}}(x'|x'')}{\partial \mathbf{c}(x)} \overline{\mathbf{P}}(x'').$$

Now, however, differentiating (B.7),

$$\begin{aligned}\frac{\partial \overline{\mathbf{Q}}(x'|x)}{\partial \mathbf{c}(y)} &= \sum_{\mathcal{A}\times\mathcal{A}} \frac{\partial \mathbf{Q}(x',\alpha'|x,\alpha)}{\partial \mathbf{c}(y)} \frac{\mathbf{P}(x,\alpha)}{\overline{\mathbf{P}}(x)} \\ &\quad + \sum_{\mathcal{A}\times\mathcal{A}} \{\mathbf{Q}(x',\alpha'|x,\alpha) - \overline{\mathbf{Q}}(x'|x)\} \frac{1}{\overline{\mathbf{P}}(x)} \frac{\partial \mathbf{P}(x,\alpha)}{\partial \mathbf{c}(y)}.\end{aligned} \tag{B.9}$$

In the long run, a consumption perturbation modifies the aggregate one step transition matrix $\overline{\text{Q}}$ by perturbing the type-specific transitions $\text{Q}(\cdot|\cdot,\alpha)$ (the first right hand side term in (B.9)) and the stocks $\text{P}(\cdot,\alpha)$ of different agent types with different flow rates in each state (the second right hand side term in (B.9)). While the first of these responses is instantaneous, the second is not. The term $\overline{\Phi}$ provides the long run response of the net flow of workers into j induced by a consumption perturbation at k. Absent type heterogeneity, this response is immediate. With such heterogeneity, it takes time to realize.

Plugging (B.8) into the optimal tax equation gives:

$$\mathbf{D}_{\overline{\mathbf{P}}} - \frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^{\top} = \frac{\partial \overline{\mathbf{P}}}{\partial \mathbf{c}}^{\top}\mathbf{T} = \overline{\mathbf{\Phi}}^{\top}(\mathbf{I} - \overline{\mathbf{Q}}^{\top})^{\#}\mathbf{T}. \tag{B.10}$$

This equation is similar to that previously obtained in the setting without persistent heterogeneity except that $\overline{\mathbf{P}}$, $\overline{\mathbf{Q}}$ and $\overline{\mathbf{\Phi}}$ now relate to the aggregated process over private types.

In our earlier analysis we obtained an inverted optimal tax equation by further factoring the analogue of $\overline{\mathbf{\Phi}}$ and applying a generalized inverse to the matrix pre-multiplying taxes. In the current setting such a factorization is not available. However,

$$\mathbf{1}^{\top}\overline{\mathbf{\Phi}}(\cdot, y) = \sum_{x' \in \mathcal{X}} \sum_{x \in \mathcal{X}} \frac{\partial \overline{\mathbf{Q}}(x', x)}{\partial \mathbf{c}(y)} \overline{\mathbf{P}}(x) = 0.$$

This motivates us to define:

$$\overline{\mathbf{S}} := \mathbf{I} - \overline{\mathbf{\Phi}}.$$

Then, $\overline{\mathbf{\Phi}} = \mathbf{I} - \overline{\mathbf{S}}$ and $\mathbf{1}^{\top}\overline{\mathbf{\Phi}} = 0$ implies $\mathbf{1}^{\top} = \mathbf{1}^{\top}\overline{\mathbf{S}}$ and, hence, each column of $\overline{\mathbf{S}}$ sums to one. After possible renormalization of utilities, all elements of $\overline{\mathbf{S}}$ are non-negative. Hence, $\overline{\mathbf{S}}$ is a Markov matrix. Replacing $\overline{\mathbf{\Phi}}^{\top}$ with $(\mathbf{I} - \overline{\mathbf{S}})$ in (B.10) and applying similar arguments to that used in the model without heterogeneity and the definition of $\overline{\mathbf{S}}$ yields:

$$\mathbf{T} = (\mathbf{I} - \overline{\mathbf{Q}}^{\top})(\mathbf{I} - \overline{\mathbf{S}}^{\top})^{\#}\overline{\mathbf{C}} + G\mathbf{1},$$

with $\overline{\mathbf{C}} := \mathbf{D}_{\overline{\mathbf{P}}} - \frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^{\top}$. The interpretation is modified from that in the main text. Now, $(\mathbf{I} - \overline{\mathbf{S}}^{\top})^{\#}$ is the consumption reallocation needed to induce a "unit flow of population" towards each state in each period. Precisely, if consumption is reallocated according to $\{(\mathbf{I} - \overline{\mathbf{S}}^{\top})^{\#}(j, i)\}_{i \in \mathcal{I}}$, then in the induced steady state an additional 1% of workers from each state will flow every period towards j. Again $\mathbf{I} - \overline{\mathbf{Q}}$ accounts for subsequent propagation of each inflowing wave. Together each row of $(\mathbf{I} - \overline{\mathbf{Q}}^{\top})(\mathbf{I} - \overline{\mathbf{S}}^{\top})^{\#}$ gives the consumption reallocation needed to induce a long run population shift to each state. The vector $\overline{\mathbf{C}}$ costs this consumption reallocation (taking into account redistribution costs or benefits from the reallocation).

B.4.3 The mobile-immobile model

This appendix section describes a simple persistent types economy: the "mobile-immobile" economy. Such a model was previously proposed by Artuç et al (2010). The model is consistent with evidence that a workers' probability of leaving an occupation state is declining in their occupational tenure. In the model, the probability of being a mobile type declines conditional on occupational tenure. This section extends the mobile/immobile formulation of Artuç et al (2010) to accommodate perpetual youth and provides long run distributional sensitivity formulas for it. We describe its quantification and give quantitative results for this environment in Appendix D.8.

In this economy, a worker is either mobile or immobile (non-mobile): $\alpha \in \mathcal{A} := \{m, n\}$. Mobile types receive choice state-contingent Gumbel shocks and are able to re-optimize over choice states, immobile types cannot re-optimize. Entrants are born mobile: $\rho_0 = \mathbf{e}_m = (1\ 0)^\top$. The mobility type of survivors evolves according to an ergodic Markov chain ρ on $\mathcal{A}$. Thus, the Markov chain over types inclusive of replacement by an entrant is:

$$\tilde{\rho} = (1-\delta)\rho + \delta \mathbf{\Pi}_{\mathbf{e}_m}.$$

Let μ denote the stationary mass of mobile types:

$$\mu = \frac{\tilde{\rho}(m|n)}{\tilde{\rho}(m|n) + \tilde{\rho}(n|m)} = \frac{\delta + (1-\delta)\rho(m|n)}{\delta + (1-\delta)\rho(m|n) + (1-\delta)\rho(n|m)}. \tag{B.11}$$

The transition matrix formula (B.4) specializes to:

$$\mathbf{Q} = (1-\delta)\begin{pmatrix} \boldsymbol{\mathcal{Q}} & 0 \\ 0 & \mathbf{I} \end{pmatrix}(\rho \otimes \mathbf{I}) + \delta \begin{pmatrix} \mathbf{\Pi}_{\boldsymbol{\mathcal{P}}_0} & 0 \\ 0 & \mathbf{I} \end{pmatrix}(\mathbf{\Pi}_{\mathbf{I}_m} \otimes \mathbf{I})$$

where $\boldsymbol{\mathcal{Q}}$ is the transition matrix of mobile survivors, $\mathbf{I}$ has the same dimension as $\mathcal{X}$ and $\boldsymbol{\mathcal{P}}_0$ is the choice vector of entrants. Element-wise this is:

$$\mathbf{Q}(x', m|x, \alpha) = (1-\delta)\boldsymbol{\mathcal{Q}}(x'|x)\rho(m|\alpha) + \delta\boldsymbol{\mathcal{P}}_0(x'|x) \quad \text{and} \quad \mathbf{Q}(x', n|x, \alpha) = (1-\delta)\mathbf{I}(x'|x)\rho(n|\alpha).$$

The special structure of this case, relative to the more general model with persistent types, permits simplified formulas for long run responses. For each α, let $\mathbf{P}_\alpha = \mathbf{P}(\cdot, \alpha)$ denote the measure of the type α over choice states. Then, in steady state:

$$\mathbf{P}_n = (1-\delta)\{\rho(n|m)\mathbf{P}_m + \rho(n|n)\mathbf{P}_n\} \implies \mathbf{P}_n = \frac{(1-\delta)\rho(n|m)}{1-(1-\delta)\rho(n|n)}\mathbf{P}_m = \frac{1-\mu}{\mu}\mathbf{P}_m. \tag{B.12}$$

It follows from (B.12) that it is sufficient to compute the long run response of $\mathbf{P}_m$ (or $\mathbf{P}(\cdot|m) = \frac{\mathbf{P}_m}{\mu}$) to obtain long run responses for $\mathbf{P}_n$ and, hence, $\overline{\mathbf{P}} = \mathbf{P}_m + \mathbf{P}_n$, the marginal over choice states.

Substituting (B.12) into the steady state expression for $\mathbf{P}_m$ gives:

$$\mathbf{P}_m = (1-\delta)\boldsymbol{\mathcal{Q}}\{\rho(m|m)\mathbf{P}_m + \rho(m|n)\mathbf{P}_n\} + \delta\boldsymbol{\mathcal{P}}_0 = \nu(1-\delta)\boldsymbol{\mathcal{Q}}\mathbf{P}_m + \delta\boldsymbol{\mathcal{P}}_0,$$

where

$$\nu = \frac{\rho(m|m)\mu + \rho(m|n)(1-\mu)}{\mu}.$$

This is converted into a conditional distribution of mobile types over choice states by dividing by μ:

$$\mathbf{P}(\cdot|m) = \nu(1-\delta)\boldsymbol{\mathcal{Q}}\mathbf{P}(\cdot|m) + \frac{\delta}{\mu}\boldsymbol{\mathcal{P}}_0, \tag{B.13}$$

The sensitivity of $\mathbf{P}(\cdot|m)$ to $\mathbf{c}$ can be calculated from (B.13):

$$\frac{\partial \mathbf{P}(\cdot|m)}{\partial \mathbf{c}} = (\mathbf{I} - \nu(1-\delta)\boldsymbol{\mathcal{Q}})^{-1}\boldsymbol{\Phi}^m,$$

where:

$$\boldsymbol{\Phi}^m(x', x) := \sum_{x'' \in \mathcal{X}} \left\{ \nu(1-\delta)\frac{\partial \boldsymbol{\mathcal{Q}}(x', x'')}{\partial \mathbf{c}(x)} + \frac{\delta}{\mu}\frac{\partial \boldsymbol{\mathcal{P}}_0(x')}{\partial \mathbf{c}(x)} \right\} \mathbf{P}(x''|m). \tag{B.14}$$

Assuming Gumbel shocks and a logit structure for mobile workers:

$$\begin{aligned}\boldsymbol{\Phi}^m(x', x) =& \xi \sum_{x'' \in \mathcal{X}} \Big\{ \mathbf{I}(x', x'')\mathbf{P}(x'|m) \\ & - \sum_{x''' \in \mathcal{X}} \{\theta \boldsymbol{\mathcal{Q}}(x', x''')\boldsymbol{\mathcal{Q}}(x'', x''') + (1-\theta)\boldsymbol{\mathcal{P}}_0(x')\boldsymbol{\mathcal{P}}_0(x'')\}\mathbf{P}(x'''|m) \Big\} \frac{\partial \mathbf{V}(x'', m)}{\partial \mathbf{c}(x)},\end{aligned}$$

where $\xi = \nu(1-\delta) + \frac{\delta}{\mu}$ and $\theta = \nu(1-\delta)/\xi$. In matrix form:

$$\boldsymbol{\Phi}^m = \xi(\mathbf{D}_{\mathbf{P}^m} - \theta \boldsymbol{\mathcal{Q}}\mathbf{D}_{\mathbf{P}^m}\boldsymbol{\mathcal{Q}}^\top - (1-\theta)\boldsymbol{\Pi}_{\boldsymbol{\mathcal{P}}_0}\mathbf{D}_{\mathbf{P}^m}\boldsymbol{\Pi}_{\boldsymbol{\mathcal{P}}_0}^\top)\frac{\partial \mathbf{V}^m}{\partial \mathbf{c}} = \xi(\mathbf{I} - \mathbf{S}^m)\mathbf{D}_{\mathbf{P}^m}\frac{\partial \mathbf{V}^m}{\partial \mathbf{c}},$$

where $\frac{\partial \mathbf{V}^m}{\partial \mathbf{c}} = \frac{\partial \mathbf{V}(\cdot, m)}{\partial \mathbf{c}}$, $\frac{\partial \mathbf{V}}{\partial \mathbf{c}} = (\mathbf{I} - \beta\hat{\mathbf{Q}})^{-1}\mathbf{N}^\top\mathbf{D}_{\partial \mathbf{u}}$, $\hat{\mathbf{Q}} = \begin{pmatrix} \boldsymbol{\mathcal{Q}} & 0 \\ 0 & \mathbf{I} \end{pmatrix}(\rho \otimes \mathbf{I})$ and:

$$\mathbf{S}^m = \theta\boldsymbol{\mathcal{Q}}\mathbf{D}_{\mathbf{P}^m}\boldsymbol{\mathcal{Q}}^\top\mathbf{D}_{\mathbf{P}^m}^{-1} + (1-\theta)\boldsymbol{\Pi}_{\boldsymbol{\mathcal{P}}_0}\mathbf{D}_{\mathbf{P}^m}\boldsymbol{\Pi}_{\boldsymbol{\mathcal{P}}_0}^\top\mathbf{D}_{\mathbf{P}^m}^{-1}.$$

The overall sensitivity is then recovered as:

$$\frac{\partial \mathbf{P}}{\partial \mathbf{c}} = \begin{bmatrix} \mu\mathbf{I} \\ (1-\mu)\mathbf{I} \end{bmatrix} \frac{\partial \mathbf{P}(\cdot|m)}{\partial \mathbf{c}},$$

where $\mathbf{I}$ is an $X \times X$ identity matrix.

B.5 Educational types case

In the quantitative section of the paper, we consider a model in which workers make an initial educational and occupational choice and then in subsequent periods of life make only occupational choices. This subsection describes the behavioral model used and its implications for distributional sensitivities.

Assume that a worker's state has two components, $i = (o, s) \in \mathcal{O} \times \mathcal{S} = \mathcal{I}$, where $o \in \mathcal{O} = \{1, \ldots, O\}$ denotes an occupation and $s = \{1, \ldots, S\}$ an education type. Suppose that s is chosen in the first period of life and thereafter is fixed; o is updated in each period as in the benchmark model. Then, survivor payoffs conditional on education types evolve as, for each $s \in \mathcal{S}$,

$$\mathbf{V}(\mathbf{c}|s) = u(\mathbf{c}, s) + \beta\boldsymbol{\mathcal{K}}[\mathbf{V}(\mathbf{c}|s), s], \tag{B.15}$$

where $\mathbf{c} \in \mathbb{R}_+^I = \mathbb{R}_+^{OS}$, $u(\mathbf{c}, s) = \{u(\mathbf{c}(o, s), s)\}_{o \in \mathcal{O}}$ and for each $s \in \mathcal{S}$, $\boldsymbol{\mathcal{K}}[\cdot, s] : \mathbb{R}^I \to \mathbb{R}^I$ is a payoff aggregator satisfying:

$$\forall o \in \mathcal{O}, \qquad \mathcal{K}[\mathbf{v}(s), s](o) := E\left[\max_{o' \in \mathcal{O}} \{\mathbf{v}(o', s) - \boldsymbol{\kappa}(o', o, s) + \boldsymbol{\varepsilon}(o')\}\right]. \tag{B.16}$$

Assuming Gumbel shocks and a logit structure, the corresponding education type-specific Markov matrix given $\mathbf{V}(\mathbf{c}|s)$ is:

$$\mathcal{Q}[\mathbf{V}(\mathbf{c}|s), s](o', o) = \frac{\boldsymbol{\omega}(o', o, s) \exp^{\mathbf{V}(\mathbf{c}|s)(o')}}{\sum_{r \in \mathcal{O}} \boldsymbol{\omega}(r, o, s) \exp^{\mathbf{V}(\mathbf{c}|s)(r)}},$$

with $\boldsymbol{\omega}(o', o, s) := \exp^{-\boldsymbol{\kappa}(o', o, s)}$. Let $\mathbf{V}(\mathbf{c}) = \{\mathbf{V}(\mathbf{c}|s)\}$ and let $\mathcal{Q}[\mathbf{V}(\mathbf{c})]$ denote the block diagonal matrix formed from placing the matrices $\{\mathcal{Q}[\mathbf{V}(\mathbf{c}|s), s]\}$ on the diagonal.

Entrants select an education state and an initial occupation. Their expected maximized value is:

$$\mathcal{K}_0[\mathbf{V}(\mathbf{c})] := E\left[\max_{s \in \mathcal{S}, o \in \mathcal{O}} \{\mathbf{V}(\mathbf{c}|s)(o) - \boldsymbol{\kappa}_0(o, s) + \boldsymbol{\varepsilon}(o, s)\}\right]. \tag{B.17}$$

Again assuming a logit structure:

$$\mathcal{P}_0(\mathbf{V}(\mathbf{c}))(o, s) = \frac{\boldsymbol{\omega}_0(o, s) \exp^{\mathbf{V}(\mathbf{c}|s)(o)}}{\sum_{o' \in \mathcal{O}, s' \in \mathcal{S}} \boldsymbol{\omega}_0(o', s') \exp^{\mathbf{V}(\mathbf{c}|s')(o')}}, \quad \text{with} \quad \boldsymbol{\omega}_0(o, s) := \exp^{-\boldsymbol{\kappa}_0(o, s)}. \tag{B.18}$$

The complete transition is then assembled as:

$$\mathbf{Q}(\mathbf{c}) = (1 - \delta)\mathcal{Q}(\mathbf{V}(\mathbf{c})) + \delta \boldsymbol{\Pi}_{\mathcal{P}_0(\mathbf{V}(\mathbf{c}))}.$$

Remark B.1. The preceding can be modified and generalized by assuming an initial nested logit structure in which workers first select an educational level conditional on a payoff shock and then select an initial occupation contingent on a subsequent occupation conditional payoff shock.

Formulas for sensitivities can be constructed as in the main paper on the state space $\mathcal{I} = \mathcal{O} \times \mathcal{S}$. Explicitly,

$$\frac{\partial \mathbf{P}}{\partial \mathbf{c}} = \sum_{m=0}^{\infty} \mathbf{Q}(\mathbf{c})^m \boldsymbol{\Phi}$$

with:

$$\boldsymbol{\Phi}(j, k) = \sum_{i \in \mathcal{I}} \left\{(1 - \delta)\frac{\partial \mathcal{Q}(j, i)}{\partial \mathbf{c}(k)} + \delta \frac{\partial \mathcal{P}_0(j)}{\partial \mathbf{c}(k)}\right\} \mathbf{P}(i). \tag{B.19}$$

Under the logit assumption:

$$\frac{\partial \mathcal{Q}(j, i)}{\partial \mathbf{c}(k)} = \sum_{l \in \mathcal{I}} \{\mathbf{I}(j, l) - \mathcal{Q}(j, i)\} \mathcal{Q}(l, i) \frac{\partial \mathbf{V}(l)}{\partial \mathbf{c}(k)} \tag{B.20}$$

and

$$\frac{\partial \boldsymbol{\mathcal{P}}_0(j)}{\partial \mathbf{c}(k)} = \sum_{l\in\mathcal{I}} \{\mathbf{I}(j,l) - \boldsymbol{\mathcal{P}}_0(j)\}\boldsymbol{\mathcal{P}}_0(l)\frac{\partial \mathbf{V}(l)}{\partial \mathbf{c}(k)}. \tag{B.21}$$

Substituting (B.20) and (B.21) into (B.19):

$$\mathbf{\Phi}(j,k) = \sum_{l\in\mathcal{I}}\sum_{i\in\mathcal{I}} \Big\{\mathbf{I}(j,l)\mathbf{P}(j) - \{(1-\delta)\boldsymbol{\mathcal{Q}}(j,i)\boldsymbol{\mathcal{Q}}(l,i) + \delta\boldsymbol{\mathcal{P}}_0(j)\boldsymbol{\mathcal{P}}_0(l)\}\mathbf{P}(i)\Big\}\frac{\partial \mathbf{V}(l)}{\partial \mathbf{c}(k)}.$$

In matrix form:

$$\mathbf{\Phi} = (\mathbf{D_P} - (1-\delta)\boldsymbol{\mathcal{Q}}\mathbf{D_P}\boldsymbol{\mathcal{Q}}^\top - \delta\mathbf{\Pi}_{\boldsymbol{\mathcal{P}}_0}\mathbf{D_P}\mathbf{\Pi}^\top_{\boldsymbol{\mathcal{P}}_0})\frac{\partial \mathbf{V}}{\partial \mathbf{c}} = (\mathbf{I} - \mathbf{S})\mathbf{D_P}\frac{\partial \mathbf{V}}{\partial \mathbf{c}},$$

where:

$$\mathbf{S} = (1-\delta)\boldsymbol{\mathcal{Q}}\mathbf{D_P}\boldsymbol{\mathcal{Q}}^\top\mathbf{D_P}^{-1} + \delta\mathbf{\Pi}_{\boldsymbol{\mathcal{P}}_0}\mathbf{D_P}\mathbf{\Pi}^\top_{\boldsymbol{\mathcal{P}}_0}\mathbf{D_P}^{-1}.$$

C Connecting to data

Estimating stationary distribution sensitivities at an observed equilibrium Evaluation of prevailing tax systems using (17) requires calculation of $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ at an observed equilibrium. Equation (22) implies that, in the absence of direct evidence, $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ can be constructed from estimates of Q, P, S, and $\frac{\partial \mathbf{V}}{\partial \mathbf{c}}$. Non-parametric estimates of Q and P may be obtained directly from data. The matrix S can be constructed from estimates of $\boldsymbol{\mathcal{Q}}$, P, $\mathbf{P_0}$ and δ given an assumption regarding preference shocks and re-optimization opportunities. We assume a dynamic logit with perpetual youth in quantitative work. While lifetime payoff sensitivities $\frac{\partial \mathbf{V}}{\partial \mathbf{c}}$ are not directly observable, it follows from $\frac{\partial \mathbf{V}}{\partial \mathbf{c}} = (\mathbf{I} - \beta\boldsymbol{\mathcal{Q}})^{-1}\mathbf{D}_{\partial \mathbf{u}}$ that they may be constructed from $\boldsymbol{\mathcal{Q}}(\mathbf{V})$ and parameters describing marginal utilities of consumption and the discount factor. In particular, if $u(c) = a\log c$, then only the two parameters a and β are needed (along with observed $\boldsymbol{\mathcal{Q}}$) to build $\frac{\partial \mathbf{V}}{\partial \mathbf{c}}$. Costs of adjustment $\boldsymbol{\kappa}$, which may be difficult to identify, do not need to be separately estimated. All information about these parameters relevant for stationary distribution sensitivities is embedded in observed Q, $\boldsymbol{\mathcal{Q}}$, P and $\mathbf{P_0}$. Marginal utility and discount parameters may be estimated by applying the approach of Artuç et al (2010), which combines a procedure for identifying flow payoff differences inclusive of adjustment cost terms from observed Q together with an IV strategy for estimating sensitivity of payoffs to consumption variation.[42]

Estimating stationary distribution sensitivities at a counterfactual equilibrium Evaluation of optimal taxes at a fixed welfare criterion using (17) requires calculation

[42] Hotz and Miller (1993) originate a procedure for inverting conditional choice probabilities to obtain flow payoff differences. Variations on such procedures are applied in structural IO and trade, see Artuç et al (2010).

of P and $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ at counterfactual equilibria. One approach is to assume a dynamic logit and structurally estimate all preference parameters and use these parameters to build maps from consumption allocations to Q, P, $\mathcal{Q}$, $\mathbf{P_0}$,and $\frac{\partial \mathbf{V}}{\partial \mathbf{c}}$. These permit evaluation of $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ via (22). Alternatively, redirecting an approach of Caliendo et al. (2019) towards stationary tax reform, if $u = a \log$, then the map from counterfactual consumption allocations c to $\mathcal{Q}$ and to $\mathbf{P_0}$ can be constructed from estimates of a and the discount factor β in combination with a prevailing consumption allocation, transition for survivors and distribution for entrants. Combined with an estimate of δ, counterfactual Q may then be constructed. Corresponding maps from c to P and to $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$ follow from (9) and (22). Estimates of cost of adjustment parameters are again not needed. Thus, the only preference parameters that require explicit estimation for tax evaluation are a and β. We refer to these parameters as *structurally sufficient* for optimal tax analysis. The result is formalized in Proposition C.1, which for simplicity we state for the case without perpetual youth.

Proposition C.1. *Assume a dynamic logit model without perpetual youth in which utility from consumption is $u = a \log$. Let* c *denote a consumption allocation and* Q *the corresponding transition matrix. Let* $\tilde{\mathbf{c}}$ *denote an alternative (counterfactual) consumption allocation and define $\boldsymbol{\Delta c} := \frac{\tilde{\mathbf{c}}}{\mathbf{c}}$. Then the corresponding counterfactual transition matrix* $\tilde{\mathbf{Q}}$ *satisfies:*

$$\tilde{\mathbf{Q}}(j,i) = \frac{\boldsymbol{\Delta U}(j)\mathbf{Q}(j,i)}{\sum_{k\in\mathcal{I}} \boldsymbol{\Delta U}(k)\mathbf{Q}(k,i)}, \tag{C.1}$$

with $\boldsymbol{\Delta U}$ the unique solution to $\boldsymbol{\Delta U} = \{(\boldsymbol{\Delta c}(j))^a(\sum_{k\in\mathcal{I}} \exp^{\log \boldsymbol{\Delta U}(k)} \mathbf{Q}(k,j))^\beta\}_{j\in\mathcal{I}}$. The corresponding counterfactual distribution $\tilde{\mathbf{P}}$ is the unique solution to $\tilde{\mathbf{P}} = \tilde{\mathbf{Q}}\tilde{\mathbf{P}}$.

Proof. The lifetime payoffs associated with moving to j from i net of current Gumbel shocks at c are: $\mathbf{V}(j) - \boldsymbol{\kappa}(j,i) = a \log \mathbf{c}(j) - \boldsymbol{\kappa}(j,i) + \beta \log \sum_{k\in\mathcal{I}} \exp^{\mathbf{V}(k)-\boldsymbol{\kappa}(k,j)}$, with $\boldsymbol{\kappa}(i,i) = 0$. Let $\tilde{\mathbf{V}}$ be the corresponding lifetime payoff function at the new stationary allocation $\tilde{\mathbf{c}}$. Then: $\frac{\tilde{\mathbf{Q}}(j,i)}{\mathbf{Q}(j,i)} = \frac{\frac{\exp^{\tilde{\mathbf{V}}(j)-\boldsymbol{\kappa}(j,i)}}{\sum_{k\in\mathcal{I}}\exp^{\tilde{\mathbf{V}}(k)-\boldsymbol{\kappa}(k,i)}}}{\frac{\exp^{\mathbf{V}(j)-\boldsymbol{\kappa}(j,i)}}{\sum_{k\in\mathcal{I}}\exp^{\mathbf{V}(k)-\boldsymbol{\kappa}(k,i)}}} = \frac{\exp^{\tilde{\mathbf{V}}(j)-\mathbf{V}(j)}}{\sum_{k\in\mathcal{I}} \frac{\exp^{\tilde{\mathbf{V}}(k)-\boldsymbol{\kappa}(k,i)}}{\sum_{l\in\mathcal{I}}\exp^{\mathbf{V}(l)-\boldsymbol{\kappa}(l,i)}}} = \frac{\exp^{\tilde{\mathbf{V}}(j)-\mathbf{V}(j)}}{\sum_{k\in\mathcal{I}}\exp^{\tilde{\mathbf{V}}(k)-\mathbf{V}(k)} \frac{\exp^{\mathbf{V}(k)-\boldsymbol{\kappa}(k,i)}}{\sum_{l\in\mathcal{I}}\exp^{\mathbf{V}(l)-\boldsymbol{\kappa}(l.i)}}}$ $= \frac{\exp^{\tilde{\mathbf{V}}(j)-\mathbf{V}(j)}}{\sum_{k\in\mathcal{I}}\exp^{\tilde{\mathbf{V}}(k)-\mathbf{V}(k)}\mathbf{Q}(k,i)}$. Define $\boldsymbol{\Delta U}(j) := \exp^{\tilde{\mathbf{V}}(j)-\mathbf{V}(j)}$. Substituting this into the previous formula gives: $\tilde{\mathbf{Q}}(j,i) = \frac{\boldsymbol{\Delta U}(j)\mathbf{Q}(j,i)}{\sum_{k\in\mathcal{I}}\boldsymbol{\Delta U}(k)\mathbf{Q}(k,i)}$. Also: $\boldsymbol{\Delta U}(j) = \exp^{\tilde{\mathbf{V}}(j)-\mathbf{V}(j)} = \exp^{a\log\tilde{\mathbf{c}}(j) - a\log\mathbf{c}(j) + \beta\{\log\sum_{k\in\mathcal{I}}\exp^{\tilde{\mathbf{V}}(k)-\boldsymbol{\kappa}(k,j)} - \log\sum_{k\in\mathcal{I}}\exp^{\mathbf{V}(k)-\boldsymbol{\kappa}(k,j)}\}}$. And so, $\boldsymbol{\Delta U}(j) = (\boldsymbol{\Delta c}(j))^a \times \exp^{\beta\{\log\sum_{k\in\mathcal{I}}\exp^{\tilde{\mathbf{V}}(k)-\boldsymbol{\kappa}(k,j)} - \log\sum_{k\in\mathcal{I}}\exp^{\mathbf{V}(k)-\boldsymbol{\kappa}(k,j)}\}} = (\boldsymbol{\Delta c}(j))^a(\frac{\sum_{k\in\mathcal{I}}\exp^{\tilde{\mathbf{V}}(k)-\boldsymbol{\kappa}(k,j)}}{\sum_{k\in\mathcal{I}}\exp^{\mathbf{V}(k)-\boldsymbol{\kappa}(k,j)}})^\beta$ $= (\boldsymbol{\Delta c}(j))^a(\sum_{k\in\mathcal{I}}\exp^{\tilde{\mathbf{V}}(k)-\mathbf{V}(k)} \frac{\exp^{\mathbf{V}(k)-\boldsymbol{\kappa}(k,j)}}{\sum_{k\in\mathcal{I}}\exp^{\mathbf{V}(k)-\boldsymbol{\kappa}(k,j)}})^\beta = (\boldsymbol{\Delta c}(j))^a \left(\sum_{k\in\mathcal{I}}\boldsymbol{\Delta U}(k)\mathbf{Q}(k,j)\right)^\beta$. Thus, $\tilde{\mathbf{Q}}$ satisfies (C.1) for a $\boldsymbol{\Delta U}$ satisfying the preceding equation. It is readily established that the map $\mathcal{T}(\mathbf{f}) = \{a\log(\boldsymbol{\Delta c}(j)) + \beta\log\left(\sum_{k\in\mathcal{I}}\exp^{\mathbf{f}(k,j)}\mathbf{Q}(k,j)\right)\}_{j\in\mathcal{I}}$ is a contraction (on the space $\mathbb{R}^{2I}$). Thus, $\log\boldsymbol{\Delta U}$ is the unique solution to $\mathbf{f} = \mathcal{T}(\mathbf{f})$ (given $\boldsymbol{\Delta c}$ and Q) and $\tilde{\mathbf{Q}}$ satisfies (C.1) for the unique $\boldsymbol{\Delta U}$ satisfying $\log\boldsymbol{\Delta U} = \mathcal{T}(\log\boldsymbol{\Delta U})$. □

D Quantitative Application

This appendix gives additional details on the quantitative application.

D.1 Data Selection

Our primary source of data is the March Supplement of the Current Population Survey (ASEC-CPS) for the years 2003 to 2020. This data set allows us to identify transitions by comparing the reported longest-held job in the previous calendar year to the job held at the time of the survey, in March. We restrict our analysis to full-time wage-earners aged 25 to 65 at the beginning of their occupational transition and drop individuals who spent more than one stretch of time looking for work in the previous year or who moved their place of residence for reasons of retirement, job loss, or college attendance. These restrictions eliminate or reduce retirement transitions, student employment and involuntary separations. We drop self-employed individuals, those who did not work at least 30 hours a week for at least 26 weeks in the previous year and those not working full time at the time of the interview. As a measure of wage income, we use reported wage income earned in the calendar year previous to the survey year, across all jobs held.[43] We drop individuals whose implied hourly wage is less than the minimum wage, whose annual wage income is less than 1,000 times the minimum hourly wage, whose real weekly earnings are less than $75 per week or above $750,000 per year (in 2019 dollars), who file taxes jointly and whose spouse has a real income greater than $750,000 and those whose wage and salary income contributes less than 80% of their total income.

We calculate individual statutory taxes using TAXSIM. Our tax notion is federal, state, and FICA taxes. Hence, we obtain individual after-tax labor earnings. We then calculate average occupation-specific after-tax labor incomes for each year as the average after-tax wage income across all individuals in the same occupation in a given year. We use this as a proxy for the consumption available in an occupation.

D.2 Estimation

Q estimation Transition probabilities for survivors $\{\mathcal{Q}_t\}_{t=2002}^{2019}$ are estimated from ASEC-CPS data using a cell estimator and the appropriate survey weights. Because transition probabilities enter the estimation in logarithms, estimates that are equal to zero are problematic. We address this issue in two alternative ways: through imputation of a very low value and through Pseudo-Poisson Maximum Likelihood regression. The first method is straightforward. When an estimated survivor transition $\widetilde{\mathcal{Q}}_t(j,i)$ is equal to zero we substitute it with 1e-03 and rescale all estimated probabilities in $\{\widetilde{\mathcal{Q}}_t(k,i)\}_{k\in\mathcal{I}}$ so that they sum up to 1. We use this procedure to generate our baseline estimates of $\{\mathcal{Q}_t\}$.

[43] A caveat is that this makes the measurement of occupational wages noisy, since it might contain wage income from occupations different than the reported longest-held occupation in the previous calendar year.

The alternative PPML approach uses the Poisson regression equation:

$$\widetilde{\mathcal{Q}}_t(j,i) = \exp\Bigg(\sum_{d=1,2} \beta^1_d \mathbf{1}[c_t(j) \geq c_t(i)](c_t(j)-c_t(i))^d + \sum_{d=1,2} \beta^2_d \mathbf{1}[c_t(j) < c_t(i)](c_t(j)-c_t(i))^d + \sum_{s=1,2,3}\sum_{d=1,2} \beta^3_{s,d}\mathbf{1}[\sigma_s(j) \geq \sigma_s(i)][\sigma_s(j)-\sigma_s(i)]^d + \sum_{d=1,2} \beta^4_{s,d}\mathbf{1}[\sigma_s(j) < \sigma_s(i)](\sigma_s(j)-\sigma_s(i))^d + \sum_{t'}\sum_{i'} \beta^5_{t'}(i')\mathbf{1}(i'=i,t'=t) + \sum_{t'}\sum_{j'} \beta^6_{t'}(j')\mathbf{1}(j'=j,t'=t) + \beta^7\mathbf{1}(j \neq i) + \beta_0 \Bigg) + \xi_t(j,i).$$

where $s = 1,2,3$ indexes, respectively, cognitive, physical, and social skills and $\sigma_s(i)$ is an index describing the amount of skill s required in occupation i. To obtain these indices we start from O*NET data, which provides a large number of occupation characteristics and skill requirements. Following Yamaguchi (2010), we use principal component analysis to reduce the O*NET variables to three main components. In estimations with distinct observable worker types, all coefficients are allowed to differ across types. The predicted values $\widehat{\mathcal{Q}}_t(j,i)$ from this regression are rescaled so that $\sum_{k\in\mathcal{I}} \widehat{\mathcal{Q}}_t(k,i) = 1$ and then substituted for $\widetilde{\mathcal{Q}}_t(j,i)$. We apply a strategy described by Artuç et al (2010) to annualize the survivor transitions we obtain from CPS data. We utilize survivor transitions corrected in these ways for zeros and frequency in our estimation of the utility parameter a (described below).

To obtain a stationary measure of $\mathbf{Q}$, the transition inclusive of replacement and re-entry, for use in our dynamic hat algebra evaluations of counterfactuals, we first combine nonparametric survivor transition estimates $\{\widetilde{\mathcal{Q}}_t\}_{t=2002}^{2019}$ (obtained via imputation), with birth/death probabilities δ and $1-\delta$ and estimates of the empirical entrant distributions $\{\widetilde{\mathbf{P}}_{0,t}\}$ to generate a sequence of empirical transitions $\{\widetilde{\mathbf{Q}}_t\}$. Entrant distributions are identified with the distributions of 25 year old workers over occupations. The sequence $\{\widetilde{\mathbf{Q}}_t\}$ is averaged over all time periods to form a proxy for the stationary transition:

$$\mathbf{Q} = \frac{1}{2019-2002+1}\sum_{t=2002}^{2019} \widetilde{\mathbf{Q}}_t.$$

Calibration and Structural Estimation The benchmark model's structural preference parameters are, respectively, the sensitivity of utility to $\log$ consumption a, the discount factor b and the survival probability $1-\delta$. Also, the Cobb-Douglas production function parameters A and ϕ are needed to compute optimal tax equilibria. Following Heathcote, Storesletten and Violante (2017), we set $b = .96$ and $\delta = .029$ implying $\beta = (1-\delta)b = .93$. Cobb Douglas production function parameters are set to be consistent with occupational income shares. Table D.1 reports average incomes by occupation (deflated to 2019 dollars for each year and averaged over sample years), the empirical distribution of 25 year olds over occupations averaged over sample years (denoted $\mathbf{P_0}^{\text{data}}$ in the table), the distribution of all workers over occupations averaged over sample years (denoted $\mathbf{P}^{\text{data}}$ in the table), the stationary distribution of our estimated Q (labeled P) and the estimated Cobb-Douglas parameter values ϕ. We note that $\mathbf{P}^{\text{data}}$ and $\mathbf{P}$ are reasonably close giving reassurance that our stationary occupation distribution assumption is plausible for the time period we consider.

Occupation	Average Income	$\mathbf{P_0}^{\text{data}}$	$\mathbf{P}^{\text{data}}$	$\mathbf{P}$	ϕ
Legal	$127594	0.009	0.012	0.023	0.044
Management	$101749	0.076	0.125	0.111	0.168
Architecture and engineering	$ 95814	0.028	0.028	0.033	0.048
Computer and mathematical	$ 94815	0.041	0.039	0.039	0.055
Life, physical, and social	$ 85813	0.012	0.012	0.019	0.025
Healthcare practitioner and technical	$ 85081	0.061	0.062	0.054	0.068
Business and financial	$ 81698	0.068	0.057	0.058	0.07
Arts, design, entertainment, sports, and media	$ 72533	0.023	0.015	0.023	0.025
Sales and related	$ 69466	0.104	0.087	0.081	0.083
Protective service	$ 67133	0.025	0.026	0.029	0.029
Education, training, and library	$ 59914	0.063	0.068	0.06	0.053
Installation, maintenance, and repair	$ 58263	0.037	0.041	0.037	0.032
Community and social services	$ 54796	0.019	0.019	0.023	0.019
Construction and extraction	$ 54246	0.059	0.052	0.05	0.04
Transportation and material moving	$ 49531	0.056	0.059	0.055	0.04
Production	$ 48562	0.067	0.075	0.066	0.048
Office and administrative support	$ 47556	0.145	0.137	0.139	0.098
Personal care and service	$ 38962	0.021	0.016	0.021	0.012
Healthcare support	$ 36931	0.024	0.019	0.026	0.014
Building and grounds cleaning and maintenance	$ 35941	0.015	0.023	0.024	0.013
Food preparation and serving	$ 33241	0.048	0.027	0.029	0.014

Data averaged across sample years. Incomes are converted to 2019 USD prior to averaging. Mean income is $66,974 in 2019 USD. $\mathbf{P_0}^{\text{data}}$ and $\mathbf{P}^{\text{data}}$ represent the distribution of young and all workers at the data. $\mathbf{P}$ refers to the stationary distribution of workers implied by the estimated transition, $\mathbf{Q}$. ϕ refers to the Cobb Douglas parameters which are calculated as a share of total income.

Table D.1: Occupational Incomes, Distributions, and Cobb-Douglas Parameters

The remaining parameter to be estimated is a. For this estimation step we do not impose the steady state assumption. We also modify the model slightly relative to that in the main paper and introduce a shock to consumption that is realized after agents migrate to a location. This aligns with the formulation in Artuç et al (2010). Given transitions for survivors $\{\mathcal{Q}_t\}$, the discount adjusted probability for survivors of moving early from i to j rather than moving later is defined as:

$$\mathbf{y}_t(j,i) = [\log \mathcal{Q}_t(j,i) - \log \mathcal{Q}_t(i,i)] + \beta[\log \mathcal{Q}_{t+1}(j,j) - \log \mathcal{Q}_{t+1}(j,i)]. \tag{D.1}$$

The next proposition uses the model to relate this variable to worker consumption and to a.

Proposition D.1. *Assume that payoffs are as in the benchmark dynamic discrete choice model. Given an intertemporal consumption allocation $\{\mathbf{c}_t\}_{t=1}^{\infty}$ and corresponding transitions for survivors $\{\mathcal{Q}_t\}_{t=0}^{\infty}$:*

$$\mathbf{y}_t(j,i) = a[\log \mathbf{c}_t(j) - \log \mathbf{c}_t(i)] - \boldsymbol{\kappa}(j,i) + \mu_t, \tag{D.2}$$

where: $\mu_t := \hat{\mathbb{E}}[a\{\log \mathbf{c}_t(j) - \log \mathbf{c}_t(i)\} + \log \boldsymbol{\mathcal{Q}}_{t+1}(j,i) - \log \boldsymbol{\mathcal{Q}}_{t+1}(j,j)|I_t]$, $\hat{\mathbb{E}}[x|I_t] = \mathbb{E}[x|I_t] - x$ *and* $\mathbb{E}[x|I_t]$ *denotes the expectation of* x *conditional on an information set that includes period* t *Gumbel shocks, but excludes period* t *consumption shocks.* $\boldsymbol{\kappa}$ *is normalized by* $1-\beta$ *relative to the text.*

Proof. Transition matrix for survivors is: $\boldsymbol{\mathcal{Q}}_t(j,i) = \frac{\exp^{\mathbf{V}_t(j)-\boldsymbol{\kappa}(j,i)}}{\sum_{k\in\mathcal{I}}\exp^{\mathbf{V}_t(k)-\boldsymbol{\kappa}(k,i)}}$, with lifetime payoffs: $\mathbf{V}_t(j) = \mathbb{E}[u(\mathbf{c}_t(j)) + \beta \log \sum_{k\in\mathcal{I}} \exp^{\mathbf{V}_{t+1}(k)-\boldsymbol{\kappa}(k,j)}|I_t]$, where the conditioning information set I_t includes the period t Gumbel shock, but excludes shocks to the consumption allocation that occur at t. Hence, for each j, i: $\mathbf{V}_t(j) - \boldsymbol{\kappa}(j,i) - \log \boldsymbol{\mathcal{Q}}_t(j,i) = \overline{\mathbf{V}}_t(i) := \log \sum_{k\in\mathcal{I}} \exp^{\mathbf{V}_t(k)-\boldsymbol{\kappa}(k,i)}$. So, $\overline{\mathbf{V}}_t(i) = \mathbf{V}_t(j) - \boldsymbol{\kappa}(j,i) - \log \boldsymbol{\mathcal{Q}}_t(j,i) = \mathbb{E}[u(\mathbf{c}_t(j)) + \beta\overline{\mathbf{V}}_{t+1}(j)|I_t] - \boldsymbol{\kappa}(j,i) - \log \boldsymbol{\mathcal{Q}}_t(j,i) = \mathbb{E}[u(\mathbf{c}_t(j)) + \beta\{\mathbf{V}_{t+1}(j) - \log \boldsymbol{\mathcal{Q}}_{t+1}(j,j)\}|I_t] - \boldsymbol{\kappa}(j,i) - \log \boldsymbol{\mathcal{Q}}_t(j,i)$, where the normalization $\boldsymbol{\kappa}(j,j) = 0$ is applied. Also, for $j = i$, $\overline{\mathbf{V}}_t(i) = \mathbf{V}_t(i) - \log \boldsymbol{\mathcal{Q}}_t(i,i) = \mathbb{E}[u(\mathbf{c}_t(i)) + \beta\overline{\mathbf{V}}_{t+1}(i)|I_t] - \log \boldsymbol{\mathcal{Q}}_t(i,i) = \mathbb{E}[u(\mathbf{c}_t(i)) + \beta\{\mathbf{V}_{t+1}(j) - \boldsymbol{\kappa}(j,i) - \log \boldsymbol{\mathcal{Q}}_{t+1}(j,i)\}|I_t] - \log \boldsymbol{\mathcal{Q}}_t(i,i)$. Combining these conditions and using the payoff parameterization $a \log c = u(c)$, renormalizing $\boldsymbol{\kappa}$ by $1-\beta$ and using the definitions in the proposition yields (D.2). □

Proposition D.1 underpins an identification strategy originated (in more general form) by Hotz and Miller (1993). In particular, it suggests the estimating equation:

$$\mathbf{y}_t(j,i) = \hat{a}\{\log \mathbf{c}_t(j) - \log \mathbf{c}_t(i)\} + \hat{\boldsymbol{\kappa}}(j,i) + \epsilon_t(j,i). \tag{D.3}$$

To construct empirical values for the dependent variable $\mathbf{y}_t$, we combine the non-parametric estimates of survivor transition probabilities $\{\boldsymbol{\mathcal{Q}}_t\}$ described previously with calibrated values for β. The term $\hat{\boldsymbol{\kappa}}(j,i)$ is a specification of transition costs. We estimate the model with two cost functions. $\hat{\boldsymbol{\kappa}}_1(j,i)$ is a quadratic polynomial of skill requirement differentials between occupations interacted with a dummy indicating whether the destination occupation has a higher skill requirement than the origin. $\hat{\boldsymbol{\kappa}}_2(j,i)$ is also quadratic in skill differentials when the skill differential is positive, but it is restricted to be a constant when the worker down skills:

$$\begin{aligned}\hat{\boldsymbol{\kappa}}_1(j,i) =& \kappa_0 + \sum_{s=1,2,3}\sum_{d=1,2} \kappa_{d,s}^{upskill} \times \mathbf{1}(\sigma_s(j) \geq \sigma_s(i)) \times (\sigma_s(j) - \sigma s(i))^d \\ &+ \sum_{s=1,2,3}\sum_{d=1,2} \kappa_{d,s}^{downskill} \times \mathbf{1}(\sigma_s(j) < \sigma_s(i)) \times (\sigma_s(j) - \sigma_s(i))^d, \end{aligned} \tag{D.4}$$

$$\begin{aligned}\hat{\boldsymbol{\kappa}}_2(j,i) =& \kappa_0 + \sum_{s=1,2,3}\sum_{d=1,2} \kappa_{d,s}^{upskill} \times \mathbf{1}(\sigma_s(j) \geq \sigma_s(i)) \times (\sigma_s(j) - \sigma_s(i))^d \\ &+ \sum_{s=1,2,3}\sum_{d=1,2} \kappa_{s}^{downskill} \times \mathbf{1}(\sigma_s(j) < \sigma_s(i)). \end{aligned} \tag{D.5}$$

As before, $\sigma_s(j)$ indicates occupation ij's type s skill requirement. In estimations with distinct observable worker types, all coefficients in the transition costs are allowed to differ across types.

The error $\epsilon_t(j,i)$ in (D.3) is interpreted as a sum of sampling, measurement, and expectation error. We then estimate the parameters in (D.3) and, hence, a via IV regression. Following Artuç et al (2010), we use $\log[\mathbf{c}_{t-2}(j)/\mathbf{c}_{t-2}(i)]$ as instrument for $\log[\mathbf{c}_t(j)/\mathbf{c}_t(i)]$.

D.3 Additional Results for the Benchmark Model

This section provides additional quantitative results that supplement those in the main text.

Empirical Equilibrium Figure D.1 displays the matrix of mean first passage times $\mathbf{m_Q}$ implied by (the non-parametrically estimated stationary) Q at the empirical policy as a heat map. Rows show mean first passage times to an occupation, columns show mean first passage times from an occupation. They are computed

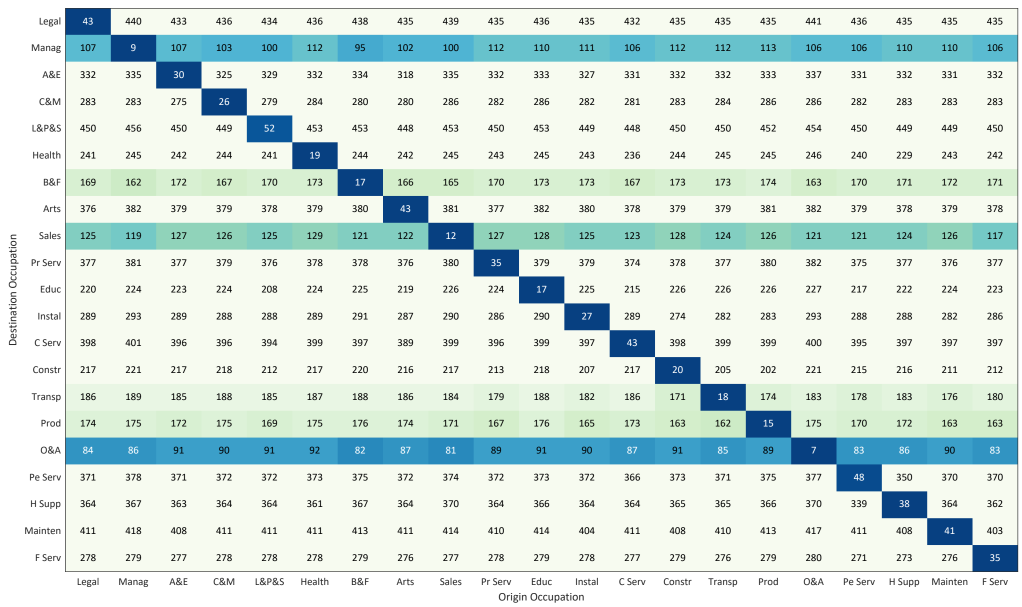
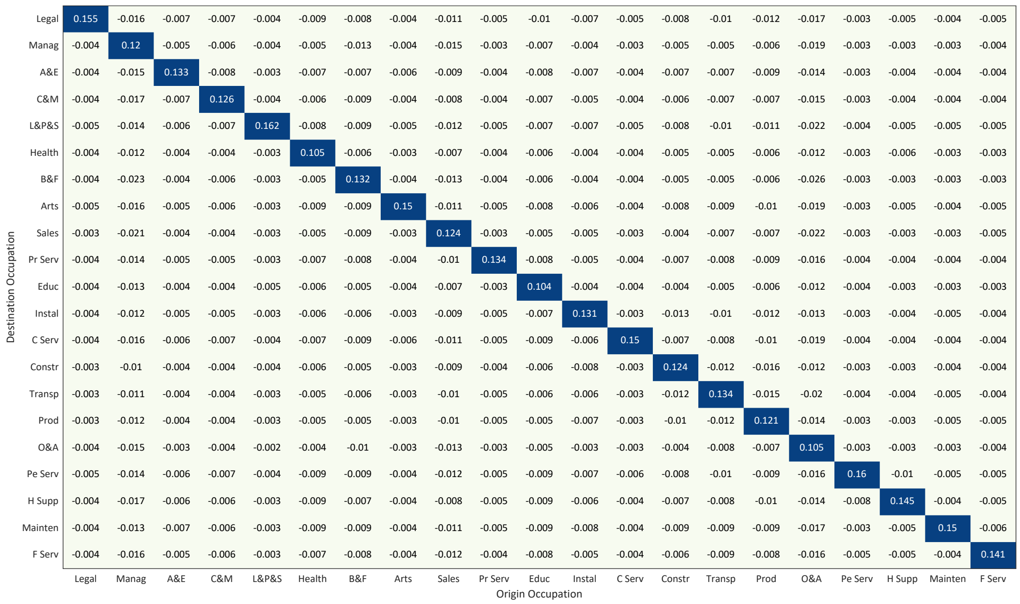


Figure D.1: Mean first passage matrix, $\mathbf{m_Q}$, in the data.

Notes: Darker colors imply higher smaller mean first passage times and faster travel between occupations.

from the estimated stationary Q and its associated stationary distribution using formulas available in Kemeny and Snell (1976), Theorem 4.4.7, p.79. Elements on the diagonal of $\mathbf{m_Q}$ (mean first return times) are lower than off-diagonal elements reflecting the persistence of the chain. Mean first passage times to management, sales and office and administration are lower than other occupations. These are occupations that workers migrate to from both lower and higher paid activities. There is slight relative reduction of mean first passage times between transport, production and construction indicating higher substitutability between these occupations. Mean first passage times to food services (last row in the figure) are in the 300's, significantly below those to maintenance or health services. In addition mean first passage times from food services (last column in the figure) to sales, office and administration and management are below those for other low income occupations.

Figures D.2a and D.2b display the short and long run distributional elasticities computed using (21) and (22) and a dynamic logit assumption at the policy in the data.



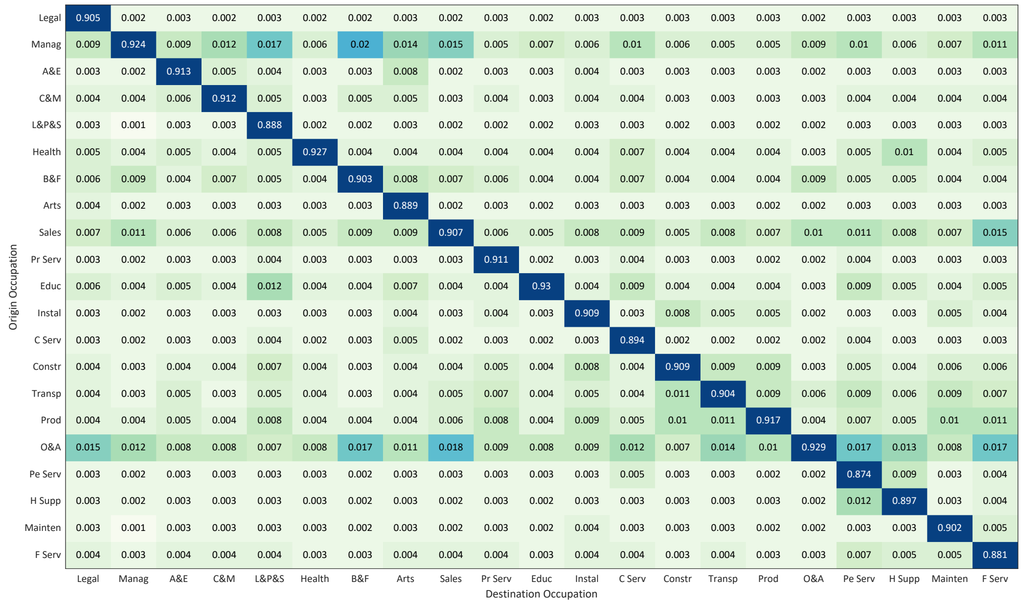
(a) Short run elasticities

(b) Long run elasticities

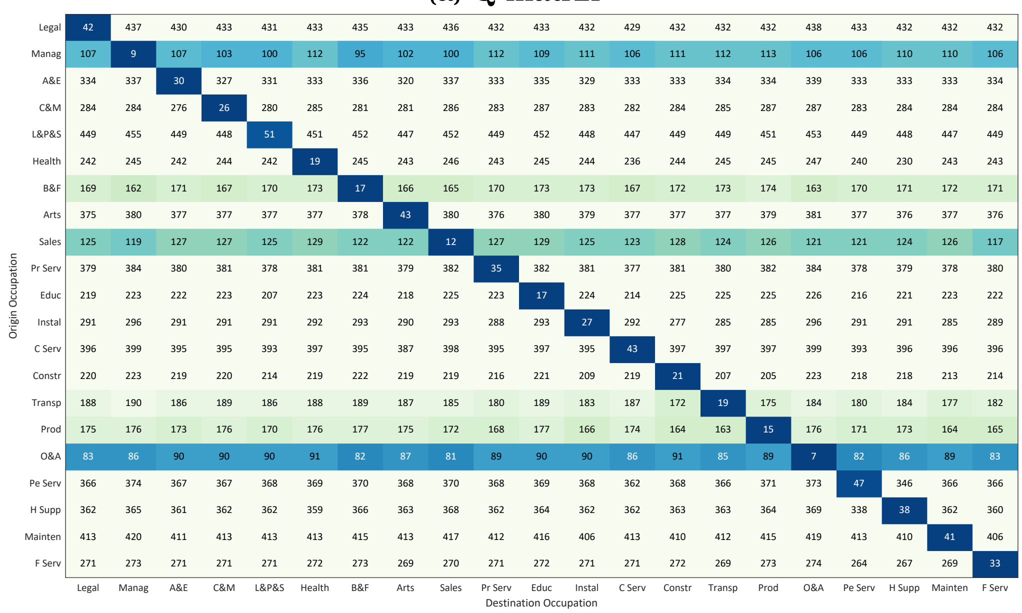
Figure D.2: Short and long run elasticities at the empirical policy.

Notes: Darker colors imply higher elasticities between occupations.

Optimum, Benchmark Social Criterion Figures D.3 and D.4 displays the transition Q, mean first passage m_Q, short and long run elasticity matrices for the optimum with benchmark social objective. These are not structural and are modified by policy. However, the modifications are small and they are similar to their estimated values at the data equilibrium.



(a) Q matrix



(b) m_Q matrix

Figure D.3: Q and mean first passage matrices at the optimum.

Origin Occupation \ Destination Occupation	Legal	Manag	A&E	C&M	L&P&S	Health	B&F	Arts	Sales	Pr Serv	Educ	Instal	C Serv	Constr	Transp	Prod	O&A	Pe Serv	H Supp	Mainten	F Serv
Legal	0.155	-0.016	-0.007	-0.006	-0.004	-0.009	-0.008	-0.004	-0.011	-0.005	-0.01	-0.006	-0.005	-0.008	-0.009	-0.012	-0.017	-0.004	-0.005	-0.004	-0.005
Manag	-0.004	0.12	-0.005	-0.006	-0.004	-0.005	-0.013	-0.004	-0.015	-0.003	-0.007	-0.004	-0.003	-0.005	-0.005	-0.006	-0.019	-0.003	-0.003	-0.003	-0.004
A&E	-0.004	-0.015	0.133	-0.008	-0.003	-0.007	-0.007	-0.006	-0.009	-0.004	-0.008	-0.007	-0.004	-0.007	-0.007	-0.009	-0.014	-0.003	-0.004	-0.004	-0.004
C&M	-0.004	-0.017	-0.007	0.126	-0.004	-0.006	-0.009	-0.004	-0.008	-0.004	-0.007	-0.005	-0.004	-0.006	-0.007	-0.007	-0.015	-0.003	-0.004	-0.004	-0.004
L&P&S	-0.005	-0.014	-0.006	-0.007	0.162	-0.008	-0.009	-0.005	-0.012	-0.005	-0.008	-0.007	-0.005	-0.008	-0.01	-0.011	-0.022	-0.004	-0.005	-0.005	-0.005
Health	-0.004	-0.012	-0.004	-0.004	-0.003	0.105	-0.006	-0.003	-0.007	-0.004	-0.006	-0.004	-0.004	-0.005	-0.005	-0.006	-0.012	-0.003	-0.006	-0.003	-0.003
B&F	-0.004	-0.023	-0.004	-0.006	-0.003	-0.005	0.132	-0.004	-0.013	-0.004	-0.006	-0.004	-0.004	-0.005	-0.005	-0.006	-0.027	-0.003	-0.003	-0.003	-0.003
Arts	-0.005	-0.016	-0.005	-0.006	-0.003	-0.009	-0.009	0.15	-0.011	-0.005	-0.008	-0.006	-0.004	-0.008	-0.009	-0.01	-0.019	-0.004	-0.005	-0.004	-0.005
Sales	-0.003	-0.021	-0.004	-0.004	-0.003	-0.005	-0.009	-0.003	0.124	-0.003	-0.005	-0.004	-0.003	-0.004	-0.007	-0.007	-0.022	-0.003	-0.003	-0.003	-0.006
Pr Serv	-0.004	-0.014	-0.005	-0.005	-0.003	-0.007	-0.008	-0.004	-0.01	0.135	-0.008	-0.005	-0.004	-0.007	-0.008	-0.009	-0.016	-0.004	-0.004	-0.004	-0.004
Educ	-0.004	-0.013	-0.004	-0.004	-0.005	-0.006	-0.005	-0.004	-0.007	-0.003	0.104	-0.004	-0.004	-0.004	-0.005	-0.006	-0.012	-0.004	-0.003	-0.003	-0.003
Instal	-0.004	-0.012	-0.005	-0.005	-0.003	-0.006	-0.006	-0.004	-0.009	-0.005	-0.007	0.131	-0.003	-0.012	-0.01	-0.012	-0.013	-0.003	-0.004	-0.005	-0.004
C Serv	-0.004	-0.016	-0.006	-0.007	-0.004	-0.007	-0.009	-0.006	-0.011	-0.005	-0.009	-0.006	0.15	-0.007	-0.008	-0.01	-0.019	-0.004	-0.004	-0.004	-0.004
Constr	-0.003	-0.01	-0.004	-0.004	-0.004	-0.006	-0.005	-0.003	-0.009	-0.004	-0.006	-0.008	-0.003	0.124	-0.012	-0.016	-0.012	-0.003	-0.003	-0.004	-0.004
Transp	-0.003	-0.011	-0.004	-0.004	-0.003	-0.005	-0.006	-0.003	-0.01	-0.005	-0.006	-0.005	-0.003	-0.012	0.134	-0.015	-0.021	-0.004	-0.004	-0.005	-0.004
Prod	-0.003	-0.012	-0.004	-0.004	-0.003	-0.005	-0.005	-0.003	-0.01	-0.005	-0.006	-0.007	-0.003	-0.01	-0.012	0.121	-0.014	-0.003	-0.003	-0.005	-0.005
O&A	-0.004	-0.015	-0.003	-0.004	-0.002	-0.004	-0.01	-0.003	-0.013	-0.003	-0.005	-0.003	-0.003	-0.004	-0.007	-0.007	0.105	-0.003	-0.003	-0.003	-0.004
Pe Serv	-0.005	-0.014	-0.006	-0.007	-0.004	-0.009	-0.009	-0.004	-0.012	-0.005	-0.009	-0.007	-0.006	-0.008	-0.01	-0.009	-0.016	0.16	-0.01	-0.005	-0.005
H Supp	-0.004	-0.017	-0.006	-0.006	-0.003	-0.009	-0.007	-0.004	-0.008	-0.005	-0.009	-0.006	-0.004	-0.007	-0.008	-0.01	-0.014	-0.008	0.145	-0.004	-0.005
Mainten	-0.004	-0.013	-0.006	-0.006	-0.003	-0.009	-0.009	-0.004	-0.011	-0.005	-0.009	-0.008	-0.004	-0.009	-0.009	-0.009	-0.017	-0.004	-0.005	0.15	-0.007
F Serv	-0.004	-0.016	-0.005	-0.006	-0.003	-0.007	-0.008	-0.004	-0.012	-0.004	-0.008	-0.005	-0.004	-0.006	-0.009	-0.008	-0.016	-0.005	-0.005	-0.004	0.14

(a) Short run elasticity matrix

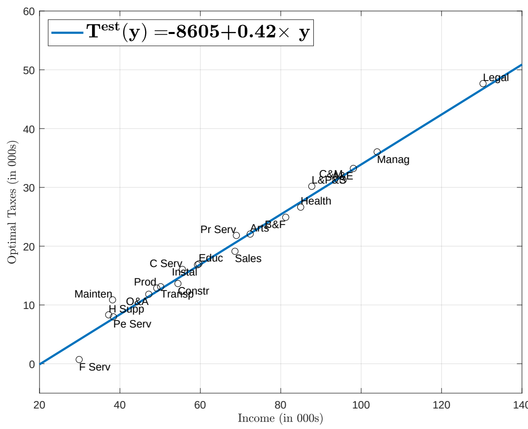
Origin Occupation \ Destination Occupation	Legal	Manag	A&E	C&M	L&P&S	Health	B&F	Arts	Sales	Pr Serv	Educ	Instal	C Serv	Constr	Transp	Prod	O&A	Pe Serv	H Supp	Mainten	F Serv
Legal	1.594	-0.207	-0.052	-0.062	-0.03	-0.086	-0.092	-0.035	-0.128	-0.045	-0.099	-0.058	-0.033	-0.077	-0.087	-0.108	-0.238	-0.032	-0.041	-0.041	-0.041
Manag	-0.043	1.408	-0.056	-0.057	-0.024	-0.088	-0.066	-0.031	-0.106	-0.054	-0.101	-0.062	-0.038	-0.077	-0.094	-0.116	-0.229	-0.033	-0.047	-0.046	-0.042
A&E	-0.041	-0.192	1.471	-0.048	-0.029	-0.079	-0.087	-0.017	-0.119	-0.045	-0.095	-0.049	-0.037	-0.071	-0.08	-0.102	-0.227	-0.032	-0.041	-0.039	-0.042
C&M	-0.039	-0.175	-0.04	1.387	-0.026	-0.076	-0.074	-0.03	-0.115	-0.042	-0.091	-0.05	-0.033	-0.066	-0.077	-0.101	-0.208	-0.029	-0.039	-0.039	-0.039
L&P&S	-0.036	-0.179	-0.047	-0.05	1.419	-0.08	-0.085	-0.029	-0.114	-0.04	-0.077	-0.051	-0.028	-0.065	-0.078	-0.097	-0.228	-0.029	-0.035	-0.034	-0.038
Health	-0.039	-0.172	-0.051	-0.057	-0.03	1.37	-0.079	-0.036	-0.109	-0.044	-0.09	-0.053	-0.027	-0.068	-0.079	-0.096	-0.203	-0.03	-0.025	-0.041	-0.04
B&F	-0.036	-0.138	-0.05	-0.049	-0.03	-0.075	1.278	-0.028	-0.098	-0.041	-0.088	-0.053	-0.031	-0.065	-0.08	-0.098	-0.171	-0.029	-0.038	-0.039	-0.039
Arts	-0.031	-0.165	-0.04	-0.05	-0.026	-0.072	-0.074	1.326	-0.105	-0.037	-0.085	-0.051	-0.028	-0.064	-0.073	-0.096	-0.203	-0.027	-0.034	-0.034	-0.033
Sales	-0.037	-0.144	-0.049	-0.055	-0.029	-0.074	-0.073	-0.029	1.227	-0.043	-0.086	-0.048	-0.031	-0.066	-0.071	-0.092	-0.18	-0.024	-0.035	-0.037	-0.026
Pr Serv	-0.038	-0.192	-0.05	-0.059	-0.028	-0.079	-0.081	-0.033	-0.114	1.464	-0.093	-0.055	-0.031	-0.07	-0.077	-0.102	-0.222	-0.028	-0.038	-0.036	-0.039
Educ	-0.039	-0.183	-0.053	-0.059	-0.013	-0.079	-0.085	-0.03	-0.116	-0.046	1.4	-0.056	-0.027	-0.072	-0.082	-0.099	-0.212	-0.025	-0.041	-0.041	-0.041
Instal	-0.041	-0.186	-0.051	-0.055	-0.031	-0.076	-0.081	-0.033	-0.109	-0.04	-0.088	1.387	-0.037	-0.048	-0.068	-0.089	-0.215	-0.032	-0.04	-0.031	-0.036
C Serv	-0.037	-0.186	-0.048	-0.052	-0.024	-0.073	-0.075	-0.021	-0.11	-0.039	-0.085	-0.051	1.392	-0.069	-0.079	-0.098	-0.21	-0.024	-0.037	-0.036	-0.037
Constr	-0.042	-0.178	-0.051	-0.055	-0.026	-0.072	-0.08	-0.034	-0.103	-0.039	-0.085	-0.043	-0.036	1.305	-0.057	-0.069	-0.205	-0.03	-0.039	-0.032	-0.032
Transp	-0.043	-0.189	-0.05	-0.059	-0.032	-0.078	-0.087	-0.036	-0.106	-0.036	-0.09	-0.052	-0.037	-0.048	1.326	-0.074	-0.189	-0.023	-0.037	-0.026	-0.034
Prod	-0.046	-0.182	-0.052	-0.06	-0.029	-0.079	-0.087	-0.038	-0.107	-0.038	-0.092	-0.049	-0.038	-0.063	-0.063	1.366	-0.213	-0.03	-0.042	-0.029	-0.029
O&A	-0.034	-0.183	-0.054	-0.06	-0.036	-0.079	-0.078	-0.035	-0.101	-0.046	-0.09	-0.055	-0.036	-0.072	-0.074	-0.099	1.269	-0.026	-0.037	-0.042	-0.034
Pe Serv	-0.032	-0.179	-0.044	-0.049	-0.026	-0.073	-0.076	-0.029	-0.097	-0.037	-0.077	-0.048	-0.021	-0.064	-0.06	-0.093	-0.195	1.245	0.01	-0.031	-0.027
H Supp	-0.036	-0.178	-0.046	-0.053	-0.028	-0.067	-0.082	-0.032	-0.119	-0.041	-0.09	-0.051	-0.032	-0.067	-0.076	-0.097	-0.218	0.009	1.373	-0.035	-0.033
Mainten	-0.04	-0.211	-0.049	-0.058	-0.03	-0.081	-0.085	-0.034	-0.122	-0.042	-0.098	-0.043	-0.036	-0.066	-0.074	-0.101	-0.237	-0.031	-0.036	1.503	-0.029
F Serv	-0.031	-0.147	-0.041	-0.046	-0.025	-0.064	-0.069	-0.025	-0.082	-0.035	-0.075	-0.043	-0.028	-0.057	-0.058	-0.079	-0.175	-0.016	-0.026	-0.026	1.146

(b) Long run elasticity matrix

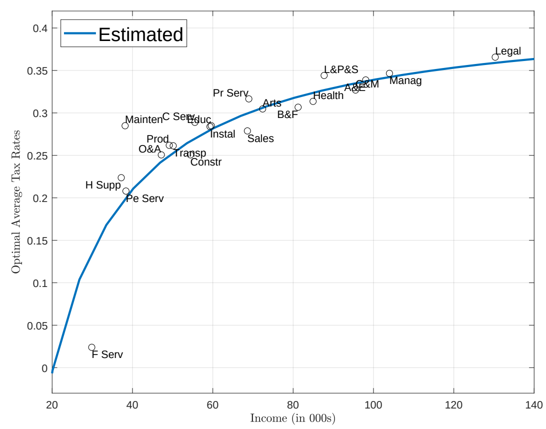
Figure D.4: Short and long run elasticity matrices at the optimum.

D.4 Results under alternate welfare criteria

We next report optimal tax schedules under a social criterion different from the paper. Specifically, we identify social welfare with the expected utility of a new entrant to the labor market distributed over occupations according to P_0 at steady state. Since workers discount the future, this welfare criterion emphasizes occupations that are inhabited earlier in life. Figure D.5 illustrates the implied optimal taxes and



(a) Taxes: Optimal



(b) Average Income Tax Rates: Optimal

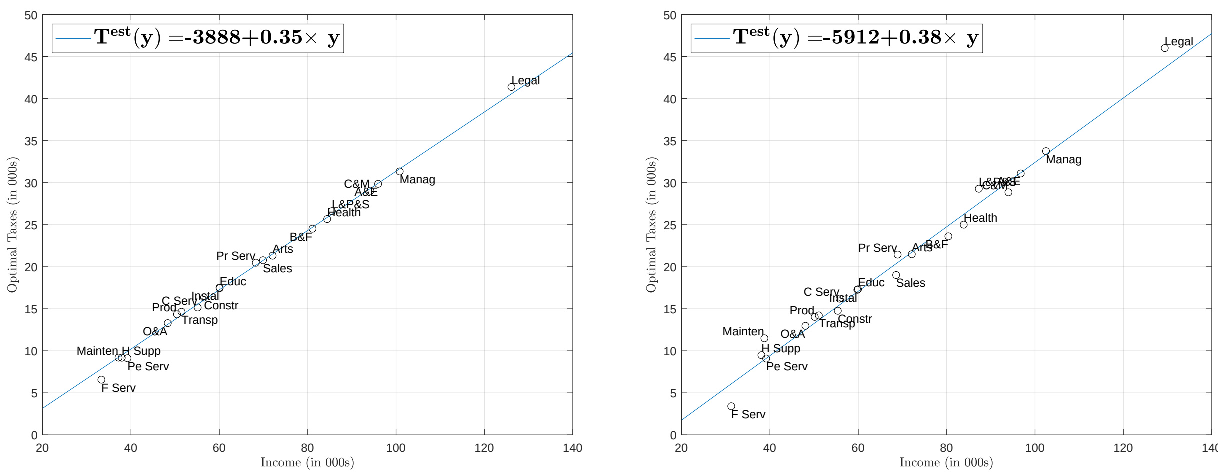
Figure D.5: Taxes and Average Income Tax Rates by Occupation at the Optimum and in the Data. Alternate welfare criterion.

average income taxes by occupation. Relative to the previous optimum, taxes are reduced in lower paid occupations inhabited disproportionately by younger workers and are raised in higher paid ones inhabited disproportionately by older workers. The intercept of the approximated tax falls and the slope rises. Again food services stands out, with taxes essentially reduced to zero. Concern for younger workers and recognition that workers transition into and out of food services relatively quickly encourages redistribution towards those in this occupation. In contrast, those in the legal occupation see a tax hike relative to the earlier optimum.

D.5 Results with entrant occupational choice

Our benchmark model assumes that the occupations of entrants is fixed exogenously. Our subsequent educational types model allows entrant workers to pick both their initial education and occupation. In this subsection, we give results for the case without educational types, but with entrant selection of occupations. Specifically, we allow entrants to choose their initial occupation according to the logit formulation in (B.18). We apply dynamic hat algebra to generate responses of entrants to policy variation.

Figure D.6 displays optimal taxes for the model with entrant choice under two social welfare criteria. In the first, effective Pareto weights are set to be one, as in the main text. In the second, we set Pareto weights to equal the distribution of entrants' states. Optimal marginal income taxes are depressed relative the ones in the main text or under the alternative optimality criterion in the appendix because the government does not want to distort entrant choices.

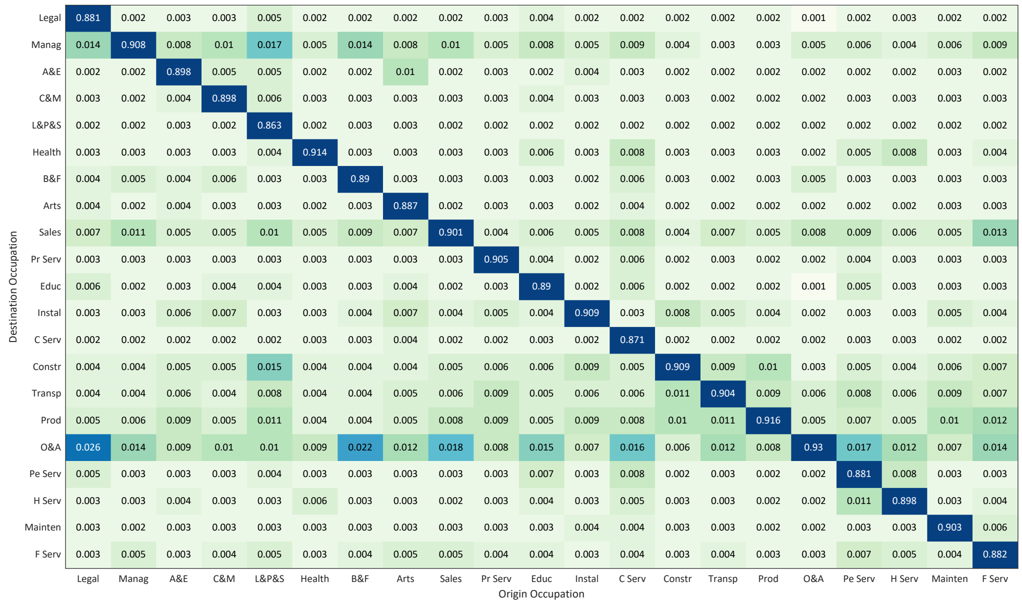


(a) Optimal taxes in benchmark criteria (b) Optimal taxes in alternative criteria

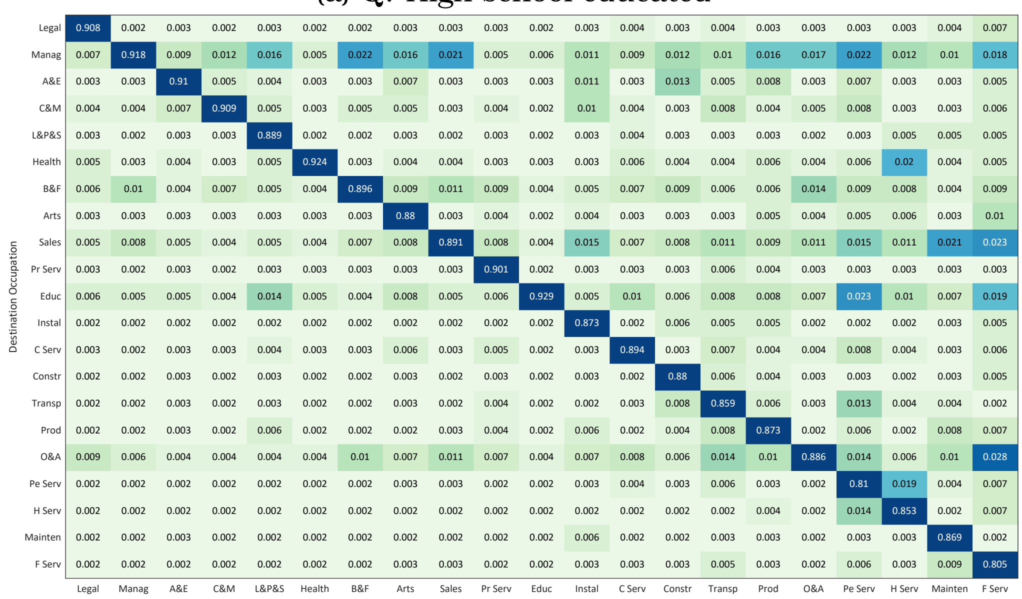
Figure D.6: Optimal taxes for entrant choice case.

D.6 Markov transition matrices for the educational type model

This section supplements results in the main text for the model with educational types. Figure D.7 shows Q matrices for the two education groups. The high school Q



(a) Q: High school educated



(b) Q: College educated.

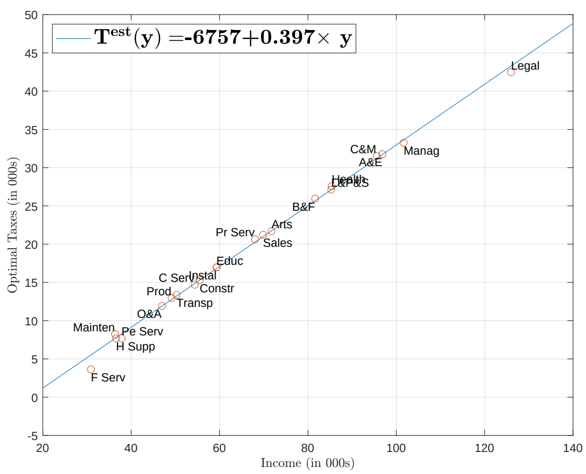
Figure D.7: Q matrices for different education groups at the empirical policy.

matrix shows significantly less upward mobility towards higher paying white collar occupations than does the college Q matrix. (Compare the upper triangles of the respective transition matrices and the lower retention rates for college educated workers in low paid occupations relative to high school educated workers). Instead, the high school Q matrix shows greater mobility to and from blue collar occupations like construction, transportation and production and the lower paid office and administration occupation.

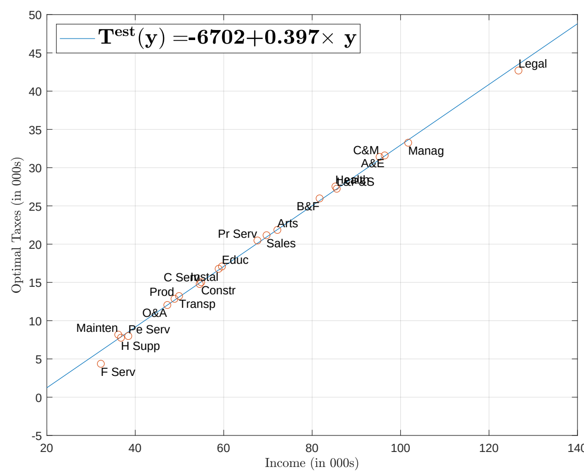
D.7 Robustness exercises

This section describes sensitivity of the benchmark results to various robustness exercises.

Elasticity of Substitution of Occupational Labor In the main text, we assume a Cobb Douglas technology. We now consider implications of alternative degrees of substitutability across occupations. Suppose a CES production function: $F(\mathbf{P}) = \sum_{i \in \mathcal{I}} \left(\phi(i) \mathbf{P}(i)^{\frac{\epsilon}{\epsilon - 1}} \right)^{\frac{\epsilon - 1}{\epsilon}}$ where $\phi(i)$ is the share of occupation i and ϵ is the elasticity of substitution of labor across occupations. Figure D.8 displays optimal taxes for the cases $\epsilon = 0.5$ and $\epsilon = 2$. These variations have negligible impacts on optimal taxes.



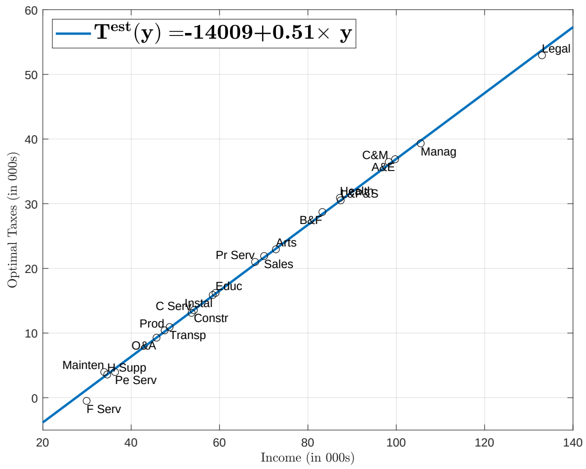
(a) Taxes: Optimal, $\epsilon = 0.5$.



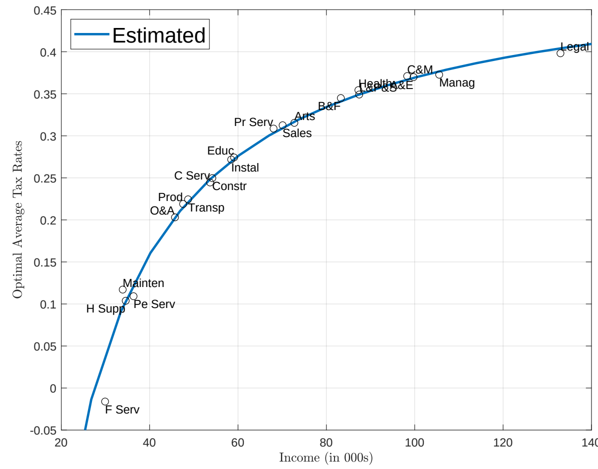
(b) Taxes: Optimal, $\epsilon = 2$.

Figure D.8: Optimal taxes under alternative elasticities of substitution.

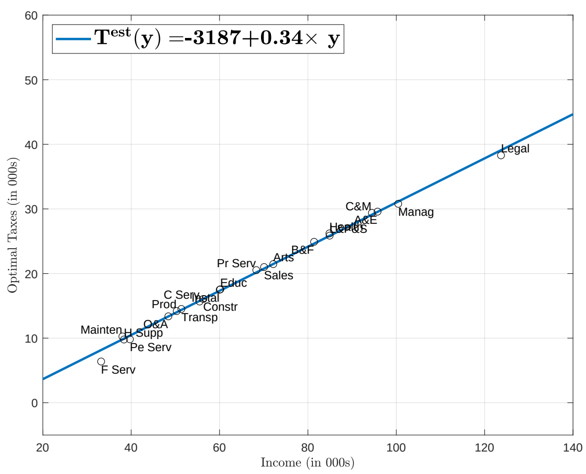
Variation of the a parameter The parameter a controls the sensitivity of utility to log consumption and is an important parameter. It is the target of our structural estimation (and the target of many discrete choice estimations).



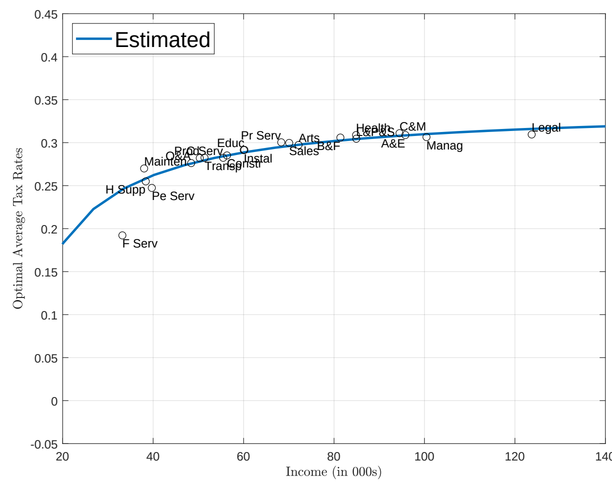
(a) Taxes: Optimal, $a = 0.09$.



(b) Average Tax Rates: Optimal, $a = 0.09$.



(c) Taxes: Optimal, $a = 0.18$.



(d) Average Tax Rates: Optimal, $a = 0.18$.

Figure D.9: Taxes and Average Income Tax Rates by Occupation at the Optimum. $a = 0.09$ and $a = 0.18$.

Figure D.9 shows the optimal tax function for two values of a: $a = 0.09$ and $a = 0.18$, that are, respectively, lower and higher than our benchmark estimate of $a = 0.142$. Given these modified values, we continue to compute counterfactual Q values according to Proposition C.1. Although variation in a leaves the broad affine structure of optimal taxes intact (along with the particular treatment of low income/high churn food services), it significantly impacts the intercept and slope of the affine approximation. In particular, higher values of a are associated with larger long run stationary distribution elasticities and long run marginal excess burdens (in absolute value) and an optimal affine tax approximation with a correspondingly larger intercept and lower marginal tax rate. As a increases from 0.09 to 0.18, the tax intercept rises from -$14,009 to -$3,187 (both 2019 USD) and the marginal rate falls from 0.51 to 0.34. Thus, a is a central parameter in determining optimal tax code redistributiveness.

Decomposing Management In the main text we decompose occupations to 21 aggregates corresponding to 2-digit 2010 SOC codes. This decomposition represents a compromise between reliable estimation of flows between occupations, which argues for greater aggregation, and isolation of differently paying options for workers, which motivates a finer decomposition. Management is a large occupation at the main occupation level that encompasses an array of differently paying roles. To accommodate this and assess the extent to which aggregation of different management roles affects results we decompose the management occupation into five categories labeled M1, M2, M3, M4, and M5. These categories group finer management occupations with similar mean incomes. Table D.2 details the decomposition.

Label	Income Range	Finer Management Occupations
M1	> 2	CEO's, legislators, public administrators
M2	$1.66 - 2$	computer & information systems managers, architectural & engineering managers, natural science managers
M3	$1.33 - 1.66$	general & operations managers, marketing managers, financial managers, human resources managers, education administrators, medical & health services managers
M4	$1.1 - 1.33$	administrative services managers, industrial production managers, purchasing managers, construction managers, funeral directors, property, real estate, community association managers, social & community service managers
M5	< 1.1	All others

Note: Finer management occupations grouped into categories by their mean income, expressed as a multiple of the mean income across the entire population.

Table D.2: Management Categories Definition

Figure D.10 shows the structure of long run elasticities in this case. Several of the management categories are long run complements. M5 has the lowest long run own elasticity and, in contrast, one the highest short run own elasticities of 0.185. Like food services, M5 has significant churn of workers relative to other similarly earning occupations. In particular, there are high flow rates to and, to a lesser extent, from other higher earning management categories. This lowest paid management occupation displays characteristics of a stepping stone occupation, with relatively large flows up to M3 and M4.

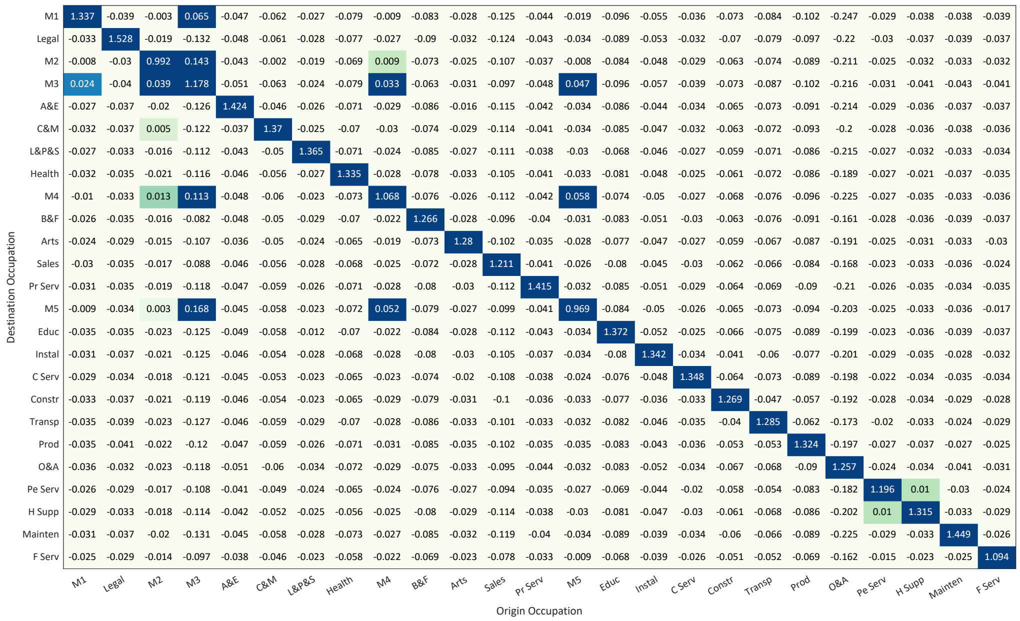
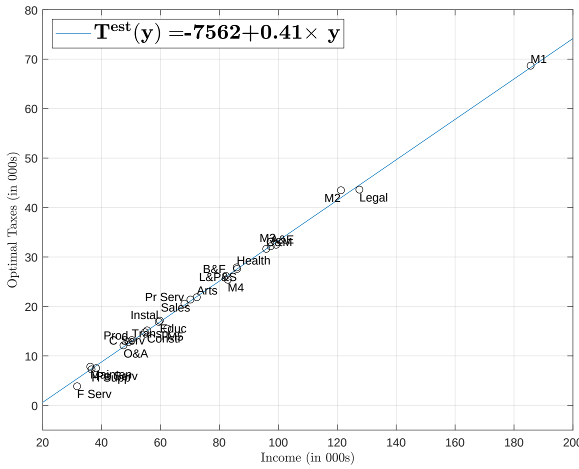
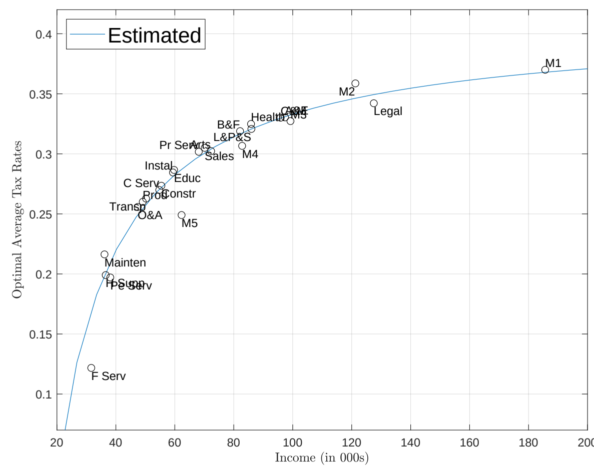


Figure D.10: Long Run Elasticities at the Data

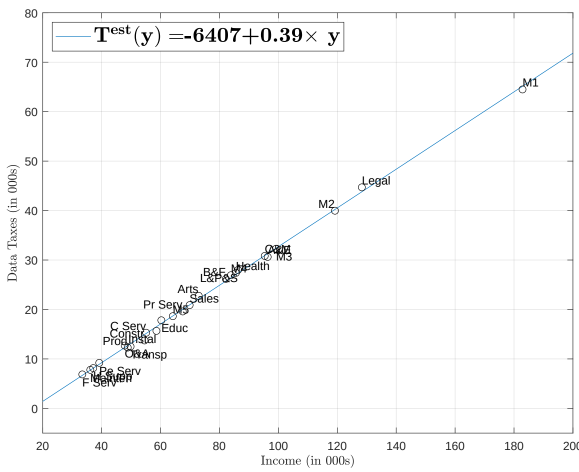
Optimal taxes are illustrated in Figure D.11. The broad structure retains the approximate affine form of the benchmark case. Now, however, M5 management occupation joins food services in receiving a relatively more favorable optimal tax treatment. In particular, the flows from occupation M5 to higher income occupations are higher compared to the flows from similarly earning occupations education and installation. Therefore, the optimal tax rate for M5 occupation is 4% lower compared to the optimal rates of the these other occupations.



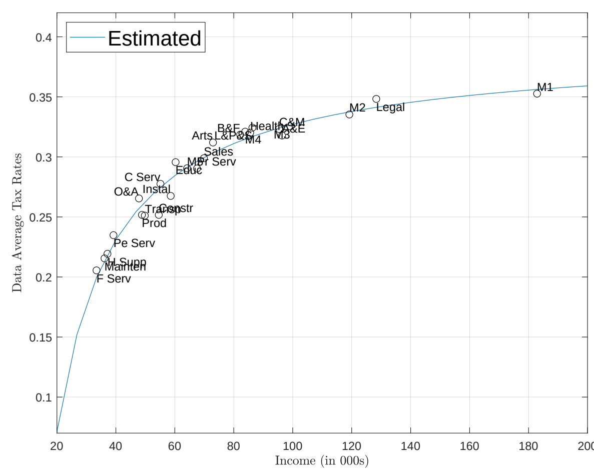
(a) Taxes: Optimal.



(b) Average Income Tax Rates: Optimal.



(c) Taxes: Data.



(d) Average Income Tax Rates: Data.

Figure D.11: Taxes and Average Income Tax Rates by Occupation at the Optimum and in the Data under Decomposition of Management.

D.8 Mobile/Immobile type case

We now provide results for the model with latent mobile/immobile types described in Appendix B.4.3. We first describe how to take this extension to the data and then give the results.

D.8.1 Taking the mobile/immobile types case to the data

To implement formulas for the case with persistent mobile and immobile types quantitatively, it is necessary to recover estimates of ρ, $\mathcal{Q}$ (and a) from the data.

Identifying ρ. Define the gross flow rate of survivor workers as:

$$g = \{\mu(1-\rho(n|m)) + (1-\mu)\rho(m|n)\}\, z, \tag{D.6}$$

where $z = \sum_{x' \neq x, x} \mathcal{Q}(x', x)\mathbf{P}(x|m)$ gives the fraction of mobile types that choose to move. If a worker changes its choice state at t, then they are a mover at t and, conditional on surviving, change their choice state at $t+1$ and $t+2$ with probabilities:

$$\upsilon_1 = z(1-\rho(n|m)) \quad \text{and} \quad \upsilon_2 = \{(1-\rho(n|m))^2 + \rho(m|n)\rho(n|m)\}z. \tag{D.7}$$

Since δ, g, υ_1 and υ_2 are observable (in panels of two period duration), (B.11), (D.6) and (D.7) reduce to three equations in the three unknowns: $\{\mu, \rho(m|n), \rho(n|m), z\}$.

Identifying $\mathcal{Q}$ In the data, at a stationary equilibrium, the distribution of workers over chosen states unconditional on type, $\overline{\mathbf{P}}(x) = \mathbf{P}(x,m) + \mathbf{P}(x,n)$, and the Markov transition for survivors unconditional on mobility type $\overline{\mathbf{Q}}$ satisfy:

$$\begin{aligned}\overline{\mathbf{Q}}(x'|x) =& \mathcal{Q}(x'|x)\frac{(1-\rho(n|m))\mathbf{P}(x,m) + \rho(m|n)\mathbf{P}(x,n)}{\overline{\mathbf{P}}(x)} \\ &+ \mathbf{I}(x'|x)\frac{\rho(n|m)\mathbf{P}(x,m) + (1-\rho(m|n))\mathbf{P}(x,n)}{\overline{\mathbf{P}}(x)},\end{aligned} \tag{D.8}$$

Or:

$$\begin{aligned}\overline{\mathbf{Q}}(x'|x) =& \mathcal{Q}(x'|x)\{(1-\rho(n|m))\mu + \rho(m|n)(1-\mu)\} \\ &+ \mathbf{I}(x'|x)\{\rho(n|m)\mu + (1-\rho(m|n))(1-\mu)\},\end{aligned} \tag{D.9}$$

Then, $\mathcal{Q}$ is recoverable using (D.9) and the estimates of ρ and μ.

Estimation of a Artuç et al (2010) describe how to estimate a given a recovered sequence of Markov transitions for survivors and knowledge of ρ. For completeness, we reproduce their result. Estimation of a utilizes equation (D.10) derived in Proposition D.2. This proposition extends Proposition D.1 to the setting with hidden mobility types.

Proposition D.2. *Assume a model with persistent mobile and immobile types. Given an inter-temporal consumption allocation $\{\mathbf{c}_t\}_{t=1}^{\infty}$ and corresponding transitions for mobile survivors $\{\mathcal{Q}_t\}_{t=0}^{\infty}$:*

$$\begin{aligned}\log \mathcal{Q}_t(x',x) - \log \mathcal{Q}_t(x,x) =& \beta\rho(m|m)\mathbb{E}[\log \mathcal{Q}_{t+1}(x',x) - \log \mathcal{Q}_{t+1}(x',x')|I_t] \\ &+ \beta\rho(n|n)\mathbb{E}[\log \mathcal{Q}_{t+1}(x',x) - \log \mathcal{Q}_{t+1}(x,x)|I_t] \\ &+ \beta^2\{\rho(m|n)\rho(n|m) + \rho(n|n)\rho(m|m)\}\mathbb{E}[\log \mathcal{Q}_{t+2}(x',x) - \log \mathcal{Q}_{t+2}(x',x')|I_t] \\ &+ a\mathbb{E}[\{\log \mathbf{c}_t(x') - \log \mathbf{c}_t(x)\} + \beta\{\rho(n|m) - \rho(n|n)\}\{\log \mathbf{c}_{t+1}(x') - \log \mathbf{c}_{t+1}(x)\}|I_t] \\ &+ H\boldsymbol{\kappa}(x',x).\end{aligned} \tag{D.10}$$

where H is defined in the proof of the proposition, *Proof.* By definition, $\mathcal{Q}_t(x',x) = \frac{\exp^{\mathbf{V}_t(x',m)-\boldsymbol{\kappa}(x',x)}}{\sum_{x''\in\mathcal{X}}\exp^{\mathbf{V}_t(x'',m)-\boldsymbol{\kappa}(x'',x)}}$, with lifetime payoffs: $\mathbf{V}_t(x'',m) = \mathbb{E}[u(\mathbf{c}_t(x'')) + \rho(m|m)\beta\log\sum_{x'''\in\mathcal{X}}\exp^{\mathbf{V}_{t+1}(x''')-\boldsymbol{\kappa}(x''',x'')} + \rho(n|m)\beta\mathbf{V}_{t+1}(x'',n)|I_t]$ and $u = a\log$. So, for all x, x': $\mathbf{V}_t(x',m)-\boldsymbol{\kappa}(x',x)-\log\mathcal{Q}_t(x',x) = \overline{\mathbf{V}}_t(x,m) := \log\sum_{x''\in\mathcal{X}}\exp^{\mathbf{V}_t(x'',m)-\boldsymbol{\kappa}(x'',x)}$. Consequently, we have: $\overline{\mathbf{V}}_t(x,m) = \mathbf{V}_t(x',m) - \boldsymbol{\kappa}(x',x) - \log\mathcal{Q}_t(x',x) = \mathbb{E}[u(\mathbf{c}_t(x')) + \beta\rho(m|m)\ \overline{\mathbf{V}}_{t+1}(x',m) + \beta\rho(n|m)\mathbf{V}_{t+1}(x',n)|I_t] - \boldsymbol{\kappa}(x',x) - \log\mathcal{Q}_t(x',x) = \mathbb{E}[u(\mathbf{c}_t(x'))+ \beta\rho(m|m)\ \mathbf{V}_{t+1}(x',m) - \log\mathcal{Q}_{t+1}(x',x') + \beta\rho(n|m)\ \mathbf{V}_{t+1}(x',n)|I_t] - \boldsymbol{\kappa}(x',x) - \log\mathcal{Q}_t(x',x)$, where $\boldsymbol{\kappa}(j,j) = 0$ is applied. And for $x' = x$, $\overline{\mathbf{V}}_t(x,m) = \mathbf{V}_t(x,m) - \log\mathcal{Q}_t(x,x) = \mathbb{E}[u(\mathbf{c}_t(x)) + \beta\rho(m|m)\ \overline{\mathbf{V}}_{t+1}(x,m) + \beta\rho(n|m)\mathbf{V}_{t+1}(x,n)|I_t] - \log\mathcal{Q}_t(x,x) = \mathbb{E}[u(\mathbf{c}_t(x)) + \beta\rho(m|m)\mathbf{V}_{t+1}(x',m) - \boldsymbol{\kappa}(x',x) - \log\mathcal{Q}_{t+1}(x',x) + \beta\rho(n|m)\mathbf{V}_{t+1}(x,n)|I_t] - \log\mathcal{Q}_t(x,x)$. Combining these conditions gives:

$$\begin{aligned}\log\mathcal{Q}_t(x',x) - \log\mathcal{Q}_t(x,x) &= \\ &\mathbb{E}[\Delta u_t(x',x) - \beta\rho(m|m)\{\log\mathcal{Q}_{t+1}(x',x') - \log\mathcal{Q}_{t+1}(x',x) - \boldsymbol{\kappa}(x',x)\} \\ &\quad + \beta\rho(n|m)\{\mathbf{V}_{t+1}(x',n) - \mathbf{V}_{t+1}(x,n)\}|I_t] - \boldsymbol{\kappa}(x',x),\end{aligned} \tag{D.11}$$

where: $\Delta u_t(x',x) := u(\mathbf{c}_t(x')) - u(\mathbf{c}_t(x))$. Now, using the equations for the evolution of the non-mobile type's payoffs and the fact that, as obtained above, $\overline{\mathbf{V}}_{t+1}(x',m) = \mathbf{V}_{t+1}(x',m) - \log\mathcal{Q}_{t+1}(x',x')$ and $\overline{\mathbf{V}}_{t+1}(x,m) = \mathbf{V}_{t+1}(x',m) - \log\mathcal{Q}_{t+1}(x',x) - \boldsymbol{\kappa}(x',x)$,

$$\begin{aligned}\mathbf{V}_t(x'|n) - \mathbf{V}_t(x|n) &= \\ &\mathbb{E}[\Delta u_t(x',x) - \beta\rho(m|n)\{\log\mathcal{Q}_{t+1}(x',x') - \log\mathcal{Q}_{t+1}(x',x) - \boldsymbol{\kappa}(x',x)\} \\ &\quad + \beta\rho(n|n)\{\mathbf{V}_{t+1}(x',n) - \mathbf{V}_{t+1}(x,n)\}|I_t].\end{aligned} \tag{D.12}$$

Pushing (D.12) forward one period and substituting into (D.11) gives:

$$\begin{aligned}\log\mathcal{Q}_t(x',x) - \log\mathcal{Q}_t(x,x) &= \\ &\mathbb{E}[\Delta u_t(x',x) - \beta\rho(m|m)\{\log\mathcal{Q}_{t+1}(x',x') - \log\mathcal{Q}_{t+1}(x',x) - \boldsymbol{\kappa}(x',x)\} \\ &\quad + \beta\rho(n|m)\Delta u_{t+1}(x',x) \\ &\quad - \beta^2\rho(m|n)\rho(n|m)\{\log\mathcal{Q}_{t+2}(x',x') - \log\mathcal{Q}_{t+2}(x',x) - \boldsymbol{\kappa}(x',x)\} \\ &\quad + \beta^2\rho(n|n)\rho(n|m)\{\mathbf{V}_{t+2}(x',n) - \mathbf{V}_{t+2}(x,n)\}|I_t] - \boldsymbol{\kappa}(x',x).\end{aligned} \tag{D.13}$$

Next push (D.11) forward one period and rearrange to get:

$$\begin{aligned}\beta\rho(n|m)\mathbb{E}[\mathbf{V}_{t+2}(x',n) - \mathbf{V}_{t+2}(x,n)|I_{t+1}] &= \\ &\log\mathcal{Q}_{t+1}(x',x) - \log\mathcal{Q}_{t+1}(x,x) - \mathbb{E}[\Delta u_{t+1}(x',x) \\ &+ \beta\rho(m|m)\{\log\mathcal{Q}_{t+2}(x',x') - \log\mathcal{Q}_{t+2}(x',x) - \kappa(x',x)\}|I_{t+1}] + \boldsymbol{\kappa}(x',x).\end{aligned} \tag{D.14}$$

Finally, substitute (D.14) into (D.13):

$$
\begin{aligned}
\log \boldsymbol{\mathcal{Q}}_t(x',x) - \log \boldsymbol{\mathcal{Q}}_t(x,x) = \quad & \\
\mathbb{E}[\Delta u_t(x',x) - \beta\rho(m|m)\{\log \boldsymbol{\mathcal{Q}}_{t+1}(x',x') - \log \boldsymbol{\mathcal{Q}}_{t+1}(x',x) - \boldsymbol{\kappa}(x',x)\}|I_t] & \\
+ \beta\rho(n|m)\mathbb{E}[\Delta u_{t+1}(x',x)|I_t] & \\
- \beta^2\rho(m|n)\rho(n|m)\mathbb{E}[\log \boldsymbol{\mathcal{Q}}_{t+2}(x',x') - \log \boldsymbol{\mathcal{Q}}_{t+2}(x',x) - \boldsymbol{\kappa}(x',x)|I_t] & \\
+ \beta\rho(n|n)\mathbb{E}[\log \boldsymbol{\mathcal{Q}}_{t+1}(x',x) - \log \boldsymbol{\mathcal{Q}}_{t+1}(x,x) - \Delta u_{t+1}(x',x)|I_t] & \\
+ \beta^2\rho(n|n)\rho(m|m)\mathbb{E}[\log \boldsymbol{\mathcal{Q}}_{t+2}(x',x') - \log \boldsymbol{\mathcal{Q}}_{t+2}(x',x) - \boldsymbol{\kappa}(x',x)|I_t] & \\
+ \beta\rho(n|n)\boldsymbol{\kappa}(x',x) - \boldsymbol{\kappa}(x',x). &
\end{aligned} \tag{D.15}
$$

Reorganize this as:

$$
\begin{aligned}
\log \boldsymbol{\mathcal{Q}}_t(x',x) - \log \boldsymbol{\mathcal{Q}}_t(x,x) = {} & \beta\rho(m|m)\mathbb{E}[\log \boldsymbol{\mathcal{Q}}_{t+1}(x',x) - \log \boldsymbol{\mathcal{Q}}_{t+1}(x',x')|I_t] \\
& + \beta\rho(n|n)\mathbb{E}[\log \boldsymbol{\mathcal{Q}}_{t+1}(x',x) - \log \boldsymbol{\mathcal{Q}}_{t+1}(x,x)|I_t] \\
& - \beta^2\{\rho(m|n)\rho(n|m) + \rho(n|n)\rho(m|m)\}\mathbb{E}[\log \boldsymbol{\mathcal{Q}}_{t+2}(x',x') - \log \boldsymbol{\mathcal{Q}}_{t+2}(x',x)|I_t] \\
& + \mathbb{E}[\Delta u_t(x',x)|I_t] + \beta\{\rho(n|m) - \rho(n|n)\}\mathbb{E}[\Delta u_{t+1}(x',x)|I_t] \\
& + \{-1 + \beta\{\rho(m|m) + \rho(n|n)\} + \beta^2\{\rho(m|n)\rho(n|m) - \rho(n|n)\rho(m|m)\}\}\boldsymbol{\kappa}(x',x).
\end{aligned}
$$

□

Equation (D.10) can be estimated via GMM to recover a. Given an estimated a and values for ρ, we use dynamic hat algebra to construct responses to counterfactual policy variation.

D.8.2 Results for the mobile/immobile types model

We use estimated values in Artuç et al (2010) to construct transition matrix over mobility types, i.e. $\rho(n|m) = 0.441$ and $\rho(m|n) = 0.146$. Together with (B.11), these values imply that 28% of population consists of mobile workers.

We first set $a = 0.142$ as in the main text and calculate the long run marginal excess burdens implied by the model in Appendix B.4.3 at the empirical policy. Figure D.12 shows that the marginal excess burden values are very close to the ones in the main text.

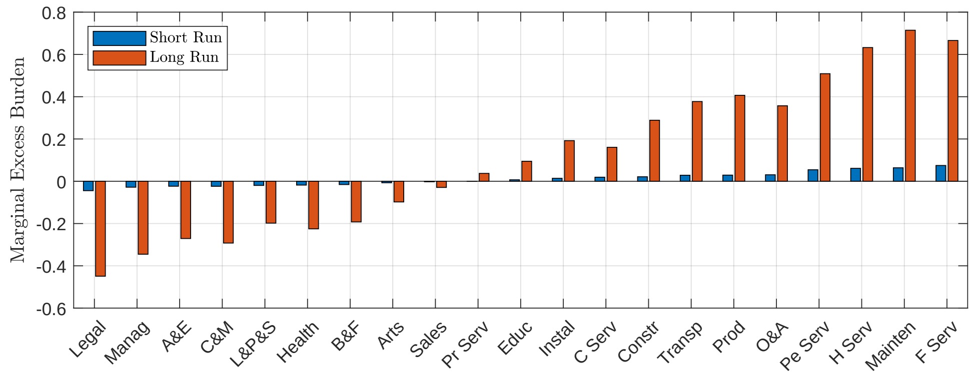
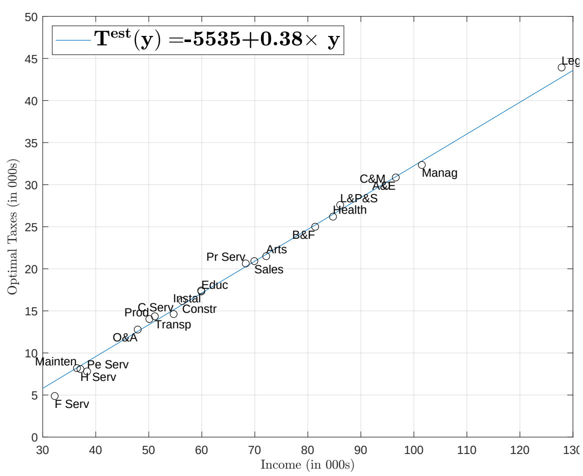
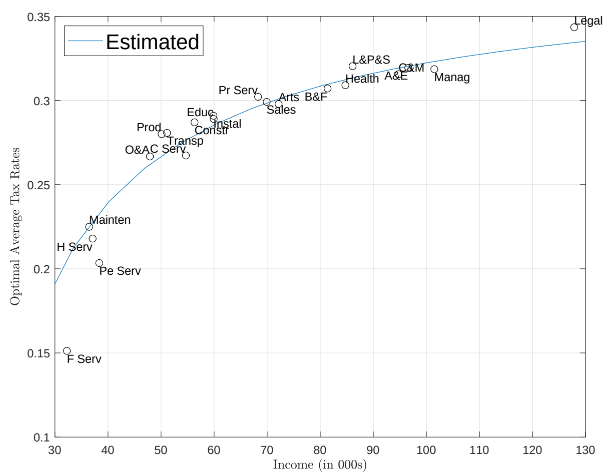


Figure D.12: Short and long run marginal excess burdens in the data when $a = 0.142$

We then calculate optimal taxes for this case. Figure D.13 shows that optimal taxes follow almost an affine tax system. Both lump-sum transfers ($5,534) and optimal marginal income tax rates (0.38) are very close to the optimal taxes in the main text. We continue to observe that the Food Services workers receive a better tax treatment- $1,771 less taxes compared to the implied taxes by affine tax system- as in the main text.



(a) Taxes: Optimal.



(b) Average Income Tax Rates: Optimal.

Figure D.13: Taxes and Average Income Tax Rates by Occupation at the Optimum with Unobservable Heterogeneity when $a = 0.142$.

Next, we re-estimate a using the estimation strategy in Artuç et al (2010) and Proposition D.2. When we use imputation, i.e. replacing zeros with 1e-3, for the transition matrix, we find $a = 0.30$. When we use PPML for smoothing the transition matrix, we find $a = 0.05$. We take the average of these and set $a = 0.175$ which is in the 95% confidence interval for both estimated a value. We again show marginal excess burden in the data. The values are slightly higher due to the increase in a.

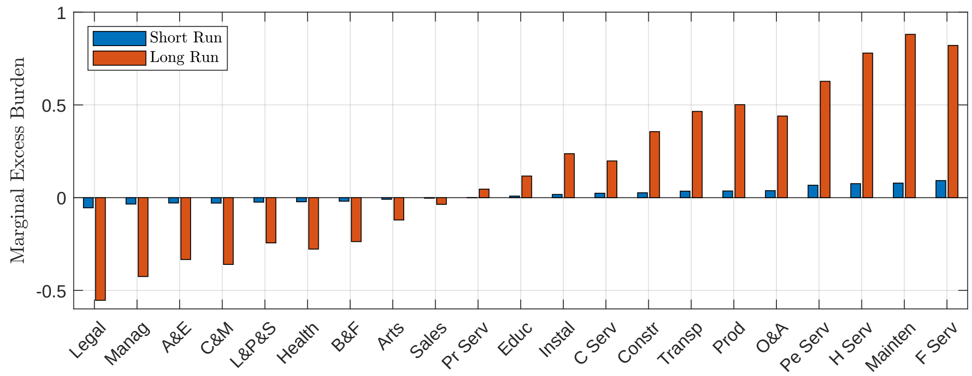
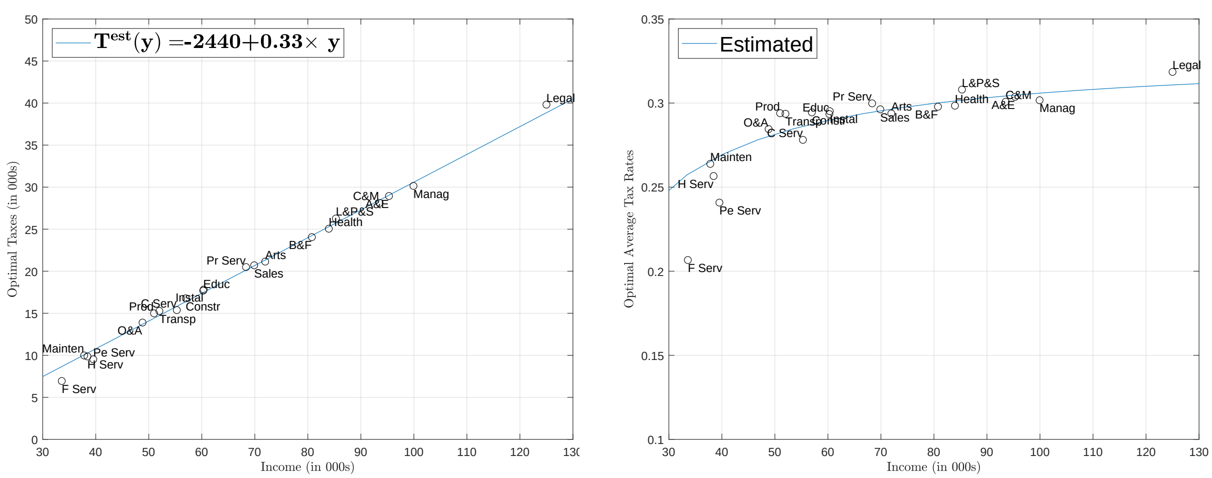


Figure D.14: Short and Long Run Marginal Excess Burdens in the Data when $a = 0.175$

Next, we calculate optimal taxes. We observe that the optimal marginal tax rate is 0.33, a lower rate compared to the previous optimal rate. Higher a implies that workers are more sensitive to consumption differentials, which increases sensitivities and marginal excess burdens. Therefore, the government sets a lower rate for this environment. On the other hand, we still observe that the generous tax treatment for Food Service workers stands. At the optimum, they pay \$1,713 less taxes than the implied optimal tax rate by the affine tax system.



(a) Taxes: Optimal.

(b) Average Income Tax Rates: Optimal.

Figure D.15: Taxes and Average Income Tax Rates by Occupation at the Optimum with Unobservable Heterogeneity when $a = 0.175$.

E Further Extensions

This appendix sketches extensions that accommodate transitions, stochastic aging, and externalities.

Transitions The theory developed in the body of the paper can be modified to accommodate optimal transitions by expanding the state space to incorporate dates as well as work states. Let $\mathbf{c} = \{\mathbf{c}(i,t)\}_{i\in\mathcal{I},t\in\mathbb{N}}$ denote an allocation of consumption across work states and time and assume a resource constraint:

$$\sum_{t=1}^{\infty} q_t \{F(\mathbf{P}_t(\mathbf{c})) - \mathbf{c}_t \cdot \mathbf{P}_t(\mathbf{c}) - G\} \geq 0, \tag{E.1}$$

with $\mathbf{q}^\infty = \{q_{t+1}\}_{t=0}^{\infty}$ an exogenous sequence of intertemporal prices normalized so that $\sum_{t=1}^{\infty} q_t = 1$ and $\mathbf{P}_t$ the distribution of agents over work states at t. Letting $M : \mathbb{R}_+^\infty \to \mathbb{R}$ denote a smooth, concave and increasing societal objective defined over (intertemporal) consumption allocations $\mathbf{c}$ gives rise to a first order condition with identical structure to (17):

$$\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^\top = \mathbf{P}^q - \left(\frac{\partial \mathbf{P}^q}{\partial \mathbf{c}}\right)^\top \mathbf{T}, \tag{E.2}$$

where $\mathbf{P}^q = \{q_t \mathbf{P}_t\}_{t=1}^{\infty}$ is the implied "distribution" of workers over dates and states that convolutes prices and state distributions, $\frac{\partial \mathbf{M}}{\partial \mathbf{c}}$ is the derivative of M at the optimum, and $\mathbf{T}$ is the family of tax wedges $\{\frac{\partial F(\mathbf{P}_t)}{\partial \mathbf{p}(i)} - \mathbf{c}_t(i)\}$. Extracting the first order condition for a specific $\mathbf{c}_t$ from (E.2) and expressing it in terms of the component $\{\mathbf{P}_s\}$ gives:

$$\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}_t}^\top = q_t \mathbf{P}_t - \sum_{s=1}^{\infty} q_s \left(\frac{\partial \mathbf{P}_s}{\partial \mathbf{c}_t}\right)^\top \mathbf{T}_s, \tag{E.3}$$

with the final term on the right hand side of (E.3) giving the dynamic marginal excess burden associated with a consumption perturbation at t. Now, a consumption perturbation at t impacts the forward-looking behavior of workers and, hence, the transitions $\mathbf{Q}_s$ in all periods prior to t. It affects state distributions in periods both before and after t via the impact of these transitions. Using the law of motion for probability distributions $\mathbf{P}_t$ and the chain rule for matrix derivatives, the cross-sensitivities of population shares at date s with respect to consumption at date t may be computed. Lemma E.1 gives the formula.

Lemma E.1. *The Jacobian of* $\mathbf{P}_s$ *with respect to the consumption allocation* $\mathbf{c}_t$ *is given by:*

$$\frac{\partial \mathbf{P}_s}{\partial \mathbf{c}_t} = \sum_{r=1}^{\min(s,t)} \mathbf{Q}_{r+1}^s \mathbf{\Phi}_{r,t}, \tag{E.4}$$

where $\mathbf{Q}_{r+1}^s = \prod_{r+1}^{s} \mathbf{Q}_m$ *and* $\mathbf{\Phi}_{r,t} = \left[(\mathbf{P}_{r-1})^\top \otimes \mathbf{I}\right] \frac{\partial(\text{vec}\, \mathbf{Q}_r)}{\partial \mathbf{c}_t}$.

In (E.4), $\mathbf{Q}_{r+1}^s$ acts as the propagation factor for the impact of the short-run (one period) sensitivity at r, $\mathbf{\Phi}_{r,t}$, on the state distribution at s. The expression in (E.4) combines all of these impacts at dates $1 \leq r \leq s$ to get the overall sensitivity at s. Note that perturbations of the consumption allocation at t can only affect behavior and the short-run transition in periods prior to t. Thus, only transitions in periods between 1 and $\min(s,t)$ are cumulated into expression (E.4). Models of short-run state distribution responses $\mathbf{\Phi}_{r,t}$ may be integrated into expressions (E.4). For example, if a dynamic logit model without perpetual youth is assumed, then:

$$\frac{\partial \mathbf{P}_s}{\partial \mathbf{c}_t} = \sum_{r=1}^{\min(s,t)} (\mathbf{Q}^s_{r+1})^\top \{\mathbf{I} - \mathbf{S}_r\} \frac{\partial \mathbf{V}_{r+1}}{\partial \mathbf{c}_t}, \tag{E.5}$$

with period r substitution matrix: $\mathbf{S}_r = \mathbf{Q}_r^\top \mathbf{D}_{\mathbf{P}_{r-1}} \mathbf{Q}_r \mathbf{D}^{-1}_{\mathbf{P}_{r-1}}$ and $\mathbf{V}_{r+1}$ the lifetime payoff from period $r+1$.[44]

Stochastic Aging and Perpetual Youth Mobility is often greatest earlier in life when workers have many future periods over which to accrue returns on any costly work state transition. To keep the state space manageable (and to capture the fact that human depreciation is random), it is useful to assume a stochastic aging process. Let $\mathcal{S} = \{1, \ldots, \overline{s}\}$ denote the set of age states. Workers in age state $s \in \{1, \ldots, \overline{s}-1\}$ remain at s with probability $1-\boldsymbol{\delta}(s)$ and enter $s+1$ with probability $\boldsymbol{\delta}(s)$. A worker with age state $s = \overline{s}$ remains at s with probability $1-\boldsymbol{\delta}(\overline{s})$ and retires and is replaced by an entrant worker with probability $\boldsymbol{\delta}(\overline{s})$. The perpetual youth model is a special case of this structure with a single age state $\overline{s} = 1$. Let $\mathbf{D}$ denote the age transition matrix that collects the elements $\boldsymbol{\delta}(s)$. The stationary distribution of workers over age states is readily computed as $\{\mathbf{L}^s\}_{s\in\mathcal{S}}$ with $\mathbf{L}^s = \frac{1/\boldsymbol{\delta}(s)}{\sum_{s'\in\mathcal{S}} 1/\boldsymbol{\delta}(s')}$. To simplify assume that entrant workers are distributed across work states according to an exogenous distribution $\mathbf{P_0}$. Worker preferences are identical to those in the benchmark dynamic discrete choice model except that utility functions and costs of adjustment are permitted to depend on age.[45] In contrast, pre-tax incomes, tax policy and consumption are assumed to depend on work state, but not age. An incumbent worker receives an age and a Gumbel preference shock at the beginning of the period and updates its work state. The lifetime utility of a worker with current age s and work state i net of current Gumbel shock is:

$$\mathbf{V}^s(i) = u(\mathbf{c}(i), i, s) + \beta(1-\boldsymbol{\delta}(s))\overline{\mathbf{V}}^s(i) + \beta\boldsymbol{\delta}(s)\overline{\mathbf{V}}^{s+1}(i),$$

with $\overline{\mathbf{V}}^s(i) = \log \sum_{j\in\mathcal{I}} \exp^{\mathbf{V}^s(j) - \boldsymbol{\kappa}(j,i,s)}$ for $s \neq \overline{s}$ and $\overline{\mathbf{V}}^{\overline{s}+1}(i) = 0$. Let $\mathbf{Q}^s$ denote the Markov transition over work states for a worker of age s. The complete transition over age

[44] Simpler expressions for transition responses and for the dynamic marginal excess burden emerge if the policymaker is constrained to make a time invariant tax policy choice, wages are time invariant, and intertemporal prices are geometric: $q_t = (1-q)q^{t-1}$ for some $0 < q < 1$. Then, if $\mathbf{P_0} = \mathbf{P}$, we have that: $\frac{\partial \mathbf{P_s}}{\partial \mathbf{c}} = \sum_{r=0}^{s-1} \mathbf{Q}^r \mathbf{\Phi}$, and the dynamic excess burden becomes: $(1-q)\sum_{t=1}^{\infty} q^{t-1} (\frac{\partial \mathbf{P}_t}{\partial \mathbf{c}})^\top \mathbf{T} = (1-q)(\mathbf{\Phi})^\top (\mathbf{I} - q\mathbf{Q}^\top)^{-1}\mathbf{T}$, where $\mathbf{T}$ denotes the vector of stationary equilibrium taxes. In comparison to (18), the propagation factor $(\mathbf{I}-\mathbf{Q})^\#$ is replaced by (the resolvent) $(1-q)(\mathbf{I}-q\mathbf{Q})^{-1}$, which convolutes the price q with the transition $\mathbf{Q}$. In this case q parameterizes the policymaker's concern with intertemporal resource allocation. In the limiting case, $q \to 0$, the policymaker is concerned only with the short-run resource consequences of its policy choices and $\lim_{q\to 0}(1-q)(\mathbf{I}-q\mathbf{Q})^{-1} = \mathbf{I}$. Conversely, $\lim_{q\to 1}(1-q)(\mathbf{I}-q\mathbf{Q})^{-1} = (1-q)\sum_{t=0}^{\infty} q^t \mathbf{Q}^t \to (\mathbf{I}-\mathbf{Q})^\#$, the long-run propagation factor.

[45] And recall that amenity values are folded into costs of adjustment, so that these are implicitly age-state dependent.

and work states is assembled from D and $\{\mathbf{Q}^s\}$. The policymaker's problem may be formulated as in (16) with $\mathbf{P} = \sum_{s\in\mathcal{S}} \mathbf{P}^s\mathbf{L}^s$ the stationary distribution of workers over work states and each $\mathbf{P}^s$ the stationary distribution over work states conditional on age. The first order condition (17) and marginal excess burdens require evaluation of $\frac{\partial \mathbf{P}}{\partial \mathbf{c}}$. Exploiting the structure of this problem, define for $s = 1, \ldots, \overline{s}$, $\mathbf{P}^s = \mathbf{Q}^s\overline{\mathbf{P}}^s$, with $\overline{\mathbf{P}}^s = (1 - \boldsymbol{\delta}(s))\mathbf{P}^s + \boldsymbol{\delta}(s)\mathbf{P}^{s-1}$. Totally differentiating and rearranging then gives:

$$\frac{\partial \mathbf{P}^s}{\partial \mathbf{c}} = \sum_{n=0}^{\infty}(1 - \boldsymbol{\delta}(s))^n(\mathbf{Q}^s)^n \left\{ \mathbf{\Phi}^s + \boldsymbol{\delta}(s)\mathbf{Q}^s \frac{\partial \mathbf{P}^{s-1}}{\partial \mathbf{c}} \right\}, \tag{E.6}$$

where $\mathbf{\Phi}^s(j,k) = \sum_{i\in\mathcal{I}} \frac{\partial \mathbf{Q}^s(j,i)}{\partial \mathbf{c}(k)}\overline{\mathbf{P}}^s(i)$. Expression (E.6) resembles that from the basic model up to the inclusion of the $(1 - \boldsymbol{\delta}(s))^n$ terms in the propagation factor (which act as a dampening factor on propagation) and the respecification of the bracketed term as a sum of a short run within age group s sensitivity and a long run sensitivity of those aged $s-1$. The short run sensitivity $\mathbf{\Phi}^s = \{\mathbf{I} - \mathbf{S}^s\}\mathbf{D}_{\mathbf{P_1}^s}\frac{\partial \mathbf{V}}{\partial \mathbf{c}}^s$, where $\mathbf{P_1}^s = \mathbf{Q}^s\overline{\mathbf{P}}^s$ and $\mathbf{S}^s = \mathbf{Q}^s\mathbf{D}_{\overline{\mathbf{P}}^s}(\mathbf{Q}^s)^\top \mathbf{D}_{\mathbf{P_1}^s}{}^{-1}$, has a similar form to earlier sections. This system of sensitivities may be solved recursively starting with $s = 1$.

Externalities Production externalities are easily added by returning to the policymaker's problem (16) and deriving the first order condition:

$$\mathbf{D_P}^{-1}\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^\top = \mathbf{1} - \mathbf{D_P}^{-1}\left(\frac{\partial \mathbf{P}}{\partial \mathbf{c}}\right)^\top \left\{\frac{\partial F(\mathbf{P})}{\partial \mathbf{p}} - \mathbf{c}\right\} \tag{E.7}$$

and then substituting for taxes:

$$\mathbf{D_P}^{-1}\frac{1}{\Upsilon}\frac{\partial \mathbf{M}}{\partial \mathbf{c}}^\top = \mathbf{1} - \mathbf{D_P}^{-1}\left(\frac{\partial \mathbf{P}}{\partial \mathbf{c}}\right)^\top \{\mathbf{E} + \mathbf{T}\}, \tag{E.8}$$

with $\mathbf{E} = \frac{\partial F(\mathbf{P})}{\partial \mathbf{p}} - \mathbf{w}$ an externality term giving the difference between the social and private marginal products of employment in each work state. The right hand side externality term may be consolidated with the left hand side marginal social welfare weight term to give an externality adjusted marginal social welfare weight. In the context of static optimal occupational taxation, Lockwood et al. (2017) provide a quantification of the externalities associated with occupations pursued by skilled workers. However, the magnitude of occupational externalities remains highly uncertain.

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Identification and estimation of dynamic matching models with unobserved heterogeneity.

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Abstract

We introduce a dynamic model of matching within a competitive market framework that explicitly incorporates time-invariant unobserved heterogeneity on both sides of the market along with endogenous separation and re-matching. In each period, agents make two sequential, forward-looking discrete choices. First, they decide whether to match with a partner or remain unmatched. In the second stage, they are committed to their matching status and face other choice margins. We establish the model's identification under the assumpton of stationarity in panel datasets where the sampling unit is the household but the measurement unit is the individual, such as the German SOEP.We propose an Expectation-Maximization (EM) algorithm for estimation. Monte Carlo simulations support our theoretical identification results and demonstrate that our model offers a computationally viable framework for studying the economic incentives underlying matching, separation, and re-matching, particularly in settings where unobserved characteristics play a critical role.

1 Introduction

The economic, empirical literature on household formation has continually drawn from the literature on discrete choices. The random utility model (RUM) of discrete choice (McFadden, 1973) served as a foundation to model one-to-one matching in a competitive market without search frictions (Choo and Siow, 2006). Dynamic discrete choice (DDC) models represent the extension of the RUM to dynamic settings and were pioneered by Rust (1987). Coupled with the identification and estimation strategy developed by Hotz and Miller (1993), which exploit nonparametrically identified conditional choice probabilities (CCPs) as nuisance parameters to identify the structural parameters of the model, they enabled the empirical study of a wide range of economic settings where agents are forward-looking and their choice sets are discrete. More recently, Choo (2015) and Gayle and Shephard (2019) extended the discrete-choice matching model to dynamic environments where agents are forward looking and need to take into account the possibility of separation in the future. Their identification and estimation strategy are based on CCPs and provide researchers with a tractable and computationally cheap way to study matching market where separation is possible. Though, they took separation as exogenous, thus limiting the analysis of its underlying economic incentives. In this paper, we build on that literature by allowing for endogenous separation and re-matching. Such events are important in the study of household formation, given that divorce and re-marriage are increasingly prevalent in many western countries (e.g. Livingston (2014)).

We study the identification of our model in the presence of time-invariant, individual characteristics on both sides of the market that are relevant for matching but unobserved to the econometrician. This is motivated by findings in the literature on the marital wage premium that suggest that selection on unobservables into marriage might account for the positive correlation between marital status and income that can be observed in the data Ludwig and Brüderl (2018). Because agents choose their match based on their own unobserved characteristics as well as those of their (potential) matches, identification is complicated by unobserved covariates and unobserved choices simultaneously. We first discuss the nonparametric identification of the CCPs and the distribution of the unobserved heterogeneity, then the identification of the structural utility parameters. Our nonparametric identification argument builds on Kasahara and Shimotsu (2009). It relies on multiple marginal distributions of observables providing a sufficient number of independent restrictions to identify the distribution of unobservable heterogeneity in the underlying population. In our model, agents make two sequential, forward-looking discrete choices at each discrete period of time. In the first stage, they decide whether to match with a partner or remain unmatched. In the second, they are committed to their matching status and face other choice margins. Second-stage CCPs are identified from the exogenous sampling process over match outcomes, adopted by panel datasets such as the German SOEP by sampling over households, and variation in match composition at the moment of sampling, together with the assumption of a stationary environment. Lastly, once second-stage CCPs are identified, first-stage CCPs can be calculated from a linear system of restrictions derived by integrating unobservable heterogeneity out of products of first- and second-stage CCPs.

To estimate the model, we adapt the Expectation-Maximization (EM) algorithm developed by Dempster et al. (1977) and applied to DDC models by Arcidiacono and Miller (2011). The required adjustments are minimal. In addition to calculating the posterior type distribution of each sampled agent, it is sufficient to compute the posterior type distribution of each of their partners in their observed history, conditional on the sampled agent's type. These posterior probabilities are then used as weights in the maximization of an expected likelihood function, where the expectation is taken over the unobserved heterogeneity of sampled agents and their partners, while unobserved heterogeneity is treated as observed inside the expectation .

We conclude this article with a Monte Carlo exercise to showcase the model and the estimation algorithm.

2 Model

Agents live T periods with $T \in \mathbb{N} \cup \infty$. Each male individual i and female individual j enters each period t endowed with their respective individual characteristics $\phi_{i,t} = (x_{i,t}, s_i)$,$\phi_{i,t} \in \Phi$, $x_{i,t} \in \mathscr{X}$, $s_i \in \mathscr{S}$, $\Phi = \mathscr{X} \times \mathscr{S}$. [1] The state x is observable to the econometrician while s indicates the unobserved characteristics. [2] Let X, and S denote the finite sizes of $\mathscr{X}$, and $\mathscr{S}$ respectively. The unobserved characteristics s are exogenous and time-invariant. Agents enter enter each period t as single or matched to their respective partners $l(i, t-1)$ and $r(j, t-1)$, whom they matched with in period $t-1$. To lighten the notation, denote $l(i,t) = \emptyset$ and $r(j,t) = \emptyset$ the case when i and j respectively do not match with any partner in period t. At each period, two stages of decision-making take place. In the first stage, agents decide whether to match with a partner and their partner's characteristics by maximizing their own expected lifetime utility in a competitive matching market. In the second stage, agents have committed to their match status and make another choice. Single agents make choices maximizing their own individual expected lifetime utility, while couples make choices maximizing a convex combination of each partner's expected lifetime utility. We use a tilde to group variables pertaining to a couple, e.g. $\tilde{s}_{ij} \equiv (s_i, s_j)$,$\tilde{x}_{ij,t} \equiv (x_{i,t}, x_{j,t})$, $\tilde{\phi}_{ij,t} \equiv (x_{i,t}, s_i, x_{j,t}, s_j)$. If $l(i,t) = \emptyset$ $(r(j,t) = \emptyset)$, then $\tilde{\phi}_{il(i,t),t} = \phi_{i,t}$ $(\tilde{\phi}_{r(j,t)j,t} = \phi_{j,t})$.

First stage. In the first stage, a male i (female j) draws a vector of preference shocks $(\theta_{i,t}(\phi), \theta_{i,t}(\emptyset)) \in \mathbb{R}^{\Phi+1}$ $((\theta_{j,t}(\phi), \theta_{j,t}(\emptyset))$ $\in \mathbb{R}^{\Phi+1})$ that are independent and identically Gumbel distributed and are independent across choices, time periods, and from any other agent's draw. Given these preference shocks, each agent solves their respective utility maximization problem:

$$\mathring{U}_t^f\left(\tilde{\phi}_{r(j,t-1)j,t}\right) = \max\left\{\max_{\phi\in\Phi}\left\{V_t^f(\phi,\phi_{j,t}) + \kappa^f\left(\phi|\tilde{\phi}_{r(j,t-1)j,t}\right) + \theta_{j,t}(\phi)\right\}, V_{\emptyset t}^f(\phi_{j,t}) + \kappa^f\left(\emptyset|\tilde{\phi}_{r(j,t-1)j,t}\right) + \theta_{j,t}(\emptyset)\right\} \tag{1}$$

$$\mathring{U}_t^m\left(\tilde{\phi}_{il(i,t-1),t}\right) = \max\left\{\max_{\phi\in\Phi}\left\{V_t^m(\phi_{i,t}\phi) + \kappa^m\left(\phi|\tilde{\phi}_{il(i,t-1),t}\right) + \theta_{j,t}(\phi)\right\}, V_{\emptyset t}^m(\phi_{i,t}) + \kappa^m\left(\emptyset|\tilde{\phi}_{il(i,t-1),t}\right) + \theta_{i,t}(\emptyset)\right\} \tag{2}$$

Where the function $\kappa^m\left(\phi|\tilde{\phi}_{il(i,t-1),t}\right)$ indicates the transition cost for male i who enters time t with characteristics ϕ_{it}, is matched with partner $l(i, t-1)$ with characteristics $\phi_{l(i,t-1)t}$, and transitions to a partner $l(i,t)$ with characteristics ϕ. [3] With a slight abuse of notation, $\kappa^m\left(\emptyset|\tilde{\phi}_{il(i,t-1),t}\right)$ indicates the cost incurred by agent i to become single. The function κ^f for females is similarly defined. We assume that $\kappa^m\left(\emptyset|\tilde{\phi}_{il(i,t-1),t}\right) = \kappa^f\left(\emptyset|\tilde{\phi}_{r(j,t-1)j,t}\right) = 0$ for all $\tilde{\phi}_{il(i,t-1),t}$ and $\tilde{\phi}_{r(j,t-1)j,t}$. This assumption is necessary for identification and it is standard in dynamic discrete choice models. The functions V_t^f and V_t^m are continuation values defined in detail below. Denote the expected lifetime utility at the beginning of stage one, before drawing the preference shocks $\boldsymbol{\theta}$ as:

$$U_t^f\left(\tilde{\phi}_{r(j,t-1)j,t}\right) = \mathbb{E}_{\theta_{j,t}}\mathring{U}_t^f\left(\tilde{\phi}_{r(j,t-1)j,t}\right)$$
$$U_t^m\left(\tilde{\phi}_{l(i,t-1),t}\right) = \mathbb{E}_{\theta_{i,t}}\mathring{U}_t^m\left(\tilde{\phi}_{il(i,t-1),t}\right)$$

The assumption that the market does not have search frictions implies that every agent that chooses to be matched will be matched. As a consequence, when an agent is single, it is because he *chose* to be single, regardless of whether he was matched in the previous period or not. Because we assume that there is no cost to transtion from partner l to partner l' when they have identical characteristics $\phi_{l,t}^f = \phi_{l',t}^f$, it is entirely indifferent for agent i whether he is matched to l or l'. As a consequence, the conditional choice probability of choosing to stay matched to l is identical to that of chosing to divorce l and immediately match with l'. Because agents can observe each other's state s, agents' identities provide no additional information to other agents. On the other hand, as we will show below, the econometrician can gain information about the unobserved state by tracking the identity of distinct spouses.

[1]The assumption that male and female inhabit the same space of characteristics simplifies our exposition, but it is not crucial.

[2]In general, x and z can each be mapped onto a vector of covariate values, i.e. a point in a space induced by such covariates. Therefore, the exposition of the model can proceed while considering x and z as the sole covariates, without loss of generality.

[3]The use of the t index twice might seem reduntant, but it is important below for values such as $x_{l(i,t-1),t}$, which indicates the value of x at time t for the individual who *was* matched to i at $t-1$. This value is distinct from $x_{l(i,t),t}$in that individuals $l(i,t-1)$ and $l(i,t)$ might be different.

Second stage. After matching decisions have been made, matched agents face the finite choice set $\mathscr{A}(\phi_{i,t},\phi_{j,t})$ of size $A(\phi_{i,t},\phi_{j,t})$, and single agents face the choice set $\mathscr{A}(\phi_{i,t})$ of size $A(\phi_{i,t})$. They draw a vector of independent, Gumbel-dstributed preference shocks $\varepsilon_{ij,t}\in\mathbb{R}^{A(\phi_{i,t},\phi_{j,t})}$and $\varepsilon_{i,t}\in\mathbb{R}^{A(\phi_{i,t})}$ respectively, having one element for each action in their choice sets. Shocks are independent across agents, time periods, and independently from the match preference shocks drawn during the matching stage. Matched agents and couples then proceed to choose the actions $\tilde{a}_{ij,t}$ and $a_{i,t}$ that maximize their expected lifetime utility, after which they enjoy their flow utilities. We define the flow utilities of males and females in couple (i,j) derived from action $\tilde{a}$ as $u_t^m(\tilde{a}|\phi_{i,t},\phi_{j,t})$ and $u_t^f(\tilde{a}|\phi_{i,t},\phi_{j,t})$ respectively, and the utility of single male i and single female j as $u_{\emptyset t}^m(a|\phi_{i,t})$ and $u_{\emptyset t}^f(a|\phi_{j,t})$, respectively.

We define the second-stage conditional valuation functions as the sum of the agents flow utilities and the discounted expected lifetime utility from the next period onwards.

$$\begin{aligned}
v_t^f\left(\tilde{a}|\phi_{r(j,t),t},\phi_{j,t}\right) &= u_t^f\left(\tilde{a}|\phi_{r(j,t),t},\phi_{j,t}\right)+\beta\mathbb{E}_{\tilde{\phi}_{r(j,t)j,t+1}|\tilde{\phi}_{r(j,t)j,t}}U_{t+1}^f\left(\phi_{r(j,t),t+1},\phi_{j,t+1}\right)\\
v_t^m\left(\tilde{a}|\phi_{i,t},\phi_{l(i,t),t}\right) &= u_t^m\left(\tilde{a}\phi_{i,t},\phi_{l(i,t),t}\right)+\beta\mathbb{E}_{\tilde{\phi}_{il(i,t),t+1}|\tilde{\phi}_{il(i,t),t}}U_{t+1}^m\left(\phi_{i,t+1},\phi_{l(i,t),t+1}\right)\\
v_{\emptyset t}^f\left(a|\phi_{j,t}\right) &= u_{\emptyset t}^f\left(a|\phi_{j,t}\right)+\beta\mathbb{E}_{\phi_{j,t+1}|\phi_{j,t}}U_{t+1}^f\left(\phi_{j,t+1}\right)\\
v_{\emptyset t}^m\left(a|\phi_{i,t}\right) &= u_{\emptyset t}^m\left(a|\phi_{i,t}\right)+\beta\mathbb{E}_{\phi_{i,t+1}|\phi_{i,t}}U_{t+1}^m\left(\phi_{i,t+1}\right)
\end{aligned}$$

Note that the definitions of v and U incorporate the standard assumptions of additive time-spearability of preferences and independence of the preference shocks and the state conditional on the lagged state.

Define $V_t^f\left(\phi_{r(j,t)},\phi_{j,t}\right)$ as the expected lifetime utility for a female j in state $\phi_{j,t}$ matched to a male $r(j,t)$ in state $\phi_{r(j,t)}$ before the realization of the couple's preference shocks $\varepsilon_{ij,t}$. The utility of male i matched with female $l(i,t)$ is analogously defined and denoted $V_t^m\left(\phi_{i,t},\phi_{l(i,t),t}\right)$. For single males and females, the analogue values are denoted as $V_{\emptyset t}^m(\phi_{i,t})$ and $V_{\emptyset t}^f(\phi_{j,t})$. Let $\lambda\left(\tilde{\phi}_{ij,t}\right)$ denote the Pareto weight associated with the male's expected lifetime utility in couple (i,j).

$$\begin{aligned}
V_{\emptyset t}^f\left(\phi_{j,t}\right) &= \mathbb{E}_{\varepsilon_{j,t}}\max_{a\in\mathscr{A}\left(\phi_{j,t}^f\right)} v_{\emptyset t}^f\left(a|\phi_{j,t}\right)+\varepsilon_{j,t}(a)\\
V_{\emptyset t}^m\left(\phi_{i,t}\right) &= \mathbb{E}_{\varepsilon_{i,t}}\max_{a\in\mathscr{A}\left(\phi_{i,t}^m\right)} v_{\emptyset t}^m\left(a|\phi_{i,t}\right)+\varepsilon_{i,t}(a)\\
\tilde{a}_{ij,t} &= \arg\max_{\tilde{a}\in\mathscr{A}\left(\tilde{\phi}_{ij,t}\right)}\lambda\left(\tilde{\phi}_{ij,t}\right)v_t^m\left(\tilde{a}|\phi_{i,t},\phi_{j,t}\right)+\left(1-\lambda\left(\tilde{\phi}_{ij,t}\right)\right)v_t^f\left(\tilde{a}|\phi_{i,t},\phi_{j,t}\right)+\varepsilon_{ij,t}(a)\\
V_t^m\left(\phi_{i,t},\phi_{l(i,t),t}\right) &= \mathbb{E}_{\varepsilon_{il(i,t),t}}v_t^m\left(\tilde{a}_{il(i,t),t}|\phi_{i,t},\phi_{l(i,t),t}\right)+\varepsilon_{il(i,t),t}\left(\tilde{a}_{il(i,t),t}\right)\\
V_t^f\left(\phi_{r(j,t)},\phi_{j,t}\right) &= \mathbb{E}_{\varepsilon_{r(j,t)j,t}}v_t^f\left(\tilde{a}_{r(j,t)j,t}|\phi_{r(j,t)},\phi_{j,t}\right)+\varepsilon_{r(j,t)j,t}\left(\tilde{a}_{r(j,t)j,t}\right)
\end{aligned}$$

We denote $\tilde{a}_0\in\mathscr{A}\left(\tilde{\phi}_{ij,t}\right)$ and $a_0\in\mathscr{A}\left(\phi_{i,t}^m\right)$ as baseline actions such that

$$\begin{aligned}
u_t^f\left(\tilde{a}_0|\phi_{r(j,t),t},\phi_{j,t}\right) &= 0\\
u_t^m\left(\tilde{a}_0|\phi_{i,t},\phi_{l(i,t),t}\right) &= 0\\
u_{\emptyset t}^f\left(a_0|\phi_{j,t}\right) &= 0\\
u_{\emptyset t}^m\left(a_0|\phi_{i,t}\right) &= 0
\end{aligned}$$

This assumption is necessary for identification and it is standard in dynamic discrete choice models. Figure 1 represents the timing of the model.

3 Data sampling procedure and selection

Before discussing the identification of the model, it is necessary to discuss the data sampling procedure. Our model requires a panel data of individual agents and their matches. It is crucial for identification that the sampling occur over match outcomes, so that both unmatched individuals and matched pairs can be randomly drawn from the population. This is the case in any population survey where the sampling unit is the household and the measurement unit is the adult individual. Households can be comprised of a single adult individual or a couple of adults– disregarding more rare household compositions that involve more adults. The panel needs to follow over time (up to some exogenous attrition) all individuals that formed the matches originally sampled, and measure the characteristics of all their future matches as well. The German SOEP dataset fulfills these requirements.[4] In 1984, a first representative sample of West German private households was drawn. In 1990 a new representative sample of East German houeholds was added. Since then, new refresher samples are periodically drawn

[4] https://www.diw.de/en/diw_01.c.678568.en/research_data_center_soep.html

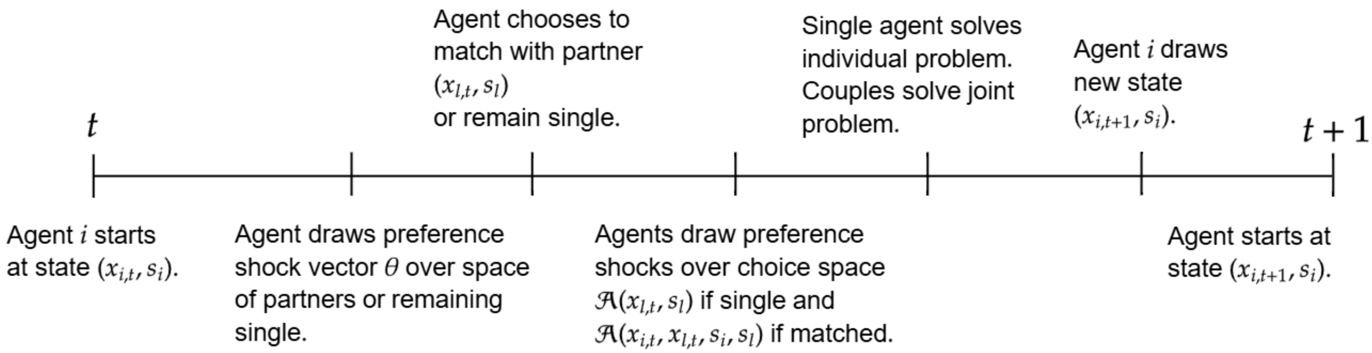

Figure 1: Timing of choices and the evolution of the state.

from the general population of German private households and included in the SOEP to counter attrition and ensure the panel remains representative. In the year following the first interview, the individuals in the household might move out, thereby forming a new household (or joining a preexisting one). Such individuals are located and interviewed, together with any adult individual who might live with them. This way, the new houshold is now part of the SOEP panel, and so are all the individuals therein. The process repeats each year, and every adult who was ever interviewed will be contacted in the following years for follow-up interviews. Similarly, any adult individual joining a household that is already in the sample will be interviewed and, upon moving out, the household they will form or join will enter the SOEP sample. For this reason, our model can be estimated on SOEP data by interpreting cohabitation as a match. Children of individuals in the panel are interviewed starting from age 16, and are followed year after year just like other panel members. To identify our model, we only need to consider households that were randomly sampled from the general population, i.e. individuals in the original samples or refresher samples. . We index these randomly drawn households by $(i,j)_{t_0}$, $i \in \mathscr{I} \cup \emptyset$, $j \in \mathscr{J} \cup \emptyset$, $i \neq j$ where $\mathscr{I}$ is the set of males in the randomly drawn households and $\mathscr{J}$ is the set of females. Denote with $t_0(i,j)$ the moment of the first survey for household $(i,j)_{t_0}$ which could be 1984, 1990, or any of the years when a refresher sample was drawn. At the moment of survey in $t_0(i,j)$, the household composition is already determined. Because of this, the household composition is simultaneously sampled together with the unobservable characteristics of its members from the general population, according to some endogenous joint probability $\pi(s_i, s_j)$. If individual i (j) is in a one-person household at the time of the first interview, the joint probability of being single and being of type s_i (s_j) is denoted $\pi(s_i, \emptyset)$ $(\pi(\emptyset, s_j))$. Notice that

$$\sum_{s,s'} \pi(s, s') + \sum_{s} \pi(s, \emptyset) + \sum_{s'} \pi(\emptyset, s') = 1$$

where P^m and .

To indicate if a household is composed of a single male, a single female, or a couple at sampling, we define the variable

$$h_{t_0}(i,j) = \begin{cases} h^m_{t_0} & if\ (i,j)_{t_0} = (i, \emptyset) \\ h^f_{t_0} & if\ (i,j)_{t_0} = (\emptyset, j) \\ \tilde{h}_{t_0} & otherwise \end{cases}$$

Suppose individuals i and j formed a match together at the time of their first interview $t_0(i,j)$. Suppose they separate at t_0+1 and i immediately forms a new household with individual $l \notin \mathscr{I}$. Then at t_0+2, i and l separate and l immediately forms a new household with $r \notin \mathscr{I}$. Then, although the household formed by l and r at t_0+2 year is included in the SOEP dataset and will be surveyed at all $t > t_0+2$ until attrition, we do not need their observations from t_0+3 onwards. Since the presence of l and r in the SOEP is determined by the matching choices of i and l, rather than by random sampling from the German population, neither l nor r are representative of the overall population because they were selected into the sample based on their characteristics (including those unobserved to the econometrician) through partner choice. Hence, including observations concerning l or r for t_0+3 onwards would require accounting for their selection on unobservables into the sample. Such endeavor is beyond the scope of this article. The only reason we use observations concerning l at time t_0+2 is to be able to condition on l's characterstics when i and l separate. [5]

[5]We only consider households formed by a single individual or by two cohabiting heterosexual partners. Households comprised of two adults that do not consider each other partners are treated as two distinct one-person households. In the case of households comprised of one or more couples living with other adults, we consider each couple as a distinct household, and the other single adults are each considered independent households as well.

4 The likelihood

We can write the sampling likelihood for each sampled household. First, the households in $\{(i,\emptyset)_{t_0}\}$. The discussion for households in $\{(\emptyset,j)_{t_0}\}$proceeds analogously.

$$L(i,\emptyset)_{t_0} \equiv P\left(\left\{\tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}, \tilde{a}_{il(i,t),t}\right\}_t |h^m_{t_0}\right) = \sum_{s_i} \pi(s_i,\emptyset) P\left(\left\{\tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}, \tilde{a}_{il(i,t),t}\right\}_t \mid s_i, h^m_{t_0}\right)$$

With some abuse of notation because $x_{l(i,t-1),t}$ is undefined for $t = 1, 2$. Define $\mathscr{L}(i) = \{l : \exists t\, s.t. l(i,t) = l\}(\mathscr{R}(j) = \{r : \exists t\, s.t. r(j,t) = r\})$ the set of individuals that ever matched to i (j) in the data. Denote $l^*(i,k)$ to indicate i's k-th spouse since sampling. Define $|\mathscr{L}(i)|$ $(|\mathscr{R}(j)|)$ to indicate the total number of spouses i had in his observed history. Define $\mathscr{T}(i, l(i,t^*)) = \{t : l(i,t) = l(i,t^*)\}$, the set of time periods when $l(i,t^*)$ and i have been matched. The likelihood can be expressed as:

$$L(i,\emptyset)_{t_0} = \sum_{s_i} \pi(s_i,\emptyset) p^m(a_{i1}, x_{i1}|s_i, h^m_{t_0}) \prod_{t=2}^{T} f\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right) \times \tag{3}$$

$$\sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)}} \prod_{t \in \mathscr{T}(i,\emptyset)}^{T} P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset | \tilde{\phi}_{il(i,t-1)t}, h^m_{t_0}\right) \prod_{l^* \in \mathscr{L}(i) \backslash j} \prod_{t \in \mathscr{T}(i,l^*)}^{T} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t} | \tilde{\phi}_{il(i,t-1)t}, h^m_{t_0}\right) \tag{4}$$

$$f\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right) = \begin{cases} f\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1),t-1}\right) \times & \text{if } t > 1, l(i,t-1) \neq \emptyset \\ f^m\left(x_{it}|\phi_{it-1}, a_{it-1}\right) & \text{if } t > 1, l(i,t-1) = \emptyset \end{cases}$$

The derivation of (3) is in B.1. Notice that $\tilde{a}_{il(i,t),t} = a_{it}$, $\tilde{x}_{il(i,t),t} = x_{it}$, and$\tilde{\phi}_{il(i,t),t} = \phi_{it}$ if $l(i,t) = \emptyset$, and similarly $\tilde{a}_{r(j,t)j,t} = a_{jt}$, $\tilde{x}_{r(j,t)j,t} = x_{jt}$, and $\tilde{\phi}_{r(j,t)j,t} = \phi_{jt}$ if $l(i,t) = \emptyset$. Because $\left(\tilde{x}_{il(i,t-1),t}, \tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right)$ are always observable, $f\left(\tilde{x}'|\tilde{x}, \tilde{a}\right)$ is always nonparametrically identified for any values of $(\tilde{x}, \tilde{x}', \tilde{a})$, and can be separated from the rest of the likelihood. For convenience, we divide (3) by

$$F\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right) \equiv \prod_{t=2}^{T} f\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right)$$

to obtain

$$\mathring{L}(i,\emptyset)_{t_0} \equiv \mathring{P}\left(\left\{\tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}, \tilde{a}_{il(i,t),t}\right\}_{t=1}^{T}, h^m_{t_0}\right) \tag{5}$$

$$= \sum_{s_i} \pi(s_i,\emptyset) p^m(a_{i1}, x_{i1}|s_i, h^m_{t_0}) \tag{6}$$

$$\sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)}} \prod_{t \in \mathscr{T}(i,\emptyset)}^{T} P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset | \tilde{\phi}_{il(i,t-1)t}, h^m_{t_0}\right) \prod_{l^* \in \mathscr{L}(i) \backslash j} \prod_{t \in \mathscr{T}(i,l^*)}^{T} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t} | \tilde{\phi}_{il(i,t-1)t}, h^m_{t_0}\right)$$

In the rest of the paper, we use the notation $\mathring{P}$ to indicate likelihood functions that exclude transition probabilities. Then, the likelihood for households in $\{(i,j)_{t_0}\}$ for $i, j \neq \emptyset$ is the following.

$$L(i,j)_{t_0} \equiv P\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_t | \tilde{h}_{t_0}\right) \tag{7}$$

$$= \sum_{s_i} \sum_{s_j} \pi(s_i, s_j) P\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\} \mid s_i, s_j, \tilde{h}_{t_0}\right)$$

$$= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1}|\tilde{s}_{ij}, \tilde{h}_{t_0}\right) \prod_{t=2}^{T} f(\tilde{x}_{il(i,t-1),t}, \tilde{x}_{r(j,t-1)j,t}|\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, \tilde{a}_{il(i,t-1),t-1}, \tilde{a}_{r(j,t-1)j,t-1}) \times$$

$$\sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i) \backslash j}} \sum_{\{s_{r^*}\}_{r^* \in \mathscr{R}(j) \backslash i}} \left(\begin{array}{c} \prod_{l^* \in \mathscr{L}(i) \backslash j} \prod_{t \in \mathscr{T}(i,l^*)} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\tilde{\phi}_{il(i,t-1)t}, \tilde{h}_{t_0}\right) \times \\ \prod_{r^* \in \mathscr{R}(j) \backslash i} \prod_{t \in \mathscr{T}(j,r^*)} P\left(\phi_{r^*,t}, \tilde{a}_{r^*j,t}|\tilde{\phi}_{r(j,t-1)j,t}, \tilde{h}_{t_0}\right) \times \\ \prod_{t \in \mathscr{T}(i,j) \backslash \{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t}|\tilde{\phi}_{r(j,t-1)j,t}, \tilde{\phi}_{il(i,t-1),t}, \tilde{h}_{t_0}\right) \times \\ \prod_{t \in \mathscr{T}(i,\emptyset)} P\left(\phi_{l,t} = \emptyset, a_{i,t}|\tilde{\phi}_{il(i,t-1)t}, \tilde{h}_{t_0}\right) \prod_{t \in \mathscr{T}(j,\emptyset)} P\left(\phi_{r,t} = \emptyset, a_{j,t}|\tilde{\phi}_{r(j,t-1)j,t}, \tilde{h}_{t_0}\right) \end{array} \right)$$

With some abuse of notation being $x_{l(i,t-1),t}$ and $x_{r(j,t-1),t}$ undefined for $t=1$. Notice that at each time t such that $l(i,t)=j$ and $r(j,t)=i$, we have $\tilde{\phi}_{il(i,t),t}=\tilde{\phi}_{r(j,t)j,t}=\tilde{\phi}_{ij,t}$.[6]
[7]

Where

$$f(\tilde{x}_{il(i,t-1),t},\tilde{x}_{r(j,t-1)j,t}|\tilde{x}_{il(i,t),t},\tilde{x}_{r(j,t)j,t},\tilde{a}_{il(i,t-1)t-1},\tilde{a}_{r(j,t-1)j,t-1})$$

$$= \begin{cases} f(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1},\tilde{a}_{il(i,t-1)t-1})\times \\ f(\tilde{x}_{r(j,t-1)j,t}|\tilde{x}_{r(j,t-1)j,t-1},\tilde{a}_{r(j,t-1)jt-1}) & if\ l(i,t-1)\neq\emptyset, r(j,t-1)\neq\emptyset \\ \\ f(\tilde{x}_{ijt}|\tilde{x}_{ijt-1},\tilde{a}_{ijt-1}) & if\ l(i,t-1)=j, r(j,t-1)=i \\ \\ f(x_{it}|x_{it-1},a_{it-1})f(\tilde{x}_{r(j,t-1)j,t}|\tilde{x}_{r(j,t-1)j,t-1},\tilde{a}_{r(j,t-1)j,t-1}) & if\, l(i,t-1)=\emptyset, r(j,t-1)\neq\emptyset \\ \\ f(\tilde{x}_{il(i,t-1)t}|\tilde{x}_{il(j,t-1)t-1},\tilde{a}_{il(i,t-1)t-1}) \\ \times f(x_{jt}|x_{jt-1},a_{jt-1}) & if\, l(i,t-1)\neq\emptyset, r(j,t-1)=\emptyset \\ \\ f(x_{it}|x_{it-1},a_{it-1})\times f(x_{jt}|x_{jt-1},a_{jt-1}) & if\, l(i,t-1)=\emptyset, r(j,t-1)=\emptyset \end{cases}$$

The derivation of (7) is in B.2.

Once again, for convenience, we divide 7 by

$$F(\tilde{x}_{il(i,t-1),t},\tilde{x}_{r(j,t-1)j,t}|\tilde{x}_{il(i,t),t},\tilde{x}_{r(j,t)j,t},\tilde{a}_{il(i,t-1)t-1},\tilde{a}_{r(j,t-1)j,t-1}) \equiv \prod_{t=2}^{T} f(\tilde{x}_{il(i,t-1),t},\tilde{x}_{r(j,t-1)j,t}|\tilde{x}_{il(i,t),t},\tilde{x}_{r(j,t)j,t},\tilde{a}_{il(i,t-1)t-1},\tilde{a}_{r(j,t-1)j,t-1})$$

to define

$$\begin{aligned} \mathring{L}(i,j)_{t_0} &\equiv \mathring{P}\left(\left\{\tilde{x}_{il(i,t),t},\tilde{x}_{r(j,t)j,t},x_{l(i,t-1),t},x_{r(j,t-1),t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_t|\tilde{h}_{t_0}\right) \\ &= \sum_{\tilde{s}_{ij}}\pi\left(\tilde{s}_{ij}\right)p\left(\tilde{a}_{ij1},\tilde{x}_{ij1}|\tilde{s}_{ij},\tilde{h}_{t_0}\right) \\ &\sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i)\setminus j}}\sum_{\{s_{r^*}\}_{r^*\in\mathscr{R}(j)\setminus i}}\left(\begin{array}{c} \prod_{l^*\in\mathscr{L}(i)\setminus j}\prod_{t\in\mathscr{T}(i,l^*)}P\left(\phi_{l^*,t},\tilde{a}_{il^*,t}|\tilde{\phi}_{il(i,t-1)t},\tilde{h}_{t_0}\right)\times \\ \prod_{r^*\in\mathscr{R}(j)\setminus i}\prod_{t\in\mathscr{T}(j,r^*)}P\left(\phi_{r^*,t},\tilde{a}_{r^*j,t}|\tilde{\phi}_{r(j,t-1)j,t},\tilde{h}_{t_0}\right)\times \\ \prod_{t\in\mathscr{T}(i,j)\setminus\{t=1\}}P\left(\phi_{r(j,t),t}=\phi_{i,t},\phi_{l(i,t),t}=\phi_{j,t},\tilde{a}_{ij,t}|\tilde{\phi}_{r(j,t-1)j,t},\tilde{\phi}_{il(i,t-1),t},\tilde{h}_{t_0}\right)\times \\ \prod_{t\in\mathscr{T}(i,\emptyset)}P\left(\phi_{l,t}=\emptyset,a_{i,t}|\tilde{\phi}_{il(i,t-1)t},\tilde{h}_{t_0}\right)\prod_{t\in\mathscr{T}(j,\emptyset)}P\left(\phi_{r,t}=\emptyset,a_{j,t}|\tilde{\phi}_{r(j,t-1)j,t},\tilde{h}_{t_0}\right) \end{array}\right) \end{aligned} \tag{8}$$

5 Nonparametric identification

The model implies that the first- and second-stage choice probabilities can be factored, conditionally on $\tilde{\phi}_{il(i,t-1),t}$, as follows

$$\begin{aligned} P\left(\phi_{l(i,t),t},\tilde{a}_{il(i,t),t}|\tilde{\phi}_{il(i,t-1)t}\right) &= \psi^m(\phi_{l(i,t),t}|\tilde{\phi}_{il(i,t-1),t})p(\tilde{a}_{il(i,t),t}|\phi_{it},\phi_{l(i,t),t}) \\ P\left(\phi_{l,t}=\emptyset,a_{i,t}|\tilde{\phi}_{il(i,t-1)t}\right) &= \psi^m(\emptyset|\tilde{\phi}_{il(i,t-1),t})p(a_{i,t}|\phi_{it}) \end{aligned}$$

Our goal is to identify $\psi^m(\phi_{l(i,t),t}|\tilde{\phi}_{il(i,t-1),t})$ and $p(\tilde{a}_{ijt}|\phi_{it},\phi_{l(i,t),t})$ for all values of $\left(\tilde{\phi}_{il(i,t-1),t},\phi_{l(i,t),t},\phi_{it},\phi_{l(i,t),t},\tilde{a}_{ijt}\right)$.

[6] The reason why the likelihoods of the histories of agents (i,j), who formed a household together at sampling, are joint despite the assumption that one's marital choices do not affect the other (i.e. an agent is always single because they chose so) is that their unobserved types are potentially correlated (e.g. if there exists sorting on unobservables) and it would be a mistake to factor $\pi(s_i,s_j)=\pi(s_i)\pi(s_j)$.

[7] Notice that $\prod_{t\in\mathscr{T}(i,j)\setminus\{t=1\}}P\left(\phi_{r(j,t),t}=\phi_{i,t},\phi_{l(i,t),t}=\phi_{j,t},\tilde{a}_{ij,t}|\tilde{\phi}_{r(j,t-1)j,t},\tilde{\phi}_{il(i,t-1),t}\right)$ cannot be factored out of the summations over $l^*(i,k)$ and $r^*(j,k)$ because of the dependency on $r(j,t-1)$ and $l(i,t-1)$, which could include $r(j,t-1)\neq i$ and $l(i,t-1)\neq j$ if the couple (i,j) separated at any point in time, matched to other partners, and then re-matched.

5.1 Identification in a special case

In this subsection, we look at identification in a simplified case, where we assume that the matching transition costs do not depend on the previous spouse's characteristics. This allows to simplify the likelihood noticing that each spouse's l^* (r^*) unobserved type only enters the choice probabilities related to the periods in $\mathscr{T}(i,l^*)$ $(\mathscr{T}(j,r^*))$.

Assumption 1.

$$\kappa^f\left(\phi|\tilde{\phi}_{r(j,t-1)j,t}\right)=\kappa^f\left(\phi|\phi_{j,t}\right) \quad (9)$$

$$\kappa^m\left(\phi|\tilde{\phi}_{il(i,t-1),t}\right)=\kappa^m\left(\phi|\phi_{i,t}\right)$$

which implies

$$\psi^m(\phi_{l(i,t),t}|\tilde{\phi}_{il(i,t-1),t})=\psi^m(\phi_{l(i,t),t}|\phi_{it})$$

$$\psi^f(\phi_{r(j,t),t}|\tilde{\phi}_{r(j,t-1)j,t})=\psi^f(\phi_{r(j,t),t}|\phi_{jt})$$

5.1.1 Unmatched individuals

Under Assumption 1,(5) simplifies to

$$\mathring{L}(i,\emptyset)_{t_0}=\sum_{s_i}\pi(s_i,\emptyset)p^m(a_{i1},x_{i1}|s_i,h^m_{t_0})\prod_{t=2}^{T}P\left(\tilde{a}_{il(i,t),t},x_{l(i,t),t}|\phi_{it},h^m_{t_0}\right)$$

Which is derived in B.3.

This shows that the sample likelihood for a single individual fulfills the property of independent marginals required by Kasahara and Shimotsu (2009) and Hall and Zhou (2003). Let $\pi^m(s|h^m_{t_0})$ indicate the distribution of unobserved types for males conditional on being single at period $t_0(i,\emptyset)$.

Then, we can write

$$\mathring{L}(i,\emptyset)_{t_0}=P(h^m_{t_0})\sum_{s_i}\pi(s_i|h^m_{t_0})p^m(a_{i1},x_{i1}|s_i,h^m_{t_0})\prod_{t=2}^{T}P\left(\tilde{a}_{il(i,t),t},\phi_{l(i,t),t},x_{l(i,t-1),t}|\phi_{it},h^m_{t_0}\right) \quad (10)$$

And notice that $P(h^m_{t_0})$ is nonparametrically identified in the data since $h^m_{t_0}$ is observable.

Assumption 2.

The conditional choice probabilities ψ and p do not depend on t.

To simplify the exposition of the identification argument, we consider a fixed choice set for every individual at every state and every match status.

Assumption 3.

$$\mathscr{A}(\phi)=\mathscr{A}\;\forall\phi\in\mathscr{X}\times\mathscr{S}$$

$$\mathscr{A}\left(\tilde{\phi}\right)=\mathscr{A}^2\;\forall\tilde{\phi}\in(\mathscr{X}\times\mathscr{S})^2$$

for all $\tilde{x},\tilde{s},\tilde{x}',\tilde{s}'$.

The following identification proof is constructive and proceeds by steps. First, we focus on households of type $(i,\emptyset)_{t_0}$at the moment of sampling and 1 shows the identification of $p^m(a_{it}|\phi_{it},h^m_{t_0})$ and $P\left(\tilde{a}_{il(i,t),t},x_{l(i,t),t}|\phi_{it},h^m_{t_0}\right)$ for every value of $\left(a_{it},\tilde{a}_{il(i,t),t},x_{l(i,t),t},\phi_{it}\right)$. The identification argument is identical for $(\emptyset,j)_{t_0}$ households. Then, we focus on households of type $(i,j)_{t_0}$, formed of two partners at the moment of sampling and 2 shows the identification of $p(\tilde{a}_{ij,t}|\tilde{\phi}_{ij,t},\tilde{h}_{t_0})$ for all values of $\left(\tilde{a}_{ij,t},\tilde{\phi}_{ij,t}\right)$. Finally, 1 proves of the identification of $\psi^m\left(\phi_{l(i,t),t}|\phi_{it}\right)$, with the argument being identical for $\psi^f\left(\phi_{r(j,t),t}|\phi_{jt}\right)$.

Lemma 1. , *Under assumptions 1, 2, and 3, the parameters $\pi(s|h^m_{t_0})$, $P\left(\tilde{a}_{il(i,t),t},x_{l(i,t),t}|\phi_{it},h^m_{t_0}\right)$, $p^m(a_{it}|\phi_{it},h^m_{t_0})$, for all $s\in\mathscr{S}$, $\phi\in\Phi$, $a\in\mathscr{A}(x,s)$ are identified from (10)*

Proof. See E.1. □

5.1.2 Matched pairs

The simplified likelihood:

$$\mathring{L}(i,\emptyset)_{t_0} = \sum_{\tilde{s}_{ij}} \pi(\tilde{s}_{ij}) p(\tilde{a}_{ij1}, \tilde{x}_{ij,1}|\tilde{s}_{ij}, \tilde{h}_{t_0}) \prod_{t=2}^{T} P\left(\tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}, x_{l(i,t),t}, x_{r(j,t),t}|\tilde{\phi}_{ij,t}, \tilde{s}_{ij}, \tilde{h}_{t_0}\right)$$

$$= P(\tilde{h}_{t_0}) \sum_{\tilde{s}_{ij}} \pi(\tilde{s}_{ij}|\tilde{h}_{t_0}) p(\tilde{a}_{ij1}, \tilde{x}_{ij,1}|\tilde{s}_{ij}, \tilde{h}_{t_0}) \prod_{t=2}^{T} P\left(\tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}, x_{l(i,t),t}, x_{r(j,t),t}|\tilde{\phi}_{ij,t}, \tilde{s}_{ij}, \tilde{h}_{t_0}\right) \quad (11)$$

Where $\pi(\tilde{s}|\tilde{h}_{t_0})$ is the distribution of unobserved heterogeneity among couples at time $t_0(i,j)$. The derivation of (11) is in B.4.

Again, it is apparent that the likelihood satisfies the indepenent marginals property.

Lemma 2. *Under assumptions 1, 2, and 3, the parameters* $\pi(\tilde{s}|\tilde{h}_{t_0}), p(\tilde{a}, \tilde{x}|\tilde{s})$ *for all* $\tilde{s} \in \mathscr{S}^2, \tilde{x} \in \mathscr{X}^2, \tilde{a} \in \mathscr{A}\left(\tilde{\phi}\right), \tilde{a}' \in \mathscr{A}\left(\tilde{\phi}\right), \phi \in \Phi, \phi' \in \Phi$ *are identified from* (11)

Proof. See E.2. □

Proposition 1. *Under assumptions 1, 2, and 3, the parameters* $\left(\psi^m(x', s'|x, s), p^m(a|x, s), p(\tilde{a}|\tilde{\phi}), \pi^m(s), \pi^f(s)\right)$ *for all* $\phi \in \Phi, \phi' \in \Phi, s \in \mathscr{S}, s' \in \mathscr{S}, x \in \mathscr{X}, x' \in \mathscr{X}, a \in \mathscr{A}(x, s), \tilde{\phi} \in \mathscr{X}^2 \times \mathscr{S}^2, \tilde{a} \in \mathscr{A}\left(\tilde{\phi}\right)$ *are identified from* (10) *and* (11).

Proof. See E.3 □

6 Identification of the structural parameters

The parameters of the model are identified from moments of the data generating process that can be consistently estimated from the data, as proven above, through their analytic expressions. The identification arguments are well known Hotz and Miller (1993) and are based on the existence of a mapping between CCPs and differences in conditional valuation functions. The first set of such moments concern the conditional choice probabilities.

Due to the type-1 extreme value distributional assumption for the preference shocks in the first and the second stage, the ex-ante lifetime utilities have a known closed form expressions in the following cases:

$$V^f_{\emptyset t}(\phi_{j,t}) = \ln \sum_{a \in \mathscr{A}(\phi_{j,t})} \exp v^f_{\emptyset t}(a|\phi_{j,t}) = v^f_{\emptyset t}(a^*|\phi) - \nu \ln p(a|\phi_{j,t}) + \mu + \nu\gamma \forall a^* \in \mathscr{A}(\phi_{j,t}) + \mu + \nu\gamma \quad (12)$$

$$V^m_{\emptyset t}(\phi_{i,t}) = \ln \sum_{a \in \mathscr{A}(\phi^m_{i,t})} \exp v^m_{\emptyset t}(a|\phi_{i,t}) = v^m_{\emptyset t}(a^*|\phi_{i,t}) - \nu \ln p(a|\phi_{i,t}) + \mu + \nu\gamma \forall a^* \in \mathscr{A}(\phi_{i,t}) + \mu + \nu\gamma \quad (13)$$

$$U^f_t\left(\tilde{\phi}_{r(j,t-1)j,t}\right) = \ln\left[\sum_{\phi \in \Phi} \exp\left[V^f_t(\phi, \phi_{j,t}) + \kappa^f\left(\phi|\tilde{\phi}_{r(j,t-1)j,t}\right)\right] + \exp\left[V^f_{\emptyset t}(\phi_{j,t}) + \kappa^f\left(\emptyset|\tilde{\phi}_{r(j,t-1)j,t}\right)\right]\right]$$

$$= V^f_t(\phi^*, \phi_{j,t}) + \kappa^f\left(\phi^*|\tilde{\phi}_{r(j,t-1)j,t}\right) - \nu \ln \psi^f(\phi^*|\phi_{j,t}) + \mu + \nu\gamma \quad \forall \phi^* \in \Phi \quad (14)$$

$$= V^f_{\emptyset t}(\phi_{j,t}) + \kappa^f\left(\emptyset|\tilde{\phi}_{r(j,t-1)j,t}\right) - \nu \ln \psi^f(\emptyset|\phi_{j,t}) + \mu + \nu\gamma$$

$$U^m_t\left(\tilde{\phi}_{il(i,t-1),t}\right) = \ln\left[\sum_{\phi \in \Phi} \exp\left[V^m_t(\phi, \phi_{i,t}) + \kappa^m\left(\phi|\tilde{\phi}_{il(i,t-1),t}\right)\right] + \exp\left[V^m_{\emptyset t}(\phi_{i,t}) + \kappa^m\left(\emptyset|\tilde{\phi}_{il(i,t-1),t}\right)\right]\right]$$

$$= V^m_t(\phi^*, \phi_{i,t}) + \kappa^m\left(\phi^*|\tilde{\phi}_{il(i,t-1),t}\right) - \nu \ln \psi^m(\phi^*|\phi_{i,t}) + \mu + \nu\gamma \quad \forall \phi^* \in \Phi \quad (15)$$

$$= V^m_{\emptyset t}(\phi_{i,t}) + \kappa^m\left(\emptyset|\tilde{\phi}_{il(i,t-1),t}\right) - \nu \ln \psi^m(\emptyset|\phi_{i,t}) + \mu + \nu\gamma$$

Because the ex-ante lifetime utility of matched individuals derives from the maximization of a joint utility function, rather than the individual's utility, the expressions for $V^f_t(\phi_{it}, \phi_{j,t})$ and $V^m_t(\phi_{it}, \phi_{j,t})$ deviate slightly from the usual closed form and they are derived in A

$$V_t^f(\phi_{it}, \phi_{j,t}) = \sum_{\tilde{a} \in \mathscr{A}(\phi_i, \phi_j)} p(\tilde{a}|\tilde{\phi}_{ij,t}) \left(v_t^f \left(\tilde{a}|\tilde{\phi}_{ij,t} \right) - \nu \ln p(\tilde{a}|\tilde{\phi}_{ij,t}) \right) + \mu + \nu\gamma$$

$$V_t^m(\phi_{it}, \phi_{j,t}) = \sum_{\tilde{a} \in \mathscr{A}(\phi_i, \phi_j)} p(\tilde{a}|\tilde{\phi}_{ij,t}) \left(v_t^m \left(\tilde{a}|\tilde{\phi}_{ij,t} \right) - \nu \ln p(\tilde{a}|\tilde{\phi}_{ij,t}) \right) + \mu + \nu\gamma$$

Given the ex-ante value functions, the marriage probabilities conditional on the observed state variables and the unobserved types take the familiar logit form:

$$\psi^f \left(\phi_{r(j,t)t}|\tilde{\phi}_{r(j,t-1)j,t} \right) = \frac{\exp V_t^f(\tilde{\phi}_{r(j,t)j,t}) + \kappa^f \left(\phi_{r(j,t),t}|\tilde{\phi}_{r(j,t-1)j,t} \right) - V_{\emptyset t}^f(\phi_{j,t}) \cancel{-\kappa^f \left(\emptyset \tilde{\phi}_{r(j,t-1)j,t} \right)}}{\sum_{\phi \in \Phi \cup \emptyset} \exp V_t^f(\phi, \phi_{j,t}) + \kappa^f \left(\phi|\tilde{\phi}_{r(j,t-1)j,t} \right) - V_{\emptyset t}^f(\phi_{j,t}) \cancel{-\kappa^f \left(\emptyset|\tilde{\phi}_{r(j,t-1)j,t} \right)}} \tag{16}$$

$$\psi^m \left(\phi_{l(i,t)t}|\tilde{\phi}_{il(i,t-1),t} \right) = \frac{\exp V_t^m(\tilde{\phi}_{il(i,t),t}) + \kappa^m \left(\phi_{l(i,t),t}|\tilde{\phi}_{il(i,t-1),t} \right) - V_{\emptyset t}^m(\phi_{i,t}) \cancel{-\kappa^m \left(\emptyset|\tilde{\phi}_{il(i,t-1),t} \right)}}{\sum_{\phi \in \Phi \cup \emptyset} \exp V_t^m(\phi_{i,t}, \phi) + \kappa^m \left(\phi|\tilde{\phi}_{il(i,t-1),t} \right) - V_{\emptyset t}^m(\phi_{i,t}) \cancel{-\kappa^m \left(\emptyset|\tilde{\phi}_{il(i,t-1),t} \right)}} \tag{17}$$

The second-stage choice probabilities for single male, single female, and couples also have well-known closed form expressions:

$$p^f(a|\phi_{jt}) = \frac{\exp v_{0t}^f(a|\phi_{j,t}) - v_{0t}^f(\emptyset, \phi_{j,t})}{\sum_{a' \in \mathscr{A}(\phi_{j,t})} \exp v_{0t}^f(a'|\phi_{j,t}) - v_{0t}^f(\emptyset|\phi_{j,t})} \tag{18}$$

$$p^m(a|\phi_{it}) = \frac{\exp v_{0t}^m(a|\phi_{j,t}) - v_{0t}^m(\emptyset|\phi_{j,t})}{\sum_{a' \in \mathscr{A}(\phi_{j,t})} \exp v_{0t}^m(a'|\phi_{j,t}) - v_{0t}^m(\emptyset|\phi_{j,t})} \tag{19}$$

$$p(\tilde{a}|\tilde{\phi}_{ij,t}) = \frac{\exp \lambda \left(\tilde{\phi}_{ij,t} \right) \begin{bmatrix} v_t^m \left(\tilde{a}|\tilde{\phi}_{il(i,t),t} \right) \\ -v_t^m \left(\emptyset|\tilde{\phi}_{il(i,t),t} \right) \end{bmatrix} + \left(1 - \lambda \left(\tilde{\phi}_{ij,t} \right) \right) \begin{bmatrix} v_t^f \left(\tilde{a}|\tilde{\phi}_{r(j,t)j,t} \right) \\ -v_t^f \left(\emptyset|\tilde{\phi}_{r(j,t)j,t} \right) \end{bmatrix}}{\sum_{\tilde{a}' \in \mathscr{A}(\tilde{\phi}_{ij,t})} \exp \lambda \left(\tilde{\phi}_{ij,t} \right) \begin{bmatrix} v_t^m \left(\tilde{a}'|\tilde{\phi}_{il(i,t),t} \right) \\ -v_t^m \left(\emptyset|\tilde{\phi}_{il(i,t),t} \right) \end{bmatrix} + \left(1 - \lambda \left(\tilde{\phi}_{ij,t} \right) \right) \begin{bmatrix} v_t^f \left(\tilde{a}'|\tilde{\phi}_{r(j,t)j,t} \right) \\ -v_t^f \left(\emptyset|\tilde{\phi}_{r(j,t)j,t} \right) \end{bmatrix}} \tag{20}$$

These moments can be used to form and minimize a loss function such as (negative) structural likelihood, a weighted sum of square residuals in a GMM estimator, or a distance function with respect to nonparametric estimates of the choice probabilities, thereby estimating the structural parameters. To calculate the value of each of these CCPs at any value of the structural parameters, we need to telescope the conditional valuation functions v and use 12,13,14, and 15 to express them as functions of conditional choice probabilities themselves. Instead of undertaking the difficult task of solving the system of nonlinear equations in the values of the CCPs, we follow Hotz and Miller (1993) and Arcidiacono and Miller (2011) by replacing the CCPs appearing in the right hand side of each structural with nonparametric estimates.

7 Estimation

7.1 The EM algorithm

The Expectation Maximization (EM) algorithm is a computationally conservative alternative to direct maximum likelihood estimation with partial data. The technique was formalized in general terms by Dempster et al. (1977) and its convergence properties were further studied by Wu (1983). Arcidiacono and Miller (2011) applied the technique to dynamic discrete choice models. Instead of maximizing a likelihood that integrates over unobserved states and choices, we iteratively maximize an expected log-likelihood. In this approach, unobserved characteristics and choices are treated as observed within the expectation, while the expectation is taken over their possible values using the ex-post type distribution given the observed data. Our model lends itself to this technique due to the presence of the unobserved characteristics s. Though, unlike the setting of Arcidiacono and Miller (2011), our model features an unobserved choice in addition the unobserved conditioning variable, because agents choose their partner's unobserved state s_l (s_r) in the first stage of each period. We extend the algorithm in Arcidiacono and Miller (2011) by estimating the posterior type probability of each chosen partner l (r) in addition to the posterior type probability of each sampled household $(i,j)_{t_0}$, and using the product of such posterior probabilities as weights in the expectation of the log-likelihood. We define

$$\begin{aligned}
q_{is} &= P\left(s_i = s | \left\{a_{i'l(i',t),t}, x_{i't}, x_{l(i',t),t}\right\}_t, h^m_{t_0}\right) \\
q_{s_l|s_i} &= P\left(s_l = s | \left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}\right\}_t, s_i, h^m_{t_0}\right) \\
q_{ij\tilde{s}} &= P\left(\tilde{s}_{ij} = \tilde{s} | \left\{\tilde{x}_{i'j't}, x_{l(i',t),t}, x_{r(j',t),t}, a_{i'l(i',t),t}, a_{r(j',t)j',t}\right\}_t, \tilde{h}_{t_0}\right) \\
q_{s_l|\tilde{s}_{ij}} &= P\left(s_l = s | \left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_t, \tilde{s}_{ij}, \tilde{h}_{t_0}\right)
\end{aligned}$$

and form the expected likelihood

$$E_{\tilde{s}_{ij}, s_{l(i,t)}, s_{r(j,t)}}\left[\ln \mathring{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{ij,t}, \left\{\tilde{s}_{ij}, s_{l(i,t)}, s_{r(j,t)}\right\}_{ij,t}\right) | \left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{ij,t}\right]$$

The algorithm proceeds as follows.

1. Initialize the values $\left(q^0_{is}, q^0_{ij\tilde{s}}, q^0_{s_l|s_i}, q^0_{s_r|s_j}\right)$ for any household (i, j) and any partner l and r, and $(\psi^{m0}(\phi'|\phi)$, $\psi^{f0}(\phi'|\phi)$, $p^0(\tilde{a}|\tilde{\phi})$, $p^{m0}(a|\phi)$, $p^{f0}(a|\phi))$ for all values of $\phi, \phi' \in \mathscr{X}$, $\tilde{\phi} \in \mathscr{X}^2$, $a \in \mathscr{A}$, and $\tilde{a} \in \mathscr{A}^2$.
2. At any iteration h, maximize the expected log-likelihood over the space of first-stage and second-stage choice probabilities, conditional on the values $\left(q^{h-1}_{is}, q^{h-1}_{ij\tilde{s}}, q^{h-1}_{s_l|s_i}, q^{h-1}_{s_r|s_j}\right)$. This step provides the values $(\psi^{mh}(\phi'|\phi)$, $\psi^{fh}(\phi'|\phi)$, $p^h(\tilde{a}|\tilde{\phi})$, $p^{mh}(a|\phi)$, $p^{fh}(a|\phi))$.
3. Calculate the values $\left(q^h_{is}, q^h_{ij\tilde{s}}, q^h_{s_l|s_i}, q^h_{s_r|s_j}\right)$ based on the values $\left(\psi^{mh}(\phi'|\phi), \psi^{fh}(\phi'|\phi), p^h(\tilde{a}|\tilde{\phi}), p^{mh}(a|\phi), p^{fh}(a|\phi)\right)$.
4. Reiterate from step 2 until convergence.

In the following subsection, we discuss the details of steps 2 and 3.

7.2 The general case

7.2.1 Unmatched individuals

In general, by definition, we have

$$q_{is} = \frac{\pi(s, \emptyset)\mathring{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{\phi}_{il(i,t),t}\right\}_{t=1}^T | s_i = s, h^m_{t_0}\right)}{\sum_{s'} \pi(s', \emptyset)\mathring{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{\phi}_{il(i,t),t}\right\}_{t=1}^T | s_i = s', h^m_{t_0}\right)}$$

$$q_{s_l|s_i} = \frac{\mathring{P}\left(s_l, \left\{\tilde{a}_{il(i,t),t}, \tilde{\phi}_{il(i,t),t}\right\}_{t=1}^T | s_i, h^m_{t_0}\right)}{\mathring{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{\phi}_{il(i,t),t}\right\}_{t=1}^T | s_i, h^m_{t_0}\right)}$$

Where

$$\begin{aligned}
&\mathring{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{\phi}_{il(i,t),t}\right\}_t | s_i = s, h^m_{t_0}\right) \\
&= p^m(a_{i1}, \phi_{i1}|s_i, h^m_{t_0})\times \\
&\sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)\setminus j}} \prod_{t \in \mathscr{T}(i,\emptyset)}^T P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset | \tilde{\phi}_{il(i,t-1)t}, h^m_{t_0}\right) \prod_{l^* \in \mathscr{L}(i)} \prod_{t \in \mathscr{T}(i,l^*)}^T P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t} | \tilde{\phi}_{il(i,t-1)t}, h^m_{t_0}\right)
\end{aligned}$$

$$\begin{aligned}
&\mathring{P}\left(s_l, \left\{\tilde{a}_{il(i,t),t}, \tilde{\phi}_{il(i,t),t}\right\}_t | s_i, h^m_{t_0}\right) \\
&= p^m(a_{i1}, \phi_{i1}|s_i, h^m_{t_0})\times \\
&\sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)\setminus\{j,l\}}} \left\{ \prod_{t \in \mathscr{T}(i,\emptyset)}^T P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset | \tilde{\phi}_{il(i,t-1)t}, h^m_{t_0}\right) \right. \\
&\left. \prod_{l^* \in \mathscr{L}(i)} \prod_{t \in \mathscr{T}(i,l^*)}^T P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t} | \tilde{\phi}_{il(i,t-1)t}, h^m_{t_0}\right) \prod_{t \in \mathscr{T}(i,l)}^T P\left(x_{l,t}, s_l, \tilde{a}_{il,t} | \tilde{\phi}_{il(i,t-1)t}, h^m_{t_0}\right) \right\}
\end{aligned}$$

7.2.2 Matched pairs

For couples, we have

$$q_{ij\tilde{s}} = \frac{\pi(\tilde{s})\mathring{P}\left(\left\{\tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{i,j,t} | \tilde{s}_{ij} = \tilde{s}\right)}{\sum_{\tilde{s}'} \pi(\tilde{s}')\mathring{P}\left(\left\{\tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{i,j,t} | \tilde{s}'\right)}$$

$$q_{s_l|\tilde{s}_{ij}} = \frac{\mathring{P}\left(s_l, \left\{\tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{i,j,t} | \tilde{s}_{ij}\right)}{\mathring{P}\left(\left\{\tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{i,j,t} | \tilde{s}_{ij}\right)}$$

Where

$$\begin{aligned}
&\mathring{P}\left(\left\{\tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{i,j,t} | \tilde{s}_{ij} = \tilde{s}\right) \\
&= p\left(\tilde{a}_{ij1}, \tilde{\phi}_{ij1} | \tilde{s}_{ij}\right) \\
&\sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)\setminus j}} \sum_{\{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i}} \prod_{l^* \in \mathscr{L}(i)\setminus j} \prod_{t \in \mathscr{T}(i,l)} P\left(\phi_{l,t}, \tilde{a}_{il,t} | \tilde{\phi}_{il(i,t-1)t}\right) \prod_{r^* \in \mathscr{R}(j)\setminus i} \prod_{t \in \mathscr{T}(j,r)} P\left(\phi_{r,t}, \tilde{a}_{rj,t} | \tilde{\phi}_{r(j,t-1)j,t}\right) \times \\
&\prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \tilde{\phi}_{r(j,t-1)j,t}, \tilde{\phi}_{il(i,t-1),t}\right)
\end{aligned}$$

$$\begin{aligned}
&\mathring{P}\left(s_l, \left\{\tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{i,j,t} | \tilde{s}_{ij}\right) \\
&= p\left(\tilde{a}_{ij1}, \tilde{\phi}_{ij1} | \tilde{s}_{ij}\right) \sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)\setminus\{j,l\}}} \sum_{\{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i}} \\
&\prod_{l^* \in \mathscr{L}(i)\setminus j} \prod_{t \in \mathscr{T}(i,l^*)} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t} | \tilde{\phi}_{il(i,t-1)t}\right) \prod_{t \in \mathscr{T}(i,l)} P\left(x_{l,t}, s_l, \tilde{a}_{il,t} | \tilde{\phi}_{il(i,t-1)t}\right) \prod_{r^* \in \mathscr{R}(j)\setminus i} \prod_{t \in \mathscr{T}(j,r)} P\left(\phi_{r,t}, \tilde{a}_{rj,t} | \tilde{\phi}_{r(j,t-1)j,t}\right) \times \\
&\prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \tilde{\phi}_{r(j,t-1)j,t}, \tilde{\phi}_{il(i,t-1),t}\right)
\end{aligned}$$

These expression are used to update $\left(q^h_{is}, q^h_{ij\tilde{s}}, q^h_{s_l|s_i}, q^h_{s_r|s_j}\right)$ at step 3 of the EM algorithm by plugging into $\mathring{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{\phi}_{il(i,t),t}\right\}_t | s_i = s, h^m_{t_0}\right)$ the values $\left(\psi^{mh}(\phi'|\phi), \psi^{fh}(\phi'|\phi), p^h(\tilde{a}|\tilde{\phi}), p^{mh}(a|\phi), p^{fh}(a|\phi)\right)$.

7.3 Special case

7.3.1 Unmatched individuals

Under Assumption 1, $q_{s_l|s_i}$ simplifies to

$$q_{s_l|s_i} = \frac{\prod_{t \in \mathscr{T}(i,l)} \psi^m(\phi_{l,t}, s_l|\phi_{it}) p(a_{il,t}|\phi_{it}, \phi_{l,t})}{\sum_s \prod_{t \in \mathscr{T}(i,l)} \psi^m(\phi_{l,t}, s|\phi_{it}) p(a_{il,t}|\phi_{it}, \phi_{l,t})}$$

Then, the expected likelihood for households in $(i, \emptyset)_{t_0}$ becomes

$$\begin{aligned}
&\max_{p,\psi} E_{\left\{s_i, \{s_{l^*}\}_{l^* \in \mathscr{L}(i)}\right\}_{i=1}^N} \left[\ln P\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{\phi}_{il(i,t),t}\right\}_{(i,t)=(1,1)}^{(N,T)}, \left\{s_i, \{s_{l^*}\}_{l^* \in \mathscr{L}(i)}\right\}_{i=1}^N\right) | \left\{\tilde{a}_{il(i,t),t}, \tilde{\phi}_{il(i,t),t}\right\}_{(i,t)=(1,1)}^{(N,T)}\right] \quad (21) \\
&= \max_{p,\psi} \sum_i \sum_s q_{is} \left(\ln p(x_{i1}|s_i) + \ln p(a_{i1}|x_{i1}, s_i) + \sum_{t>1} \sum_{s_{l(i,t)}} q_{s_{l(i,t)}|s_i} \ln P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\phi_{it}\right)\right)
\end{aligned}$$

Where

$$P\left(\phi_{l^*,t},\tilde{a}_{il^*,t}|\phi_{it}\right)=\psi^m(\phi_{l(i,t^*),t}|\phi_{it})p(\tilde{a}_{il(i,t^*)t}|\phi_{il(i,t),t})$$

The derivation of (21) is in C.1.

In C.2, we derive the solution to (21), which provides nonparametric estimators for p and ψ.

7.3.2 Matched pairs

If transition costs do not depend on the origin spouse, the expression for $q_{s_l|s_i}$ simplifies

$$q_{s_l|\tilde{s}_{ij}}=\frac{\prod_{t\in\mathscr{T}(i,l)}P\left(s_l,\phi_{l,t},\tilde{a}_{il,t}|\phi_{it}\right)}{\sum_{s_l'}\prod_{t\in\mathscr{T}(i,l)}P\left(s_l',\phi_{l,t},\tilde{a}_{il,t}|\phi_{it}\right)}$$

And the expected likelihood for households in $(i,j)_{t_0}$ becomes

$$\max_{p,\psi}E\left[\ln P\left(\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t},\left\{s_i,s_j,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)\backslash j}\{s_{r^*}\}_{r^*\in\mathscr{R}(j)\backslash i}\right\}_{(i,j)}\right)|\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t}\right]$$

$$=\max_{p,\psi}\sum_{(i,j)_{t_0}}\sum_{\tilde{s}}q_{ij\tilde{s}}\left(\begin{array}{c}\ln\pi\left(\tilde{s}_{ij}\right)+\ln p\left(\tilde{a}_{ij1},\tilde{\phi}_{ij1}|\tilde{s}_{ij}\right)\\+\sum_{t\in\mathscr{T}(i,j)\backslash\{t=1\}}\ln P\left(\phi_{r(j,t),t}=\phi_{i,t},\phi_{l(i,t),t}=\phi_{j,t},\tilde{a}_{ij,t}|\phi_{j,t},\phi_{i,t}\right)\\+\sum_{l^*\in\mathscr{L}(i)\backslash j}\sum_{s_{l^*}}\sum_{t\in\mathscr{T}(i,l^*)}\ln P\left(\phi_{l^*,t},\tilde{a}_{il^*,t}|\phi_{it}\right)q_{s_{l^*}|\tilde{s}_{ij}}+\\\sum_{r^*\in\mathscr{R}(j)\backslash i}\sum_{s_{r^*}}\sum_{t\in\mathscr{T}(j,r^*)}\ln P\left(\phi_{r^*,t},\tilde{a}_{r^*j,t}|\phi_{j,t}\right)q_{s_{r^*}|\tilde{s}_{ij}}\end{array}\right)\tag{22}$$

Where the conditional expectation in the first line is taken over $\left\{s_i,s_j,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\{s_{r^*}\}_{r^*\in\mathscr{R}(j)\backslash i}\right\}_{(i,j)}$.

The derivation of (22) is in C.3 and in C.4, we derive the solution to C.3.

7.3.3 Joint likelihood

Clearly the likelihood of the data will include both one- and two-person households.

Then, the expected joint likelihood of all histories in the data takes the form.

$$\max_{p,\psi}E_{\tilde{s}_{ij},s_{l(i,t)},s_{r(j,t)}}\left[\ln\mathring{P}\left(\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t}|\left\{\tilde{s}_{ij},s_{l(i,t)},s_{r(j,t)}\right\}_{ij,t}\right)|\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t}\right]$$

$$=\max_{p,\psi}\sum_{\{(i,j)_{t_0}\}}\sum_{\tilde{s}}q_{ij\tilde{s}}\left(\begin{array}{c}\ln p(\tilde{x}_{ij,1}|\tilde{s}_{ij})+\ln p(\tilde{a}_{ij1}|\tilde{x}_{ij,1},\tilde{s}_{ij})\\+\sum_t\sum_{s_{l(i,t)},s_{r(j,t)}}q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}}\ln P\left(\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,tij,t}|\tilde{s}_{ij},s_{l(i,t)},s_{r(j,t)}\right)\end{array}\right)\tag{23}$$

$$+\sum_{\{(i,\emptyset)_{t_0}\}}\left(\ln p^m(x_{i1}|s_i)+\ln p^m(a_{i1}|x_{i1},s_i)+\sum_s q_{is}\sum_t\sum_{s_{l(i,t)}}q_{l(i,t)s}\ln P\left(\tilde{a}_{il(i,t),t},\phi_{it},\phi_{l(i,t),t}|s_i,s_{l(i,t)}\right)\right)$$

$$+\sum_{\{(\emptyset,j)_{t_0}\}}\left(\ln p^f(x_{j1}|s_j)+\ln p^f(a_{j1}|x_{j1},s_j)+\sum_s q_{js}\sum_t\sum_{s_{r(j,t)}}q_{r(j,t)s}\ln P\left(\tilde{a}_{r(j,t)j,t},\phi_{jt},\phi_{r(j,t),t}|s_j,s_{r(j,t)}\right)\right)$$

In C.5, we derive the solution to (23). These solutions are used to update the values of ($\psi^{mh}(\phi'|\phi)$, $\psi^{fh}(\phi'|\phi)$, $p^h(\tilde{a}|\tilde{\phi})$, $p^{mh}(a|\phi)$, $p^{fh}(a|\phi)$).

At the end of the EM algorithm, we obtain values for ($\psi^{m*}(\phi'|\phi)$, $\psi^{f*}(\phi'|\phi)$, $p^*(\tilde{a}|\tilde{\phi})$, $p^{m*}(a|\phi)$, $p^{f*}(a|\phi)$) that can be used in one last maximization 23 over the space of the structural parameters and using the values of ($\psi^{mh-1}(\phi'|\phi)$, $\psi^{fh-1}(\phi'|\phi)$, $p^{h-1}(\tilde{a}|\tilde{\phi})$, $p^{mh-1}(a|\phi)$, $p^{fh-1}(a|\phi)$) in the right-hand side of 18,19, and 20 to construct the likelihood. Alternatively, 23 can be performed over the space of structural parameters from the beginning, in which case ($\psi^{m*}(\phi'|\phi)$, $\psi^{f*}(\phi'|\phi)$, $p^*(\tilde{a}|\tilde{\phi})$, $p^{m*}(a|\phi)$, $p^{f*}(a|\phi)$) are updated by plugging the solution into 18,19, and 20 at each EM iteration. Either way, the result of such maximization will yield the structural parameters estimator.

8 Monte Carlo exercise.

In this section we specify a toy matching model to illustrate the estimation procedure and the small sample performance of our estimator. We build on the standard bus engine replacement model (Rust (1987); Arcidiacono and Miller (2011)), characterized by a renewal action that resets the engine mileage regardless of the previously accumulated mileage, but we allow buses to operate either individually or in pairs. We consider a stationary environment with a two-dimensional state space $\mathscr{X} = \{(x_1, x_2) : x_1 \in \mathscr{X}_1, x_2 \in \mathscr{X}_2\}$ with $\mathscr{X}_1 = \{0 : 1 : 25\}$ being the accumulated engine mileage and $\mathscr{X}_2 = \{0.25, 0.75\}$ being an observable characteristic that affects mileage accumulation, such as bus route length, which is individual-specific and fixed in time. Unobserved types fall into the binary set $\mathscr{S} = \{0, 1\}$. At any state, the choice set is $\mathscr{A}(\phi) = \mathscr{A} = \{a_1, a_2, a_3\}$. Utilities are specfied as

$$u^f_{\emptyset t}(a|\phi) = u^m_{\emptyset t}(a|\phi) = \begin{cases} 0 & if\, a = a_1 \\ \alpha_0 + \alpha_1 x_1 + \alpha_2 s + 2\alpha_3 & if\, a = a_2 \\ \alpha_0 + \alpha_1 x_1 + \alpha_2 s + 3\alpha_3 & if\, a = a_3 \end{cases}$$
$$u^f_t((a_{it}, a_{jt})|\phi_{it}, \phi_{jt}) = u^f_{\emptyset t}(a_{jt}|\phi_{jt})$$
$$u^m_t((a_{it}, a_{jt})|\phi_{it}, \phi_{jt}) = u^m_{\emptyset t}(a_{it}|\phi_{it})$$
$$\kappa^m(\phi|\phi_i) = \kappa^f(\phi|\phi_j) = \begin{cases} 0 & if\, \phi = \emptyset \\ \kappa & otherwise \end{cases}$$

And the Pareto weights are calibrated as $\lambda\left(\tilde{\phi}_{ij,t}\right) = \lambda = 0.5$. The individual transition probabilities of $x_1$are governed by

$$f(x_1'|x_1, x_2, a) = \begin{cases} x_2 \exp(-x_2(x_1')) & if\, x_1' \geq 0\, and\, a = 1 \\ x_2 \exp(-x_2(x_1' - x_1)) & if\, x_1' \geq x_1\, and\, a = 2 \\ 2x_2 \exp(-2x_2(x_1' - x_1)) & if\, x_1' \geq x_1\, and\, a = 3 \\ 0 & otherwise \end{cases}$$
$$f(x_{1i}', x_{1j}'|x_{1i}, x_{2i}, x_{1j}, x_{2j}, (a_i, a_j)) = f(x_{1i}'|x_{1i}, x_{2i}, a_i) f(x_{1j}'|x_{1j}, x_{2j}, a_j)$$

Because action a_3 reduces the probability that x_1' will be far from x_1 compared to action a_2, we can interpret a_3 as a maintenance intervention on the engine that falls short of total engine replacement, while a_2 can be interpreted as no maintainance being done on the engine at all. Action a_1 is the renewal action, so it can be interpreted as total engine replacement. Under this specification, the model is characterized by one-period finite dependence, i.e. at each period and each stage for any x_{1t} there exists at least two sequences of first- and second-stage choices such that the distribution of x_{1t+1} is identical whether an agent enacts one sequence of choice or the other. In D we show that because of this property, the differences $v^m_t(a|\phi_{it}) - v^m_t(0|\phi_{it})$, $v^m_t\left(\tilde{a}|\tilde{\phi}_{il(i,t),t}\right) - v^m_t\left(0, 0|\tilde{\phi}_{il(i,t),t}\right)$, and $V^f_t(\phi_{r(j,t)t}, \phi_{j,t}) - V^f_{0t}(\phi_{j,t})$ only depend on objects subscripted by t and $t+1$ but no further. This greatly simplifies the computation of the structural moments discussed in 6, which would otherwise require telescoping continuation values all the way until period T.

We do not estimate transition probabilities and instead use the true transition probabilities to form the structural moments. We solve the model by backwards induction with $\beta = 0.6$ and $T = 20$ while we retain only the first 10 periods of simulated data, to approximate a infinite-horizon setting. Each simulation is comprised of 5000 households, formed initially by either a single agent or a pair. Hence, the number of individuals in each simulation is between 5000 and 10000. We simulate data and estimate the model 50 times. Denote the vector of true values of the structural parameters with $b_0 = [\kappa, \alpha_0, \alpha_1, \alpha_2, \alpha_3]$. At each simulation, we initialize the vector of structural estimates at $\hat{b}^0 = b_0 + u$ where $u \sim U(-0.5, 0.5)$ and the posteror probabilities $(q^0_{is}, q^0_{ij\tilde{s}}, q^0_{s_l|s_i}, q^0_{s_r|s_j})$. We calibrate the exogenous distribution of types conditional on sex $\pi^m(1) = 0.4$ and $\pi^f(1) = 0.4$.

b_0	κ	α_0	α_1	α_2	α_3
True value	-4	2	-0.15	1	-1
mean estimate	-3.986	2.048	-0.144	1.103	-1.051
s.d.	0.282	0.038	0.002	0.042	0.012

Table 1: Monte Carlo simulation results.

Our Monte Carlo results support our proofs of identification and demonstrate that the model can be estimated in a relatively short time given the availability of the appropriate hardware. The mean estimation time was 129 seconds with a standard deviation of 20 seconds. We programmed the routine in the Julia language and found that performing calculations through array operators on our dedicated graphic card accelerated the completion of each simulation by two orders of magnitude. Though, this approach is memory intensive and requires loading entire arrays on the dedicated GPU memory, which limits the size of the state space we can adopt.

9 Conclusion

In this article we develop a model of dynamic matching with endogenous separations and re-matching and an identification and estimation strategy that allows for time-invariant individual unobserved heterogeneity on both sides of the matching market. Our identification results are limited to stationary environments and rely on a specific sampling procedure, namely that the unit of sampling be the match itself. Our contribution is twofold. First, we provide a framework to empirically study matching markets that is computationally relatively inexpensive, by expanding on the literature on matching as a dynamic discrete choice setting and leveraging the CCP-based techniques proper of the DDC literature. Second, the tractability of this framework enables us to provide proofs that clarify the sources of identification of the structural parameters. This model can be used to study household formation and other matching markets where persistent unobserved heterogeneity is relevant.

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Appendix

A Derivation of $V^f(\tilde{\phi})$

For convenience, I define $\mathscr{A}^2 = \mathscr{A}\left(\phi^m_{it}, \phi^f_{jt}\right)$, $\lambda = \lambda\left(\phi^m_{it}, \phi^f_{jt}\right)$ and to save space I define $v^f_{\tilde{a}}(\tilde{\phi}) = v^f(\tilde{a}|\tilde{\phi})$, $v^m_{\tilde{a}}(\tilde{\phi}) = v^m(\tilde{a}|\tilde{\phi})$

$$
\begin{aligned}
&V^f(\tilde{\phi})\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\mathbb{1}\{\lambda v^f_{\tilde{a}}(\tilde{\phi})+(1-\lambda)v^m_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}} \geq \lambda v^f_k(\tilde{\phi})+(1-\lambda)v^m_k(\tilde{\phi})+\varepsilon_k,\ \forall k\in\mathscr{A}^2\backslash\{j\}\}g(\vec{\varepsilon})d\vec{\varepsilon}\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\prod_{k\in\mathscr{A}^2\backslash\tilde{a}}\mathbb{1}\{\lambda v^f_{\tilde{a}}(\tilde{\phi})+(1-\lambda)v^m_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}} \geq \lambda v^f_k(\tilde{\phi})+(1-\lambda)v^m_k(\tilde{\phi})+\varepsilon_k\}g(\vec{\varepsilon})d\vec{\varepsilon}\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\prod_{k\in\mathscr{A}^2\backslash\tilde{a}}\mathbb{1}\{\lambda[v^f_{\tilde{a}}(\tilde{\phi})-v^f_k(\tilde{\phi})]+(1-\lambda)[v^m_{\tilde{a}}(\tilde{\phi})-v^m_k(\tilde{\phi})]+\varepsilon_{\tilde{a}}\geq\varepsilon_k\}g(\vec{\varepsilon})d\vec{\varepsilon}\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\prod_{k\in\mathscr{A}^2\backslash\tilde{a}}\mathbb{1}\{m_{\tilde{a}k}(\tilde{\phi})+\varepsilon_{\tilde{a}}\geq\varepsilon_k\}g(\vec{\varepsilon})d\vec{\varepsilon}\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int\int ...\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\prod_{k\in\mathscr{A}^2\backslash\tilde{a}}\mathbb{1}\{m_{\tilde{a}k}(\tilde{\phi})+\varepsilon_{\tilde{a}}\geq\varepsilon_k\}g(\varepsilon_1)g(\varepsilon_2)...g(\varepsilon_{\tilde{a}})...g(\varepsilon_A)d\varepsilon_1 d\varepsilon_2...d\varepsilon_{\tilde{a}}...d\varepsilon_A\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\int ...\int\prod_{k\in\mathscr{A}^2\backslash\tilde{a}}\mathbb{1}\{m_{\tilde{a}k}(\tilde{\phi})+\varepsilon_{\tilde{a}}\geq\varepsilon_k\}g(\varepsilon_1)g(\varepsilon_2)...g(\varepsilon_A)d\varepsilon_1 d\varepsilon_2...d\varepsilon_A g(\varepsilon_{\tilde{a}})d\varepsilon_{\tilde{a}}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\prod_{k\in\mathscr{A}^2\backslash\tilde{a}}\int\mathbb{1}\{m_{\tilde{a}k}(\tilde{\phi})+\varepsilon_{\tilde{a}}\geq\varepsilon_k\}g(\varepsilon_k)d\varepsilon_k g(\varepsilon_{\tilde{a}})d\varepsilon_{\tilde{a}}\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\prod_{k\in\mathscr{A}^2\backslash\tilde{a}}G(m_{\tilde{a}k}(\tilde{\phi})+\varepsilon_{\tilde{a}})g(\varepsilon_{\tilde{a}})d\varepsilon_{\tilde{a}}\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\prod_{k\in\mathscr{A}^2\backslash\tilde{a}}\exp\left\{-\exp\left(-\frac{m_{\tilde{a}k}(\tilde{\phi})+\varepsilon_{\tilde{a}}-\mu}{\nu}\right)\right\}g(\varepsilon_{\tilde{a}})d\varepsilon_{\tilde{a}}\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\exp\left\{-\sum_{k\in\mathscr{A}^2\backslash\tilde{a}}\exp\left(-\frac{m_{\tilde{a}k}(\tilde{\phi})+\varepsilon_{\tilde{a}}-\mu}{\nu}\right)\right\}g(\varepsilon_{\tilde{a}})d\varepsilon_{\tilde{a}}\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\exp\left\{-\sum_{k\in\mathscr{A}^2\backslash\tilde{a}}\exp\left(-\frac{m_{\tilde{a}k}(\tilde{\phi})+\varepsilon_{\tilde{a}}-\mu}{\nu}\right)\right\}\frac{1}{\nu}\exp\{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}-\exp(-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu})\}d\varepsilon_{\tilde{a}}\quad\text{, plugging p.d.f. of }\varepsilon_{\tilde{a}}\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\exp\left\{-\sum_{k\in\mathscr{A}^2\backslash\tilde{a}}\exp\left(-\frac{m_{\tilde{a}k}(\tilde{\phi})+\varepsilon_{\tilde{a}}-\mu}{\nu}\right)-\exp\left(-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}\right)\right\}\exp(-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu})\frac{1}{\nu}d\varepsilon_{\tilde{a}}\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\int (v^f_{\tilde{a}}(\tilde{\phi})+\varepsilon_{\tilde{a}})\exp\left\{-\exp\left(-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}\right)\left(\sum_{k\in\mathscr{A}^2\backslash\tilde{a}}\exp\left(-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}\right)+1\right)\right\}\exp(-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu})\frac{1}{\nu}d\varepsilon_{\tilde{a}}\\
&= \sum_{\tilde{a}\in\mathscr{A}^2}\underbrace{\int_{-\infty}^{\infty}v^f_{\tilde{a}}(\tilde{\phi})e^{-e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}}(1+\sum_{k\in\mathscr{A}^2\backslash\tilde{a}}e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}})}e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}}\frac{1}{\nu}d\varepsilon_{\tilde{a}}}_{(1)}+\underbrace{\int_{-\infty}^{\infty}\varepsilon_{\tilde{a}}e^{-e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}}(1+\sum_{k\in\mathscr{A}^2\backslash\tilde{a}}e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}})}e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}}\frac{1}{\nu}d\varepsilon_{\tilde{a}}}_{(2)}
\end{aligned}
$$

Using the same change of variable, $t = e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}} \implies dt = -\frac{1}{\nu}e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}}$, $\varepsilon_{\tilde{a}} \to \infty \implies t \to 0$, and $\varepsilon_{\tilde{a}} \to -\infty \implies t \to \infty$, term (1) becomes

$$
\begin{aligned}
&\int_{-\infty}^{\infty} v^f_{\tilde{a}}(\tilde{\phi}) e^{-e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}}(1+\sum_{k\in\mathscr{A}^2\setminus\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}})} e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}} \frac{1}{\nu} d\varepsilon_{\tilde{a}} \\
&= v^f_{\tilde{a}}(\tilde{\phi}) \int_{\infty}^{0} e^{-t(1+\sum_{k\in\mathscr{A}^2\setminus\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}})}(-1)dt \\
&= v^f_{\tilde{a}}(\tilde{\phi}) \int_{0}^{\infty} e^{-t(1+\sum_{k\in\mathscr{A}^2\setminus\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}})} dt \\
&= v^f_{\tilde{a}}(\tilde{\phi}) \frac{-e^{-t(1+\sum_{k\in\mathscr{A}^2\setminus\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}})}}{1+\sum_{k\in\mathscr{A}^2\setminus\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}}}\Bigg|_{t=0}^{t=\infty} \\
&= \frac{v^f_{\tilde{a}}(\tilde{\phi})}{1+\sum_{k\in\mathscr{A}^2\setminus\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}}} \\
&= \frac{v^f_{\tilde{a}}(\tilde{\phi})}{1+\sum_{k\in\mathscr{A}^2\setminus\tilde{a}} e^{-\frac{\lambda[v^f_{\tilde{a}}(\tilde{\phi})-v^f_k(\tilde{\phi})]+(1-\lambda)[v^m_{\tilde{a}}(\tilde{\phi})-v^m_k(\tilde{\phi})]}{\nu}}} \\
&= v^f_{\tilde{a}}(\tilde{\phi}) \frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2} \exp(\lambda[v^f_k(\tilde{\phi})] + (1-\lambda)v^m_k(\tilde{\phi})])^{\frac{1}{\nu}}}
\end{aligned}
$$

Of course, noticing that we are taking expectation, we can simply say that (1) is

$$
\begin{aligned}
&v^f_{\tilde{a}}(\tilde{\phi})\Pr\Big(\big[\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}) + \varepsilon_{\tilde{a}} \geq \lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi}) + \varepsilon_k\big],\ \forall k \in \mathscr{A}^2\setminus\{j\}\Big) \\
&= v^f_{\tilde{a}}(\tilde{\phi}) \frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2} \exp(\lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi})])^{\frac{1}{\nu}}}
\end{aligned}
$$

We apply the same change of variable for (2) while realizing that $t = e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}} \Rightarrow -\nu\ln t + \mu = \varepsilon_{\tilde{a}}$:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \varepsilon_{\tilde{a}} e^{-e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}}(1+\sum_{k\in\mathscr{A}^2\setminus\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}})} e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}} \frac{1}{\nu} d\varepsilon_{\tilde{a}} \\
&= \int_{\infty}^{0} -(\nu\ln t - \mu)\ e^{-t(1+\sum_{k\in\mathscr{A}^2\setminus\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}})}(-1)dt \\
&= \int_{\infty}^{0} (\nu\ln t - \mu)\ e^{-t(1+\sum_{k\in\mathscr{A}^2\setminus\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}})} dt
\end{aligned}
$$

Let's denote $M \equiv (1+\sum_{k\in\mathscr{A}^2\setminus\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}})$:

$$
\begin{aligned}
&\int_{\infty}^{0} (\nu\ln t - \mu)\ e^{-tM} dt \\
&= \int_{0}^{\infty} -(\nu\ln t - \mu)\ e^{-tM} dt \\
&= \int_{0}^{\infty} -\big(\nu\ln tM\ \ - \nu\ln M - \mu\big) e^{-tM} dt \\
&= \int_{0}^{\infty} -\nu\ln(tM) e^{-tM} dt + \int_{0}^{\infty} \nu\ln M e^{-tM} dt + \mu\int_{0}^{\infty} e^{-tM} dt
\end{aligned}
$$

We use the change of variable $y = tM \implies dy = Mdt \implies dt = \frac{1}{M}dy$ and denote γ Euler's constant. Note that the derivation of $-\int_0^\infty (\ln y)e^{-y}dy = \gamma$ involves $\Gamma'(1) = \gamma$.

$$
\begin{aligned}
&= \int_0^\infty -\frac{\nu}{M}(\ln y)e^{-y}dy - \nu \ln M \frac{e^{-tM}}{M}\bigg|_0^\infty + \mu \times (1/M) \\
&= \gamma\frac{\nu}{M} + \frac{\nu \ln M}{M} + \frac{\mu}{M} \\
&= \frac{1}{M}\left(\nu\gamma + \nu \ln M + \mu\right) \\
&= \frac{1}{1+\sum_{k\in\mathscr{A}^2\backslash\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}}}\left(\nu\gamma + \nu\ln\big(1 + \sum_{k\in\mathscr{A}^2\backslash\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}}\big) + \mu\right) \\
&= \frac{1}{1+\sum_{k\in\mathscr{A}^2}\exp\{-(\lambda[v^f_{\tilde{a}}(\tilde{\phi}) - v^f_k(\tilde{\phi})] + (1-\lambda)[v^m_{\tilde{a}}(\tilde{\phi}) - v^m_k(\tilde{\phi})])/\nu\}}\left(\nu\gamma + \nu\ln\big(1 + \sum_{k\in\mathscr{A}^2\backslash\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}}\big) + \mu\right) \\
&= \frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2}\exp(\lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi}))^{\frac{1}{\nu}}}\left(\nu\gamma + \nu\ln\big(1 + \sum_{k\in\mathscr{A}^2\backslash\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}}\big) + \mu\right) \text{, then multiply } \tilde{a}\text{and divide inside the log:} \\
&= \frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2}\exp(\lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi}))^{\frac{1}{\nu}}}\left(\nu\gamma + \nu\ln\frac{\sum_{k\in\mathscr{A}^2}\exp(\lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi}))^{\frac{1}{\nu}}}{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}} + \mu\right) \\
&= \frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2}\exp(\lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi}))^{\frac{1}{\nu}}}\Bigg(\nu\gamma + \nu\ln\left(\sum_{k\in\mathscr{A}^2}\exp(\lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi}))^{\frac{1}{\nu}}\right) - \\
&\nu\ln\left(\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}\right) + \mu\Bigg) \\
&= \frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2}\exp(\lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi})])^{\frac{1}{\nu}}}\left(\nu\gamma + \nu\ln\left(\sum_{k\in\mathscr{A}^2}\exp(\lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi}))^{\frac{1}{\nu}}\right) - \lambda v^f_{\tilde{a}}(\tilde{\phi}) - (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}) + \mu\right)
\end{aligned}
$$

Knitting (1) and (2) together, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty}(v^f_{\tilde{a}}(\tilde{\phi}) + \varepsilon_{\tilde{a}})e^{-e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}}(1+\sum_{k\in\mathscr{A}^2\backslash\tilde{a}} e^{-\frac{m_{\tilde{a}k}(\tilde{\phi})}{\nu}})}e^{-\frac{\varepsilon_{\tilde{a}}-\mu}{\nu}}\frac{1}{\nu}d\varepsilon_{\tilde{a}} \\
&= \frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2}\exp(\lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi}))^{\frac{1}{\nu}}}\times \\
&\left(\underbrace{v^f_{\tilde{a}}(\tilde{\phi})}_{(1)} + \underbrace{\nu\gamma + \nu\ln\left(\sum_{k\in\mathscr{A}^2}\exp(\lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi}))^{\frac{1}{\nu}}\right) - \lambda v^f_{\tilde{a}}(\tilde{\phi}) - (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}) + \mu}_{(2)}\right)
\end{aligned}
$$

Summing this result across all $\tilde{a} \in \mathscr{A}^2$ and noticing that the first factor is a probability, we finally get $\tilde{V}^f_1(\tilde{\phi})$:

$$
\begin{aligned}
\tilde{V}^f(\tilde{\phi}) = &\sum_{\tilde{a}\in\mathscr{A}^2}\frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2}\exp(\lambda[v^f_k(\tilde{\phi})] + v^m_k(\tilde{\phi})])^{\frac{1}{\nu}}}\times \\
&\left(v^f_{\tilde{a}}(\tilde{\phi}) + \nu\gamma + \nu\ln\left(\sum_{k\in\mathscr{A}^2}\exp(\lambda[v^f_k(\tilde{\phi})] + v^m_k(\tilde{\phi})])^{\frac{1}{\nu}}\right) - \lambda v^f_{\tilde{a}}(\tilde{\phi}) - (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}) + \mu\right)
\end{aligned}
$$

From this, we can derive the expression for $\tilde{V}^f(\tilde{\phi})$ involving conditional valuation functions and CCPs:

$$\tilde{V}^f(\tilde{\phi}) = \sum_{\tilde{a}\in\mathscr{A}^2} \frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2}\exp(\lambda[v^f_k(\tilde{\phi})] + v^m_k(\tilde{\phi})])^{\frac{1}{\nu}}} \times$$
$$\left(v^f_{\tilde{a}}(\tilde{\phi}) + \nu\gamma + \nu\ln\left(\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}} \frac{\sum_{k\in\mathscr{A}^2}\exp(\lambda v^f_k(\tilde{\phi}) + (1-\lambda)v^m_k(\tilde{\phi})])^{\frac{1}{\nu}}}{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) - (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}\right) - \right.$$
$$\left. \lambda v^f_{\tilde{a}}(\tilde{\phi}) - (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}) + \mu\right)$$
$$= \sum_{\tilde{a}\in\mathscr{A}^2} \frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2}\exp(\lambda[v^f_k(\tilde{\phi})] + v^m_k(\tilde{\phi})])^{\frac{1}{\nu}}} \times$$
$$\left(v^f_{\tilde{a}}(\tilde{\phi}) + \nu\gamma + \nu\ln\left(\frac{\sum_{k\in\mathscr{A}^2}\exp(\lambda[v^f_k(\tilde{\phi})] + v^m_k(\tilde{\phi})])^{\frac{1}{\nu}}}{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) - (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}\right) + \mu\right)$$
$$= \sum_{\tilde{a}\in\mathscr{A}^2} \frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) + (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2}\exp(\lambda[v^f_k(\tilde{\phi})] + v^m_k(\tilde{\phi})])^{\frac{1}{\nu}}} \times$$
$$\left(v^f_{\tilde{a}}(\tilde{\phi}) + \nu\gamma - \nu\ln\left(\frac{\exp(\lambda v^f_{\tilde{a}}(\tilde{\phi}) - (1-\lambda)v^m_{\tilde{a}}(\tilde{\phi}))^{\frac{1}{\nu}}}{\sum_{k\in\mathscr{A}^2}\exp(\lambda[v^f_k(\tilde{\phi})] + v^m_k(\tilde{\phi})])^{\frac{1}{\nu}}}\right) + \mu\right)$$
$$= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi})\left(v^f_{\tilde{a}}(\tilde{\phi}) + \nu\gamma - \nu\ln p(\tilde{a}|\tilde{\phi}) + \mu\right)$$
$$= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi})\left(v^f_{\tilde{a}}(\tilde{\phi}) - \nu\ln p(\tilde{a}|\tilde{\phi})\right) + \mu + \nu\gamma$$

The expression for $\tilde{V}^m_1(\tilde{\phi})$ is similar.

B Derivation of the likelihood

B.1 Unmatched individuals.

I focus on households that are comprised of single individuals first and in particular those formed by a single male individual i. The same identification argument applies to single female, mutadis mutandis. The sample likelihood for male individual i is

$$\begin{aligned}
L(i,\emptyset)_{t_0} &= \mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=1}^{T} | h_{t_0}^{m}\right) \\
&= \sum_{s_i} \mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=1}^{T}, s_i | h_{t_0}^{m}\right) \frac{P(a_{i1}, x_{i1}, s_i | h_{t_0}^{m})}{P(a_{i1}, x_{i1}, s_i | h_{t_0}^{m})} \frac{\pi(s_i,\emptyset)}{\pi(s_i,\emptyset)} \\
&= \sum_{s_i} \mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=2}^{T} | a_{i1}, x_{i1}, s_i, h_{t_0}^{m}\right) P(a_{i1}, x_{i1}, s_i, h_{t_0}^{m}) \frac{\pi(s_i,\emptyset)}{\pi(s_i,\emptyset)} \\
&= \sum_{s_i} \pi(s_i,\emptyset) p^{m}(a_{i1}, x_{i1} | s_i, h_{t_0}^{m}) \mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=2}^{T} | a_{i1}, x_{i1}, s_i, h_{t_0}^{m}\right) \\
&= \sum_{s_i} \pi(s_i,\emptyset) p^{m}(a_{i1}, x_{i1} | s_i, h_{t_0}^{m}) \sum_{s_{l(i,2)}} \mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=2}^{T}, s_{l(i,2)} | a_{i1}, x_{i1}, s_i, h_{t_0}^{m}\right) \\
&= \sum_{s_i} \pi(s_i,\emptyset) p^{m}(a_{i1}, x_{i1} | s_i) \sum_{s_{l(i,2)}} \frac{\mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=2}^{T}, s_{l(i,2)}, a_{i1}, x_{i1}, s_i\right)}{P\left(a_{i1}, x_{i1}, s_i\right)} \frac{P\left(\tilde{a}_{il(i,2)}, x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, a_{i1}, x_{i1}, s_i\right)}{P\left(\tilde{a}_{il(i,2)}, x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, a_{i1}, x_{i1}, s_i\right)} \\
&= \sum_{s_i} \pi(s_i,\emptyset) p^{m}(a_{i1}, x_{i1} | s_i) \sum_{s_{l(i,2)}} P\left(\tilde{a}_{il(i,2),2}, x_{l(i,2)2}, s_{l(i,2)}, x_{i2} | a_{i1}, x_{i1}, s_i\right) \mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=3}^{T} | \tilde{a}_{il(i,2),2}, x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, \cancel{a_{i1}, x_{i1}}, s_i\right)
\end{aligned}$$

Where $a_{i1}$and x_{i1}cancel because of conditional independence. Now, decompose $P\left(\tilde{a}_{il(i,2)}, x_{l(i,2)2}, s_{l(i,2)}, x_{i2} | a_{i1}, x_{i1}, s_i\right)$

$$\begin{aligned}
P\left(\tilde{a}_{il(i,2)}, x_{l(i,2)2}, s_{l(i,2)}, x_{i2} | a_{i1}, x_{i1}, s_i\right) &= \frac{P\left(\tilde{a}_{il(i,2),2}, x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, a_{i1}, x_{i1}, s_i\right)}{P\left(a_{i1}, x_{i1}, s_i\right)} \frac{P\left(x_{i2}, a_{i1}, x_{i1}, s_i\right)}{P\left(x_{i2}, a_{i1}, x_{i1}, s_i\right)} \\
&= \frac{P\left(\tilde{a}_{il(i,2),2}, x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, a_{i1}, x_{i1}, s_i\right)}{P\left(x_{i2}, a_{i1}, x_{i1}, s_i\right)} P\left(x_{i2} | a_{i1}, x_{i1}, \cancel{s_i}\right) \text{ type doesn't matter for transitions} \\
&= \frac{P\left(\tilde{a}_{il(i,2),2}, x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, a_{i1}, x_{i1}, s_i\right)}{P\left(x_{i2}, a_{i1}, x_{i1}, s_i\right)} \frac{P\left(x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, a_{i1}, x_{i1}, s_i\right)}{P\left(x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, a_{i1}, x_{i1}, s_i\right)} f\left(x_{i2} | a_{i1}, x_{i1}\right) \\
&= \frac{P\left(\tilde{a}_{il(i,2),2}, x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, a_{i1}, x_{i1}, s_i\right)}{P\left(x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, a_{i1}, x_{i1}, s_i\right)} \frac{P\left(x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, a_{i1}, x_{i1}, s_i\right)}{P\left(x_{i2}, a_{i1}, x_{i1}, s_i\right)} f\left(x_{i2} | a_{i1}, x_{i1}\right) \\
&= P\left(\tilde{a}_{il(i,2),2} | x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, \cancel{a_{i1}, x_{i1}}, s_i\right) P\left(x_{l(i,2)2}, s_{l(i,2)} | x_{i2}, \cancel{a_{i1}, x_{i1}}, s_i\right) f\left(x_{i2} | a_{i1}, x_{i1}\right) \\
&= p\left(\tilde{a}_{il(i,2),2} | x_{l(i,2)2}, s_{l(i,2)}, x_{i2}, s_i\right) \psi^{m}\left(x_{l(i,2)2}, s_{l(i,2)} | x_{i2}, s_i\right) f\left(x_{i2} | a_{i1}, x_{i1}\right)
\end{aligned}$$

$$L(i,\emptyset)_{t_0} = \sum_{s_i} \pi(s_i,\emptyset) p^{m}(a_{i1}, x_{i1} | s_i) \sum_{s_{l(i,2)}} P\left(\tilde{a}_{il(i,2),2}, x_{i2}, x_{l(i,2)2}, s_{l(i,2)} | a_{i1}, x_{i1}, s_i\right) \mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=3}^{T} | \tilde{a}_{il(i,2),2}, x_{i2}, s_i, x_{l(i,2)2}, s_{l(i,2)}\right)$$

$$
\begin{aligned}
&\mathbb{P}\left(\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t},x_{l(i,t-1),t}\}_{t=3}^{T}|\tilde{a}_{il(i,2),2},x_{i2},s_i,x_{l(i,2)2},s_{l(i,2)}\right)\\
&=\sum_{s_{l(i,3)}}\mathbb{P}\left(\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t},x_{l(i,t-1),t}\}_{t=3}^{T},s_{l(i,3)}|\tilde{a}_{il(i,2),2},x_{i2},s_i,x_{l(i,2)2},s_{l(i,2)}\right)\\
&=\sum_{s_{l(i,3)}}\mathbb{P}\left(\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t},x_{l(i,t-1),t}\}_{t=3}^{T},s_{l(i,3)}|\tilde{a}_{il(i,2),2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)\frac{P\left(\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2),3},\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}{P\left(\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2),3},\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}\\
&=\sum_{s_{l(i,3)}}\mathbb{P}\left(\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t},x_{l(i,t-1),t}\}_{t=4}^{T}|\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2)3},\cancel{\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)}},s_i\right)\frac{P\left(\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2)3},\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}{P\left(\tilde{a}_{il(i,2),2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}\\
&=\sum_{s_{l(i,3)}}\mathbb{P}\left(\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t},x_{l(i,t-1),t}\}_{t=4}^{T}|\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2)3},s_i\right)P\left(\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2)3}|\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)
\end{aligned}
$$

Now decompose $P\left(\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2)3}|\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)$

$$
\begin{aligned}
&P\left(\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2)3}|\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)\\
&=\frac{P\left(\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2)3},\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}{P\left(\tilde{a}_{il(i,2),2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}\frac{P\left(x_{i3},x_{l(i,2)3},\tilde{a}_{il(i,2),2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}{P\left(x_{i3},x_{l(i,2)3},\tilde{a}_{il(i,2),2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}\\
&=P\left(\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)}|x_{i3},x_{l(i,2)3},\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)P\left(x_{i3},x_{l(i,2)3}|\tilde{a}_{il(i,2),2},x_{i2},x_{l(i,2)2},\cancel{s_{l(i,2)},s_i}\right)\\
&=P\left(\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)}|x_{i3},x_{l(i,2)3},\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)f\left(x_{i3},x_{l(i,2)3}|\tilde{a}_{il(i,2),2},x_{i2},x_{l(i,2)2}\right)\\
&=\frac{P\left(\tilde{a}_{il(i,3),3},x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2)3},\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}{P\left(x_{i3},x_{l(i,2)3},\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}\frac{P\left(x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2)3},\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}{P\left(x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2)3},\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)},s_i\right)}f\left(x_{i3},x_{l(i,2)3}|\tilde{a}_{il(i,2),2},x_{i2},x_{l(i,2)2}\right)\\
&=P\left(\tilde{a}_{il(i,3),3}|x_{l(i,3)3},s_{l(i,3)},x_{i3},x_{l(i,2)3},\cancel{\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2},s_{l(i,2)}},s_i\right)P\left(x_{l(i,3)3},s_{l(i,3)}|x_{i3},x_{l(i,2)3},\cancel{\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2}},s_{l(i,2)},s_i\right)\\
&=p\left(\tilde{a}_{il(i,3),3}|x_{l(i,3)3},s_{l(i,3)},x_{i3},s_i\right)P\left(x_{l(i,3)3},s_{l(i,3)}|x_{i3},x_{l(i,2)3},\cancel{\tilde{a}_{il(i,2)2},x_{i2},x_{l(i,2)2}},s_{l(i,2)},s_i\right)
\end{aligned}
$$

With

$$
P\left(x_{l(i,3)3},s_{l(i,3)}|x_{i3},x_{l(i,2)3},s_{l(i,2)},s_i\right)=\begin{cases}0 & if\; l(i,2)=l(i,3)\; and\; s_{l(i,2)}\neq s_{l(i,3)}\\ \psi^m\left(x_{l(i,3)3},s_{l(i,3)}|x_{i3},x_{l(i,2)3},s_{l(i,2)},s_i\right) & otherwise\end{cases}
$$

More generally

$$
P\left(x_{l(i,t)t},s_{l(i,t)}|x_{it-1},x_{l(i,t-1)t},s_{l(i,t-1)},s_i\right)=\begin{cases}0 & if\;\exists t^*<t\; s.t.\; l(i,t^*)=l(i,t)\; and\; s_{l(i,t)}\neq s_{l(i,t^*)}\\ \psi^m\left(x_{l(i,t)t},s_{l(i,t)}|x_{it-1},x_{l(i,t-1)t},s_{l(i,t-1)},s_i\right) & otherwise\end{cases}
$$

So if we suppose that i stays matched with the same person from $t=2$ to $t=3$.

$$
L(i,\emptyset)_{t_0}=\sum_{s_i}\pi(s_i,\emptyset)p^m(a_{i1},x_{i1}|s_i)\sum_{s_{l(i,2)}}\left(\begin{array}{c}P\left(\tilde{a}_{il(i,2),2},x_{i2},x_{l(i,2)2},s_{l(i,2)}|a_{i1},x_{i1},s_i\right)\times\\ f\left(x_{i3},x_{l(i,2)3}|\tilde{a}_{il(i,2),2},x_{i2},x_{l(i,2)2}\right)\psi^m\left(x_{l(i,2)3},s_{l(i,2)}|x_{i3},x_{l(i,2)3},s_{l(i,2)},s_i\right)p\left(\tilde{a}_{il(i,3),3}|x_{l(i,3)3},s_{l(i,2)},x_{i3},s_i\right)\times\\ \mathbb{P}\left(\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\}_{t=4}^{T}|\tilde{a}_{il(i,3),3},x_{l(i,2)3},s_{l(i,2)},x_{i3},x_{l(i,2)3},s_i\right)\end{array}\right)
$$

Now suppose i marries another person in $t=4$

$$L(i,\emptyset)_{t_0} = \sum_{s_i} \pi(s_i,\emptyset) p^m(a_{i1},x_{i1}|s_i) \left(\begin{array}{c} \sum_{s_{l(i,2)}} P\left(\tilde{a}_{il(i,2),2}, x_{i2}, x_{l(i,2)2}, s_{l(i,2)} | a_{i1}, x_{i1}, s_i\right) \\ \times P\left(\tilde{a}_{il(i,3),3}, x_{l(i,3)3} = x_{l(i,2)3}, s_{l(i,3)} = s_{l(i,2)}, x_{i3}, x_{l(i,2)3} | \tilde{a}_{il(i,2)2}, x_{i2}, x_{l(i,2)2}, s_{l(i,2)}, s_i\right) \\ \times \mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=4}^{T} | a_{il(i,3)}, x_{i3}, x_{l(i,2)3}, s_{l(i,2)}, s_i\right) \end{array} \right)$$

$$= \sum_{s_i} \pi(s_i,\emptyset) p^m(a_{i1},x_{i1}|s_i) \left(\begin{array}{c} \sum_{s_{l(i,2)}} P\left(\tilde{a}_{il(i,2),2}, x_{i2}, x_{l(i,2)2}, s_{l(i,2)} | a_{i1}, x_{i1}, s_i\right) \\ \times P\left(\tilde{a}_{il(i,3),3}, x_{l(i,3)3} = x_{l(i,2)3}, s_{l(i,3)} = s_{l(i,2)}, x_{i3}, x_{l(i,2)3} | \tilde{a}_{il(i,2)2}, x_{i2}, x_{l(i,2)2}, s_{l(i,2)}, s_i\right) \\ \times \sum_{s_{l(i,4)}} \mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=4}^{T}, s_{l(i,4)} | a_{il(i,3)}, x_{i3}, x_{l(i,2)3}, s_{l(i,2)}, s_i\right) \end{array} \right)$$

And continue telescoping

Importantly, notice that the dependence of $\mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=4}^{T}, s_{l(i,4)} | a_{il(i,3)}, x_{i3}, x_{l(i,2)3}, s_{l(i,2)}, s_i\right)$ on $s_{l(i,2)}$ prevents us from factoring $\sum_{s_{l(i,4)}} \mathbb{P}\left(\left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}, x_{l(i,t-1),t}\right\}_{t=4}^{T}, s_{l(i,4)} | a_{il(i,3)}, x_{i3}, x_{l(i,2)3}, s_{l(i,2)}, s_i\right)$ out of the summation over $s_{l(i,2)}$.

B.2 Matched pairs

Now we focus on households that are comprised of two matched individuals at the moment of sampling. The likelihood for household (i,j) is

$$L(i,j)_{t_0} = \mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=1}^{T}\right) =$$

$$= \sum_{\tilde{s}_{ij}} \mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=1}^{T}, \tilde{s}_{ij}\right) \frac{P\left(\tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)}{P\left(\tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)} \frac{\pi\left(\tilde{s}_{ij}\right)}{\pi\left(\tilde{s}_{ij}\right)}$$

$$= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=2}^{T} | \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)$$

$$\mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=2}^{T} | \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)$$

$$= \sum_{s_{l(i,2)}} \sum_{s_{r(j,2)}} \left(\begin{array}{c} \frac{\mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=3}^{T}, \tilde{a}_{il(i,2),2}, \tilde{a}_{r(j,2)j,2}, x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)}{P\left(\tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)} \times \\ \frac{P\left(\tilde{a}_{il(i,2),2}, \tilde{a}_{r(j,2)j,2}, x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)}{P\left(\tilde{a}_{il(i,2),2}, \tilde{a}_{r(j,2)j,2}, x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)} \end{array} \right)$$

$$= \sum_{s_{l(i,2)}} \sum_{s_{r(j,2)}} \left(\begin{array}{c} \mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=3}^{T} | \tilde{a}_{il(i,2)2}, \tilde{a}_{r(j,2)j,2}, x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \cancel{\tilde{a}_{ij1}, \tilde{x}_{ij1}}, \tilde{s}_{ij}\right) \times \\ P\left(\tilde{a}_{il(i,2)2}, \tilde{a}_{r(j,2)j,2}, x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2} | \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right) \end{array} \right)$$

Now decompose $P\left(\tilde{a}_{il(i,2)2}, \tilde{a}_{r(j,2)j,2}, x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2} | \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)$

$$
\begin{aligned}
& P\left(\tilde{a}_{il(i,2)2}, \tilde{a}_{r(j,2)j,2}, x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2} | \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right) \\
&= \frac{P\left(\tilde{a}_{il(i,2)2}, \tilde{a}_{r(j,2)j,2}, x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)}{P\left(\tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)} \frac{P\left(x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)}{P\left(x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)} \\
&= P\left(\tilde{a}_{il(i,2)2}, \tilde{a}_{r(j,2)j,2} | x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right) P\left(x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2} | \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right) \\
&= P\left(\tilde{a}_{il(i,2)2}, \tilde{a}_{r(j,2)j,2} | x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right) \frac{P\left(x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)}{P\left(\tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)} \frac{P\left(\tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)}{P\left(\tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right)} \\
&= P\left(\tilde{a}_{il(i,2)2}, \tilde{a}_{r(j,2)j,2} | x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right) P\left(x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)} | \tilde{x}_{ij2}, \cancel{\tilde{a}_{ij1}, \tilde{x}_{ij1}}, \tilde{s}_{ij}\right) P\left(\tilde{x}_{ij2} | \tilde{a}_{ij1}, \tilde{x}_{ij1}, \cancel{\tilde{s}_{ij}}\right) \\
&= P\left(\tilde{a}_{il(i,2)2}, \tilde{a}_{r(j,2)j,2} | x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)}, \tilde{x}_{ij2}, \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right) P\left(x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)} | \tilde{x}_{ij2}, \tilde{s}_{ij}\right) f\left(\tilde{x}_{ij2} | \tilde{a}_{ij1}, \tilde{x}_{ij1}\right)
\end{aligned}
$$

$$
P\left(x_{l(i,2)2}, x_{r(j,2)2}, s_{l(i,2)}, s_{r(j,2)} | \tilde{x}_{ij2}, \tilde{s}_{ij}\right) = \begin{cases} 0 \quad if\, l(i,2) = j \,(\text{which implies } r(j,2) = i)\, and\, s_{l(i,2)} \neq s_j \; or \; s_{r(j,2)} \neq s_i \\ \psi^m\left(x_{l(i,2),2} = x_{j2}, s_{l(i,2)} = s_j | \tilde{x}_{ij1}, \tilde{s}_{ij}\right) \psi^f\left(x_{r(ij,2),2} = x_{i2}, s_{r(j,2)} = s_i | \tilde{x}_{ij1}, \tilde{s}_{ij}\right) \quad otherwise \end{cases}
$$

Now, suppose that i and j stay matched in period 2. Then,

$$
\begin{aligned}
L(i,j)_{t_0} = & \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) P\left(\tilde{a}_{ij,2}, s_{l(i,2)} = s_i, s_{r(j,2)} = s_j, \tilde{x}_{ij2} | \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right) \\
& \times \mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=3}^{T} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)
\end{aligned}
$$

Now suppose i and j divorce at $t = 3$ and immediately remarry other people

$$
\begin{aligned}
L(i,j)_{t_0} = & \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \times \\
& \left(\begin{array}{c} P\left(\tilde{a}_{ij,2}, x_{l(i,2)2} = x_{j2}, x_{r(j,2)2} = x_{i2}, s_{l(i,2)} = s_i, s_{r(j,2)} = s_j, \tilde{x}_{ij2} | \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right) \times \\ \sum_{s_{l(i,3)}} \sum_{s_{r(j,3)}} \mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=3}^{T}, s_{l(i,3)}, s_{r(j,3)} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right) \end{array} \right)
\end{aligned}
$$

$$
\begin{aligned}
& \mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=3}^{T}, s_{l(i,3)}, s_{r(j,3)} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right) \\
&= \frac{\mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=4}^{T}, s_{l(i,3)}, s_{r(j,3)}, \tilde{a}_{il(i,3),3}, \tilde{a}_{r(j,3)j,3}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)}{P\left(\tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)} \\
& \times \frac{P\left(s_{l(i,3)}, s_{r(j,3)}, \tilde{a}_{il(i,3),3}, \tilde{a}_{r(j,3)j,3}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)}{P\left(s_{l(i,3)}, s_{r(j,3)}, \tilde{a}_{il(i,3),3}, \tilde{a}_{r(j,3)j,3}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)} \\
&= \mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=4}^{T} | s_{l(i,3)}, s_{r(j,3)}, \tilde{a}_{il(i,3),3}, \tilde{a}_{r(j,3)j,3}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3}, \cancel{\tilde{a}_{ij,2}, \tilde{x}_{ij2}}, \tilde{s}_{ij}\right) \\
& \times P\left(s_{l(i,3)}, s_{r(j,3)}, \tilde{a}_{il(i,3)3}, \tilde{a}_{r(j,3)j,3}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)
\end{aligned}
$$

Decompose $P\left(s_{l(i,3)}, s_{r(j,3)}, \tilde{a}_{il(i,3)3}, \tilde{a}_{r(j,3)j,3}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)$

$$
\begin{aligned}
& P\left(s_{l(i,3)}, s_{r(j,3)}, \tilde{a}_{il(i,3)3}, \tilde{a}_{r(j,3)j,3}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right) \\
&= \frac{P\left(s_{l(i,3)}, s_{r(j,3)}, \tilde{a}_{il(i,3)3}, \tilde{a}_{r(j,3)j,3}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)}{P\left(\tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)} \frac{P\left(s_{l(i,3)}, s_{r(j,3)}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)}{P\left(s_{l(i,3)}, s_{r(j,3)}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)} \\
&= P\left(\tilde{a}_{il(i,3)3}, \tilde{a}_{r(j,3)j,3} | s_{l(i,3)}, s_{r(j,3)}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3}, \cancel{\tilde{a}_{ij,2}, \tilde{x}_{ij2}}, \tilde{s}_{ij}\right) \frac{P\left(s_{l(i,3)}, s_{r(j,3)}, x_{l(i,3)3}, x_{r(j,3)3}, x_{i3}, x_{j3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)}{P\left(\tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)} \frac{P\left(x_{i3}, x_{j3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)}{P\left(x_{i3}, x_{j3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)} \\
&= p\left(\tilde{a}_{il(i,3)3} | s_{l(i,3)}, x_{l(i,3)3}, x_{i3}, x_{j3}, s_i\right) p\left(\tilde{a}_{r(j,3)j,3} | s_{r(j,3)}, x_{l(i,3)3}, x_{r(j,3)3}, x_{j3}, s_j\right) P\left(s_{l(i,3)}, s_{r(j,3)}, x_{l(i,3)3}, x_{r(j,3)3} | x_{i3}, x_{j3}, \cancel{\tilde{a}_{ij,2}, \tilde{x}_{ij2}}, \tilde{s}_{ij}\right) P\left(x_{i3}, x_{j3} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \cancel{\tilde{s}_{ij}}\right) \\
&= p\left(\tilde{a}_{il(i,3)3} | s_{l(i,3)}, x_{l(i,3)3}, x_{i3}, x_{j3}, s_i\right) p\left(\tilde{a}_{r(j,3)j,3} | s_{r(j,3)}, x_{l(i,3)3}, x_{r(j,3)3}, x_{j3}, s_j\right) \psi^m\left(s_{l(i,3)}, x_{l(i,3)3} | x_{i3}, s_i\right) \psi^f\left(s_{r(j,3)}, x_{r(j,3)3} | x_{j3}, s_j\right) f\left(x_{i3}, x_{j3} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}\right) \\
&= P\left(x_{l(i,3),3}, \tilde{a}_{il(i,3),3} | x_{i,3}\right) P\left(x_{r(j,3),3}, \tilde{a}_{r(j,3)j,3} | x_{j,3}\right) f\left(x_{i3}, x_{j3} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}\right)
\end{aligned}
$$

Now instead suppose that i and j divorce at $t = 3$ and i immediately remarries but j remains single.

$$
L(i,j)_{t_0} = \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \left(\begin{array}{c} P\left(\tilde{a}_{ij,2}, x_{l(i,2)2} = x_{j2}, x_{r(j,2)2} = x_{i2}, s_{l(i,2)} = s_i, s_{r(j,2)} = s_j, \tilde{x}_{ij2} | \tilde{a}_{ij1}, \tilde{x}_{ij1}, \tilde{s}_{ij}\right) \times \\ \sum_{s_{l(i,3)}} \mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=3}^{T}, s_{l(i,3)} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right) \end{array} \right)
$$

$$
\begin{aligned}
& \mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=3}^{T}, s_{l(i,3)} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right) \\
=& \frac{\mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=4}^{T}, \tilde{a}_{il(i,3),3}, s_{l(i,3)}, x_{l(i,3),3}, x_{r(j,3),3} = \emptyset, x_{i,3}, x_{j,3}, a_{j,3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)}{P\left(\tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)} \times \\
& \frac{P\left(\tilde{a}_{il(i,3),3}, s_{l(i,3)}, x_{l(i,3),3}, x_{r(j,3),3} = \emptyset, x_{i,3}, x_{j,3}, a_{j,3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)}{P\left(\tilde{a}_{il(i,3),3}, s_{l(i,3)}, x_{l(i,3),3}, x_{r(j,3),3} = \emptyset, x_{i,3}, x_{j,3}, a_{j,3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)} \\
=& \mathbb{P}\left(\left\{\tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, x_{l(i,t-1),t}, x_{r(j,t-1),t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t}\right\}_{t=4}^{T} | \tilde{a}_{il(i,3),3}, s_{l(i,3)}, x_{l(i,3),3}, x_{r(j,3),3} = \emptyset, x_{i,3}, x_{j,3}, a_{j,3}, \cancel{\tilde{a}_{ij,2}, \tilde{x}_{ij2}}, \tilde{s}_{ij}\right) \times \\
& P\left(\tilde{a}_{il(i,3),3}, s_{l(i,3)}, x_{l(i,3),3}, x_{r(j,3),3} = \emptyset, x_{i,3}, x_{j,3}, a_{j,3} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)
\end{aligned}
$$

Decompose $P\left(\tilde{a}_{il(i,3),3}, a_{j,3}, s_{l(i,3)}, x_{l(i,3),3}, x_{i,3}, x_{j,3} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)$

$$
\begin{aligned}
& P\left(\tilde{a}_{il(i,3),3}, a_{j,3}, s_{l(i,3)}, x_{l(i,3),3}, x_{r(j,3),3} = \emptyset, x_{i,3}, x_{j,3} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right) \\
&= \frac{P\left(\tilde{a}_{il(i,3),3}, a_{j,3}, s_{l(i,3)}, x_{l(i,3),3}, x_{r(j,3),3} = \emptyset, x_{i,3}, x_{j,3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)}{P\left(\tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)} \frac{P\left(s_{l(i,3)}, x_{l(i,3),3}, x_{r(j,3),3} = \emptyset, x_{i,3}, x_{j,3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)}{P\left(s_{l(i,3)}, x_{l(i,3),3}, x_{r(j,3),3} = \emptyset, x_{i,3}, x_{j,3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)} \frac{P\left(x_{i,3}, x_{j,3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)}{P\left(x_{i,3}, x_{j,3}, \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \tilde{s}_{ij}\right)} \\
&= P\left(\tilde{a}_{il(i,3),3}, a_{j,3} | s_{l(i,3)}, x_{l(i,3),3}, x_{r(j,3),3} = \emptyset, x_{i,3}, x_{j,3}, \cancel{\tilde{a}_{ij,2}, \tilde{x}_{ij2}}, \tilde{s}_{ij}\right) P\left(s_{l(i,3)}, x_{l(i,3),3}, x_{r(j,3),3} = \emptyset | x_{i,3}, x_{j,3}, \cancel{\tilde{a}_{ij,2}, \tilde{x}_{ij2}}, \tilde{s}_{ij}\right) \mathrm{P}\left(x_{i,3}, x_{j,3} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}, \cancel{\tilde{s}_{ij}}\right) \\
&= p\left(\tilde{a}_{il(i,3),3} | s_{l(i,3)}, s_i, x_{l(i,3),3}, x_{i,3}\right) p^f\left(a_{j,3} | x_{j,3}, s_j\right) \psi^m\left(s_{l(i,3)}, x_{l(i,3),3} | x_{i,3}, s_i\right) \psi^f\left(\emptyset | x_{j,3}, s_j\right) f\left(\tilde{x}_{ij,3} | \tilde{a}_{ij,2}, \tilde{x}_{ij2}\right)
\end{aligned}
$$

Ultimately, the likelihood can be written as

$$
\begin{aligned}
L(i,j)_{t_0} = &\sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \prod_{t=2}^{T} f(\tilde{x}_{il(i,t-1),t}, \tilde{x}_{r(j,t-1)j,t} | \tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, \tilde{a}_{il(i,t-1)t-1}, \tilde{a}_{r(j,t-1)j,t-1}) \times \\
& \sum_{s_{l^*(i,2)}} \sum_{s_{r^*(j,2)}} \sum_{s_{l^*(i,3)}} \sum_{s_{r^*(j,3)}} \cdots \sum_{s_{l^*(i,|\mathscr{L}(i)|)}} \sum_{s_{r^*(j,|\mathscr{R}(j)|)}} \prod_{l^* \in \mathscr{L}(i) \setminus j} \prod_{t \in \mathscr{T}(i,l^*)} P\left(x_{l^*,t}, \tilde{a}_{il^*,t} | \tilde{x}_{il(i,t-1)t}\right) \prod_{r^* \in \mathscr{R}(j) \setminus i} \prod_{t \in \mathscr{T}(j,r)} P\left(x_{r^*,t}, \tilde{a}_{r^*j,t} | \tilde{x}_{r(j,t-1)j,t}\right) \times \\
& \prod_{t \in \mathscr{T}(i,j) \setminus \{t=1\}} P\left(x_{r(j,t),t} = x_{i,t}, x_{l(i,t),t} = x_{j,t}, \tilde{a}_{ij,t} | \tilde{x}_{r(j,t-1)j,t}, \tilde{x}_{il(i,t-1),t}\right) \times \\
& \prod_{t \in \mathscr{T}(i,\emptyset)} P\left(x_{l,t} = \emptyset, a_{i,t} | \tilde{x}_{il(i,t-1)t}\right) \prod_{t \in \mathscr{T}(j,\emptyset)} P\left(x_{r,t} = \emptyset, a_{j,t} | \tilde{x}_{r(j,t-1)j,t}\right)
\end{aligned}
$$

If i and j never remarry each other after divorcing, then we can factor their individual histories like so:

$$
\begin{aligned}
L(i,j)_{t_0} = &\sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \prod_{t=2}^{T} f(\tilde{x}_{il(i,t-1),t}, \tilde{x}_{r(j,t-1)j,t} | \tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, \tilde{a}_{il(i,t-1)t-1}, \tilde{a}_{r(j,t-1)j,t-1}) \times \\
& \prod_{t \in \mathscr{T}(i,j) \setminus \{t=1\}} P\left(x_{r(j,t),t} = x_{i,t}, x_{l(i,t),t} = x_{j,t}, \tilde{a}_{ij,t} | \tilde{x}_{ij,t}\right) \times \\
& \sum_{s_{l^*(i,2)}} \sum_{s_{l^*(i,3)}} \cdots \sum_{s_{l^*(i,|\mathscr{L}(i)|)}} \prod_{l^* \in \mathscr{L}(i) \setminus j} \prod_{t \in \mathscr{T}(i,l)} P\left(x_{l,t}, \tilde{a}_{il,t} | \tilde{x}_{il(i,t-1)t}\right) \prod_{t \in \mathscr{T}(i,\emptyset)} P\left(x_{l,t} = \emptyset, a_{i,t} | \tilde{x}_{il(i,t-1)t}\right) \times \\
& \sum_{s_{r^*(j,2)}} \sum_{s_{r^*(j,3)}} \cdots \sum_{s_{r^*(j,|\mathscr{R}(j)|)}} \prod_{r^* \in \mathscr{R}(j) \setminus i} \prod_{t \in \mathscr{T}(j,r)} P\left(x_{r,t}, \tilde{a}_{rj,t} | \tilde{x}_{r(j,t-1)j,t}\right) \prod_{t \in \mathscr{T}(j,\emptyset)} P\left(x_{r,t} = \emptyset, a_{j,t} | \tilde{x}_{r(j,t-1)j,t}\right)
\end{aligned}
$$

B.3 Simplified likelihood for unmatched individuals

$$L(i,\emptyset)_{t_0} = \sum_{s_i} \pi(s_i,\emptyset) p^m(a_{i1}, x_{i1}|s_i) \prod_{t=2}^{T} f\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right) \sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}} \prod_{t\in\mathscr{T}(i,\emptyset)}^{T} P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\tilde{\phi}_{il(i,t-1)t}\right) \prod_{l^*\in\mathscr{L}(i)\setminus j} \prod_{t\in\mathscr{T}(i,l^*)}^{T} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\tilde{\phi}_{il(i,t-1)t}\right)$$

$$= \sum_{s_i} \pi(s_i,\emptyset) p^m(a_{i1}, x_{i1}|s_i) \prod_{t=2}^{T} f\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right) \sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}} \prod_{t\in\mathscr{T}(i,\emptyset)}^{T} P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\phi_{it}\right) \prod_{l^*\in\mathscr{L}(i)\setminus j} \prod_{t\in\mathscr{T}(i,l^*)}^{T} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\phi_{it}\right)$$

$$= \sum_{s_i} \pi(s_i,\emptyset) p^m(a_{i1}, x_{i1}|s_i) \prod_{t=2}^{T} f\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right) \prod_{t\in\mathscr{T}(i,\emptyset)}^{T} P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\phi_{it}\right) \prod_{l^*\in\mathscr{L}(i)\setminus j} \sum_{s_{l^*}} \prod_{t\in\mathscr{T}(i,l^*)}^{T} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\phi_{it}\right)$$

$$= \sum_{s_i} \pi(s_i,\emptyset) p^m(a_{i1}, x_{i1}|s_i) \prod_{t=2}^{T} f\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right) \prod_{t\in\mathscr{T}(i,\emptyset)}^{T} P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\phi_{it}\right) \prod_{l^*\in\mathscr{L}(i)\setminus j} \sum_{s_{l^*}} \prod_{t\in\mathscr{T}(i,l^*)}^{T} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\, \{\phi_{it}\}_{t\in\mathscr{T}(i,l^*)}\right)$$

$$= \sum\nolimits_{s_i} \pi(s_i,\emptyset) p^m(a_{i1}, \phi_{i1}|s_i) \prod\nolimits_{t=2}^{T} f\left(\tilde{\phi}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right) \prod\nolimits_{t\in\mathscr{T}(i,\emptyset)}^{T} P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\phi_{it}\right) \prod\nolimits_{l^*\in\mathscr{L}(i)\setminus j} \sum\nolimits_{s_{l^*}} P\left(\{\phi_{l^*,t}, \tilde{a}_{il^*,t}\}_{t\in\mathscr{T}(i,l^*)}\,|\, \{\phi_{it}\}_{t\in\mathscr{T}(i,l^*)}\right)$$

Because events are conditionally independent across time

$$= \sum_{s_i} \pi(s_i,\emptyset) p^m(a_{i1}, \phi_{i1}|s_i) \prod_{t=2}^{T} f\left(\tilde{\phi}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right) \prod_{t\in\mathscr{T}(i,\emptyset)}^{T} P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\phi_{it}\right) \prod_{l^*\in\mathscr{L}(i)\setminus j} P\left(\{x_{l^*,t}, \tilde{a}_{il^*,t}\}_{t\in\mathscr{T}(i,l^*)}\,|\, \{\phi_{it}\}_{t\in\mathscr{T}(i,l^*)}\right)$$

$$= \sum\nolimits_{s_i} \pi(s_i,\emptyset) p^m(a_{i1}, \phi_{i1}|s_i) \prod\nolimits_{t=2}^{T} f\left(\tilde{\phi}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right) \prod\nolimits_{t\in\mathscr{T}(i,\emptyset)}^{T} P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\phi_{it}\right) \prod\nolimits_{l^*\in\mathscr{L}(i)\setminus j} \prod\nolimits_{t\in\mathscr{T}(i,l^*)}^{T} P\left(x_{l^*,t}, \tilde{a}_{il^*,t}|\phi_{it}\right)$$

Because there is no correlation across choices due to independence of first-stage wrt previous spouse characteristics

$$= \sum_{s_i} \pi(s_i,\emptyset) p^m(a_{i1}, x_{i1}|s_i) \prod_{t=2}^{T} f\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1}, \tilde{a}_{il(i,t-1)t-1}\right) P\left(\tilde{a}_{il(i,t),t}, x_{l(i,t),t}|\phi_{it}\right)$$

B.4 Simplified likelihood for matched pairs

$$
\begin{aligned}
\mathring{L}(i,j)_{t_0} &= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)\setminus j}} \sum_{\{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i}} \left(\begin{array}{c} \prod_{l^* \in \mathscr{L}(i)\setminus j} \prod_{t \in \mathscr{T}(i,l^*)} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t} | \tilde{\phi}_{il(i,t-1)t}\right) \prod_{r^* \in \mathscr{R}(j)\setminus i} \prod_{t \in \mathscr{T}(j,r^*)} P\left(\phi_{r^*,t}, \tilde{a}_{r^*j,t} | \tilde{\phi}_{r(j,t-1)j,t}\right) \times \\ \prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \tilde{\phi}_{r(j,t-1)j,t}, \tilde{\phi}_{il(i,t-1),t}\right) \times \\ \prod_{t \in \mathscr{T}(i,\emptyset)} P\left(\phi_{l,t} = \emptyset, a_{i,t} | \tilde{\phi}_{il(i,t-1)t}\right) \prod_{t \in \mathscr{T}(j,\emptyset)} P\left(\phi_{r,t} = \emptyset, a_{j,t} | \tilde{\phi}_{r(j,t-1)j,t}\right) \end{array} \right) \\
&= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)\setminus j}} \sum_{\{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i}} \left(\begin{array}{c} \prod_{l^* \in \mathscr{L}(i)\setminus j} \prod_{t \in \mathscr{T}(i,l)} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t} | \phi_{it}\right) \prod_{r^* \in \mathscr{R}(j)\setminus i} \prod_{t \in \mathscr{T}(j,r)} P\left(\phi_{r^*,t}, \tilde{a}_{r^*j,t} | \phi_{j,t}\right) \\ \prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \times \\ \prod_{t \in \mathscr{T}(i,\emptyset)} P\left(\phi_{l,t} = \emptyset, a_{i,t} | \phi_{it}\right) \prod_{t \in \mathscr{T}(j,\emptyset)} P\left(\phi_{r,t} = \emptyset, a_{j,t} | \phi_{j,t}\right) \end{array} \right) \\
&= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \left(\begin{array}{c} \prod_{l^* \in \mathscr{L}(i)\setminus j} \sum_{s_{l^*}} \prod_{t \in \mathscr{T}(i,l^*)} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t} | \phi_{it}\right) \prod_{r^* \in \mathscr{R}(j)\setminus i} \sum_{s_{r^*}} \prod_{t \in \mathscr{T}(j,r^*)} P\left(\phi_{r^*,t}, \tilde{a}_{r^*j,t} | \phi_{j,t}\right) \times \\ \prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \times \\ \prod_{t \in \mathscr{T}(i,\emptyset)} P\left(\phi_{l,t} = \emptyset, a_{i,t} | \phi_{it}\right) \prod_{t \in \mathscr{T}(j,\emptyset)} P\left(\phi_{r,t} = \emptyset, a_{j,t} | \phi_{j,t}\right) \end{array} \right) \\
&= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \left(\begin{array}{c} \prod_{l^* \in \mathscr{L}(i)\setminus j} \sum_{s_{l^*}} \prod_{t \in \mathscr{T}(i,l^*)} P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t} | \left\{\phi_{it}\right\}_{t \in \mathscr{T}(i,l^*)}\right) \prod_{r^* \in \mathscr{R}(j)\setminus i} \sum_{s_{r^*}} \prod_{t \in \mathscr{T}(j,r^*)} P\left(\phi_{r^*,t}, \tilde{a}_{r^*j,t} | \left\{\phi_{j,t}\right\}_{t \in \mathscr{T}(j,r^*)}\right) \times \\ \prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \times \\ \prod_{t \in \mathscr{T}(i,\emptyset)} P\left(\phi_{l,t} = \emptyset, a_{i,t} | \phi_{it}\right) \prod_{t \in \mathscr{T}(j,\emptyset)} P\left(\phi_{r,t} = \emptyset, a_{j,t} | \phi_{j,t}\right) \end{array} \right) \\
&= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \left(\begin{array}{c} \prod_{l^* \in \mathscr{L}(i)\setminus j} \sum_{s_{l^*}} P\left(\left\{\phi_{l^*,t}, \tilde{a}_{il^*,t}\right\}_{t \in \mathscr{T}(i,l^*)} | \left\{\phi_{it}\right\}_{t \in \mathscr{T}(i,l^*)}\right) \prod_{r^* \in \mathscr{R}(j)\setminus i} \sum_{s_{r^*}} P\left(\left\{\phi_{r^*,t}, \tilde{a}_{r^*j,t}\right\}_{t \in \mathscr{T}(j,r^*)} | \left\{\phi_{j,t}\right\}_{t \in \mathscr{T}(j,r^*)}\right) \\ \prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \\ \prod_{t \in \mathscr{T}(i,\emptyset)} P\left(\phi_{l,t} = \emptyset, a_{i,t} | \phi_{it}\right) \prod_{t \in \mathscr{T}(j,\emptyset)} P\left(\phi_{r,t} = \emptyset, a_{j,t} | \phi_{j,t}\right) \end{array} \right) \\
&= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \left(\begin{array}{c} \prod_{l^* \in \mathscr{L}(i)\setminus j} P\left(\left\{x_{l^*,t}, \tilde{a}_{il^*,t}\right\}_{t \in \mathscr{T}(i,l^*)} | \left\{\phi_{it}\right\}_{t \in \mathscr{T}(i,l^*)}\right) \prod_{r^* \in \mathscr{R}(j)\setminus i} P\left(\left\{x_{r^*,t}, \tilde{a}_{r^*j,t}\right\}_{t \in \mathscr{T}(j,r^*)} | \left\{\phi_{j,t}\right\}_{t \in \mathscr{T}(j,r^*)}\right) \times \\ \prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \times \\ \prod_{t \in \mathscr{T}(i,\emptyset)} P\left(\phi_{l,t} = \emptyset, a_{i,t} | \phi_{it}\right) \prod_{t \in \mathscr{T}(j,\emptyset)} P\left(\phi_{r,t} = \emptyset, a_{j,t} | \phi_{j,t}\right) \end{array} \right) \\
&= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \left(\begin{array}{c} \prod_{l^* \in \mathscr{L}(i)\setminus j} \prod_{t \in \mathscr{T}(i,l^*)} P\left(x_{l^*,t}, \tilde{a}_{il^*,t} | \phi_{it}\right) \prod_{r^* \in \mathscr{R}(j)\setminus i} \prod_{t \in \mathscr{T}(j,r^*)} P\left(x_{r^*,t}, \tilde{a}_{r^*j,t} | \phi_{j,t}\right) \times \\ \prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \times \\ \prod_{t \in \mathscr{T}(i,\emptyset)} P\left(\phi_{l,t} = \emptyset, a_{i,t} | \phi_{it}\right) \prod_{t \in \mathscr{T}(j,\emptyset)} P\left(\phi_{r,t} = \emptyset, a_{j,t} | \phi_{j,t}\right) \end{array} \right) \\
&= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \left(\begin{array}{c} \prod_{l^* \in \mathscr{L}(i)\setminus j} \prod_{t \in \mathscr{T}(i,l^*)} P\left(x_{l^*,t}, \tilde{a}_{il^*,t} | \phi_{it}\right) \prod_{r^* \in \mathscr{R}(j)\setminus i} \prod_{t \in \mathscr{T}(j,r^*)} P\left(x_{r^*,t}, \tilde{a}_{r^*j,t} | \phi_{j,t}\right) \times \\ \prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \times \\ \prod_{t \in \mathscr{T}(i,\emptyset)} P\left(\phi_{l,t} = \emptyset, a_{i,t} | \phi_{it}\right) \prod_{t \in \mathscr{T}(j,\emptyset)} P\left(\phi_{r,t} = \emptyset, a_{j,t} | \phi_{j,t}\right) \end{array} \right) \\
&= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \prod_{t \notin \mathscr{T}(i,j)\setminus\{t=1\}} P\left(x_{l(i,t),t}, \tilde{a}_{il(i,t),t} | \phi_{it}\right) P\left(x_{r(i,t),t}, \tilde{a}_{r(i,t)j,t} | \phi_{j,t}\right) \prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \\
&= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \prod_{t \notin \mathscr{T}(i,j)\setminus\{t=1\}} P\left(x_{l(i,t),t}, \tilde{a}_{il(i,t),t}, x_{r(i,t),t}, \tilde{a}_{r(i,t)j,t} | \phi_{it}, \phi_{j,t}\right) \prod_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \\
&= \sum_{\tilde{s}_{ij}} \pi\left(\tilde{s}_{ij}\right) p\left(\tilde{a}_{ij1}, \tilde{x}_{ij1} | \tilde{s}_{ij}\right) \prod_{t=2}^{T} P\left(x_{l(i,t),t}, \tilde{a}_{il(i,t),t}, x_{r(i,t),t}, \tilde{a}_{r(i,t)j,t} | \phi_{it}, \phi_{j,t}\right)
\end{aligned}
$$

C EM log-likelihood

C.1 Derivation of (21)

$$\max_{p,\psi} E_{\left\{s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right\}_{(i,\emptyset)_{t_0}}}\left[\ln P\left(\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{(i,t)=(1,1)}^{(N,T)},\left\{s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right\}_{i=1}^{N}\right)\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{(i,t)=(1,1)}^{(N,T)}\right]$$

$$=\max_{p,\psi}\sum_{\left\{s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right\}_{(i,\emptyset)_{t_0}}} P\left(\left\{s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right\}_{(i,\emptyset)_{t_0}}\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{(i,t)=(1,1)}^{(N,T)}\right)\ln P\left(\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{(i,t)=(1,1)}^{(N,T)},\left\{s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right\}_{(i,\emptyset)_{t_0}}\right)$$

$$=\max_{p,\psi}\sum_{\left\{s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right\}_{(i,\emptyset)_{t_0}}} P\left(\left\{s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right\}_{(i,\emptyset)_{t_0}}\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{(i,t)=(1,1)}^{(N,T)}\right)\sum_i\ln P\left(\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t=1}^{T},s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right)$$

$$=\max_{p,\psi}\sum_i\sum_{\left\{s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right\}_{(i,\emptyset)_{t_0}}} P\left(\left\{s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right\}_{(i,\emptyset)_{t_0}}\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{(i,t)=(1,1)}^{(N,T)}\right)\ln P\left(\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t=1}^{T},s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right)$$

$$=\max_{p,\psi}\sum_i\underbrace{\sum_{s_i}\sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}}\ln P\left(\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t=1}^{T},s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right)}_{\text{For any } i \text{ you can factor } \ln P \text{ out of the summations that are not relevant to } i}\underbrace{\sum_{i'\neq i}\sum_{s_{i'}}\sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i')}}\ln P\left(\left\{s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right\}_{(i,\emptyset)_{t_0}}\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{(i,t)=(1,1)}^{(N,T)}\right)}_{\text{The remaining sums only act on the posterior probabilities that are not relevant to } i}$$

$$=\max_{p,\psi}\sum_i\sum_{s_i}\sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}}\ln P\left(\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t=1}^{T},s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right)P\left(s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{(i,t)=(1,1)}^{(N,T)}\right)$$

$$=\max_{p,\psi}\sum_i\sum_{s_i}\sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}}\ln P\left(\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t=1}^{T},s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right)P\left(s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t}\right)$$

$$=\max_{p,\psi}\sum_i\sum_{s_i}\sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}}\ln P\left(\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t=1}^{T},s_i,\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\right)P\left(\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t},s_i\right)P\left(s_i\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t}\right)$$

$$=\max_{p,\psi}\sum_i\sum_{s_i}\sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}}\ln\left[P\left(\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t=1}^{T},\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\mid s_i\right)\pi\left(s_i,\emptyset\right)\right]P\left(\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t},s_i\right)q_{s_i}$$

$$=\max_{p,\psi}\sum_i\sum_{s_i}\sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}}\left[\begin{array}{c}\ln\left(\pi\left(s_i,\emptyset\right)p^m(a_{i1},x_{i1}|s_i)F\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1},\tilde{a}_{il(i,t-1)t-1}\right)\prod_{t\in\mathscr{T}(i,\emptyset)}^{T}P\left(a_{i,t},\phi_{l(i,t)}=\emptyset|\phi_{it}\right)\prod_{l^*\in\mathscr{L}(i)}\prod_{t\in\mathscr{T}(i,l^*)}^{T}P\left(\phi_{l^*,t},\tilde{a}_{il^*,t}|\phi_{it}\right)\right)\times\\ P\left(\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t},s_i\right)q_{s_i}\end{array}\right]$$

$$=\max_{p,\psi}\sum_i\sum_{s_i}\sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}}\left[\begin{array}{c}\left(\begin{array}{c}\ln\pi\left(s_i,\emptyset\right)+\ln p^m(a_{i1},x_{i1}|s_i)+\ln F\left(\tilde{x}_{il(i,t-1),t}|\tilde{x}_{il(i,t-1),t-1},\tilde{a}_{il(i,t-1)t-1}\right)\\+\sum_{t\in\mathscr{T}(i,\emptyset)}\ln P\left(a_{i,t},\phi_{l(i,t)}=\emptyset|\phi_{it}\right)+\sum_{l^*\in\mathscr{L}(i)}\sum_{t\in\mathscr{T}(i,l^*)}\ln P\left(\phi_{l^*,t},\tilde{a}_{il^*,t}|\phi_{it}\right)\end{array}\right)\times\\ P\left(\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}\mid\left\{\tilde{a}_{il(i,t),t},\tilde{x}_{il(i,t),t}\right\}_{t},s_i\right)q_{s_i}\end{array}\right]$$

$$= \max_{p,\psi} \sum_i \sum_{s_i} q_{s_i} \left[\ln p^m(a_{i1}, x_{i1}|s_i) + \sum_{t\in\mathscr{T}(i,\emptyset)} \ln P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\phi_{it}\right) + \sum_{\{s_{l^*}\}_{l^*\in\mathscr{L}(i)}} \sum_{t\in\mathscr{T}(i,l^*)} \ln P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\phi_{it}\right) P\left(\{s_{l^*}\}_{l^*\in\mathscr{L}(i)} \mid \left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}\right\}_t, s_i\right)\right]$$

$$= \max_{p,\psi} \sum_i \sum_{s_i} q_{s_i} \left[\ln p^m(a_{i1}, x_{i1}|s_i) + \sum_{t\in\mathscr{T}(i,\emptyset)} \ln P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\phi_{it}\right) + \sum_{l^*\in\mathscr{L}(i)} \sum_{s_{l^*}} P\left(s_{l^*} \mid \left\{\tilde{a}_{il(i,t),t}, \tilde{x}_{il(i,t),t}\right\}_{(i,t)=(1,1)}^{(N,T)}, s_i\right) \sum_{t\in\mathscr{T}(i,l^*)} \ln P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\phi_{it}\right)\right]$$

$$= \max_{p,\psi} \sum_i \sum_{s_i} q_{s_i} \left[\ln p^m(a_{i1}, x_{i1}|s_i) + \sum_{t\in\mathscr{T}(i,\emptyset)} \ln P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\phi_{it}\right) + \sum_{l^*\in\mathscr{L}(i)} \sum_{s_{l^*}} q_{s_{l^*}|s_i} \sum_{t\in\mathscr{T}(i,l^*)} \ln P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\phi_{it}\right)\right]$$

$$= \max_{p,\psi} \sum_i \sum_{s_i} q_{s_i} \left[\ln p(x_{i1}|s_i) + \ln p^m(a_{i1}|x_{i1}, s_i) + \sum_{t\in\mathscr{T}(i,\emptyset)} \ln P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\phi_{it}\right) + \sum_{l^*\in\mathscr{L}(i)} \sum_{s_{l^*}} q_{s_{l^*}|s_i} \sum_{t\in\mathscr{T}(i,l^*)} \ln P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\phi_{it}\right)\right]$$

$$= \max_{p,\psi} \sum_i \sum_{s_i} q_{s_i} \left[\ln p^m(a_{i1}, x_{i1}|s_i) + \sum_{t\in\mathscr{T}(i,\emptyset)} \ln P\left(a_{i,t}, \phi_{l(i,t)} = \emptyset|\phi_{it}\right) + \sum_{l^*\in\mathscr{L}(i)} \sum_{s_{l^*}} q_{s_{l^*}|s_i} \sum_{t\in\mathscr{T}(i,l^*)} \ln P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\phi_{it}\right)\right]$$

$$= \max_{p,\psi} \sum_i \sum_{s} q_{is} \left(\ln p(x_{i1}|s_i) + \ln p(a_{i1}|x_{i1}, s_i) + \sum_{t>1} \sum_{s_{l(i,t)}} q_{s_{l(i,t)}|s_i} \ln P\left(\phi_{l^*,t}, \tilde{a}_{il^*,t}|\phi_{it}\right)\right)$$

C.2 Derivation of the solution to (21)

Take the first order conditions with respect to $p(\tilde{a}|x, x', s, s')$

$$\frac{\sum_i \sum_s q_{is} \sum_{t>1} q_{s_{l(i,t)}|s_i} \mathbb{1}\left(\tilde{a}_{il(i,t),t} = \tilde{a}, x_{it} = x, x_{l(i,t)} = x', s_i = s, s_{l(i,t)} = s'\right)}{p(\tilde{a}|x_i, x_l, s, s')} = 0$$

so for any $\tilde{a}' \neq \tilde{a}$ we have

$$\frac{\sum_i \sum_s q_{is} \sum_t q_{s_{l(i,t)}|s_i} \mathbb{1}\left(\tilde{a}_{il(i,t),t} = \tilde{a}, x_{it} = x, x_{l(i,t)} = x', s_i = s, s_{l(i,t)} = s'\right)}{p(\tilde{a}|x, x', s, s')}$$
$$= \frac{\sum_i \sum_s q_{is} \sum_t q_{s_{l(i,t)}|s_i} \mathbb{1}\left(\tilde{a}_{il(i,t),t} = \tilde{a}', x_{it} = x, x_{l(i,t)} = x', s_i = s, s_{l(i,t)} = s'\right)}{p(\tilde{a}'|x, x', s, s')}$$

Multiply both sides by $p(\tilde{a}|x, x', s, s')p(\tilde{a}'|x, x', s, s')$

$$p(\tilde{a}'|x_{it}, x_{lt}, s, s') \sum_i \sum_s q_{is} \sum_t q_{s_{l(i,t)}|s_i} \mathbb{1}\left(\tilde{a}_{il(i,t),t} = \tilde{a}, x_{it} = x, x_{l(i,t)} = x', s_i = s, s_{l(i,t)} = s'\right)$$
$$= p(\tilde{a}|x_{it}, x_{lt}, s, s') \sum_i \sum_s q_{is} \sum_t q_{s_{l(i,t)}|s_i} \mathbb{1}\left(\tilde{a}_{il(i,t),t} = \tilde{a}', x_{it} = x, x_{l(i,t)} = x', s_i = s, s_{l(i,t)} = s'\right)$$

Then, sum across all $\tilde{a}'$, $p(\tilde{a}'|x_{it}, x_{lt}, s, s')$ on the LHS sums up to 1 and $\sum_{\tilde{a}'} \mathbb{1}\left(\tilde{a}_{il(i,t),t} = \tilde{a}', x_{it} = x, x_{l(i,t)} = x', s_i = s, s_{l(i,t)} = s'\right)$ on the RHS sums up to $\mathbb{1}\left(x_{it} = x, x_{l(i,t)} = x', s_i = s, s_{l(i,t)} = s'\right)$, hence

$$p(\tilde{a}|x_{it}, x_{lt}, s, s') = \frac{\sum_i \sum_s q_{is} \sum_t q_{s_{l(i,t)}|s_i} \mathbb{1}\left(\tilde{a}_{il(i,t),t} = \tilde{a}, x_{it} = x_i, x_{l(i,t)} = x', s_i = s, s_{l(i,t)} = s'\right)}{\sum_i \sum_s q_{is} \sum_t q_{s_{l(i,t)}|s_i} \mathbb{1}\left(x_{it} = x, x_{l(i,t)} = x', s_i = s, s_{l(i,t)} = s'\right)}$$

Similarly, the foc for $p^m(a|x, s)$

$$\sum_i \sum_s q_{is} \sum_t \frac{\mathbb{1}\left(a_{it} = a, s_i = s, x_{it} = x\right)}{p^m(a|x, s)} = 0$$

Then, for any a' we have

$$\sum_i \sum_s q_{is} \sum_t \frac{\mathbb{1}\left(a_{it} = a, s_i = s, x_{it} = x_i\right)}{p^m(a|x, s)} \sum_i \sum_s q_{is} \sum_t \frac{\mathbb{1}\left(a_{it} = a', s_i = s, x_{it} = x_i\right)}{p^m(a'|x, s)}$$

Multiply both sides by $p^m(a|x, s)p^m(a'|x, s)$ and sum across a' to obtain

$$p^m(a|x, s) = \frac{\sum_i \sum_s q_{is} \sum_t \mathbb{1}\left(a_{it} = a, s_i = s, x_{it} = x_i\right)}{\sum_i \sum_s q_{is} \sum_t \mathbb{1}\left(s_i = s, x_{it} = x_i\right)}$$

The same estimator can be constructed for $p^f(a|x, s)$
Similarly, the FOC for $\psi^m(\phi', s_l|\phi_i, s_i)$

$$\frac{\sum_i q_{is} \sum_t q_{s_{l(i,t)}|s_i} \mathbb{1}\left(\phi_{it} = \phi, \phi_{l(i,t)} = \phi'\right)}{\psi^m(\phi', s_l|\phi', s_i)} = 0$$

Again, equate to any ϕ''

$$\frac{\sum_i q_{is} \sum_t q_{s_{l(i,t)}|s_i} \mathbb{1}\left(\phi_{it} = \phi, \phi_{l(i,t)} = \phi'\right)}{\psi^m(\phi', s_l|\phi, s_i)} = \frac{\sum_i q_{is} \sum_t q_{s_{l(i,t)}|s_i} \mathbb{1}\left(\phi_{it} = \phi, \phi_{l(i,t)} = \phi''\right)}{\psi^m(\phi'', s_l|\phi, s_i)}$$

Multiply both sides by $\psi(\phi', s_l|\phi, s_i)\psi(\phi'', s_l|\phi, s_i)$ and sum across ϕ'' to obtain

$$\psi(\phi', s_l|\phi_i, s_i) = \frac{\sum_i q_{is} \sum_t q_{s_{l(i,t)}|s_i} \mathbb{1}\left(\phi_{it} = \phi, \phi_{l(i,t)} = \phi''\right)}{\sum_i q_{is} \sum_t q_{s_{l(i,t)}|s_i} \mathbb{1}\left(\phi_{it} = \phi\right)}$$

The FOC for $p(x|s)$ is

$$\frac{\sum_i \sum_s q_{is} \mathbb{1}\left(x_{i1} = x, s_i = s\right)}{p(x|s)} = 0$$

Equate to any x', multiply both sides by $p(x'|s)p(x|s)$ ad sum across x'.

$$\begin{aligned}
\frac{\sum_i \sum_s q_{is}\mathbb{1}\left(x_{i1}=x, s_i=s\right)}{p(x|s)} &= \frac{\sum_i \sum_s q_{is}\mathbb{1}\left(x_{i1}=x', s_i=s\right)}{p(x'|s)} \\
p(x'|s)\sum_i \sum_s q_{is}\mathbb{1}\left(x_{i1}=x, s_i=s\right) &= \sum_i \sum_s q_{is}\mathbb{1}\left(x_{i1}=x', s_i=s\right)p(x|s) \\
\sum_i \sum_s q_{is}\mathbb{1}\left(x_{i1}=x, s_i=s\right) &= \sum_i \sum_s q_{is}\mathbb{1}\left(s_i=s\right)p(x|s) \\
\frac{\sum_i \sum_s q_{is}\mathbb{1}\left(x_{i1}=x, s_i=s\right)}{\sum_i \sum_s q_{is}\mathbb{1}\left(s_i=s\right)} &= p(x|s)
\end{aligned}$$

C.3 Derivation of (22)

$$\max_{p,\psi} E_{\left\{s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)\setminus j}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right\}_{(i,j)_{t_0}}}\left[\ln P\left(\left\{\tilde{x}_{il(i,t),t},\tilde{x}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t},\left\{s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right\}_{(i,j)_{t_0}}\right)\mid\left\{\tilde{x}_{il(i,t),t},\tilde{x}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t}\right]$$

$$=\max_{p,\psi}\sum_{\left\{s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)\setminus j}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right\}_{(i,j)_{t_0}}}\left(\begin{array}{c}P\left(\left\{s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right\}_{(i,j)_{t_0}}\mid\left\{\tilde{x}_{il(i,t),t},\tilde{x}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t}\right)\times\\ \ln P\left(\left\{\tilde{x}_{il(i,t),t},\tilde{x}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t},\left\{s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right\}_{(i,j)_{t_0}}\right)\end{array}\right)$$

$$=\max_{p,\psi}\sum_{s_i,s_j}\sum_{\{s_{l*}\}_{l*\in\mathscr{L}(i)\setminus j}}\sum_{\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}}\left(\begin{array}{c}P\left(\left\{s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right\}_{(i,j)_{t_0}}\mid\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t}\right)\times\\ \sum_{ij}\ln P\left(\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{t},s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right)\end{array}\right)$$

$$=\max_{p,\psi}\sum_{ij}\sum_{s_i,s_j}\sum_{\{s_{l*}\}_{l*\in\mathscr{L}(i)\setminus j}}\sum_{\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}}\left(\begin{array}{c}P\left(\left\{s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right\}_{(i,j)_{t_0}}\mid\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t}\right)\times\\ \ln P\left(\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{t},s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right)\end{array}\right)$$

$$=\max_{p,\psi}\sum_{ij}\left(\begin{array}{c}\underbrace{\sum_{s_i,s_j}\sum_{\{s_{l*}\}_{l*\in\mathscr{L}(i)\setminus j}}\sum_{\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}}\ln P\left(\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{t},s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right)\times}_{\text{For any } (ij) \text{ you can factor out the } \ln P \text{ out of the summations that are not relevant to } (ij)}\\ \sum_{i'j'\neq ij}\underbrace{\sum_{s_{i'},s_{j'}}\sum_{\{s_{l*}\}_{l*\in\mathscr{L}(i')\setminus j'}}\sum_{\{s_{r*}\}_{r*\in\mathscr{R}(j')\setminus i'}}P\left(\left\{s_{i'},s_{j'},\{s_{l*}\}_{l*\in\mathscr{L}(i')}\{s_{r*}\}_{r*\in\mathscr{R}(j')\setminus i'}\right\}_{(i',j')_{t_0}}\mid\left\{\tilde{\phi}_{i'l(i',t),t},\tilde{\phi}_{r(j',t)j',t},\tilde{a}_{i'l(i',t),t},\tilde{a}_{r(j',t)j',t}\right\}_{i'j',t}\right)}_{\text{The remaining sums only act on the posterior probabilities that are not relevant to } ij}\end{array}\right)$$

$$=\max_{p,\psi}\sum_{ij}\sum_{s_i,s_j}\sum_{\{s_{l*}\}_{l*\in\mathscr{L}(i)\setminus j}}\sum_{\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}}\left(\begin{array}{c}\ln P\left(\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{t},s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right)\times\\ P\left(s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\mid\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{t}\right)\end{array}\right)$$

$$=\max_{p,\psi}\sum_{ij}\sum_{s_i,s_j}\sum_{\{s_{l*}\}_{l*\in\mathscr{L}(i)\setminus j}}\sum_{\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}}\left(\begin{array}{c}\ln P\left(\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{t},s_i,s_j,\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\right)\times\\ P\left(\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\mid\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{t},s_i,s_j\right)\\ P\left(s_i,s_j\mid\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{t}\right)\end{array}\right)$$

$$=\max_{p,\psi}\sum_{ij}\sum_{s_i,s_j}\sum_{\{s_{l*}\}_{l*\in\mathscr{L}(i)\setminus j}}\sum_{\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}}\left(\begin{array}{c}\ln\left[P\left(\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{t},\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\mid\tilde{s}_{ij}\right)\pi\left(\tilde{s}_{ij}\right)\right]\times\\ P\left(\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\mid\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{t},s_i,s_j\right)\end{array}\right)q_{\tilde{s}_{ij}}$$

$$=\max_{p,\psi}\sum_{ij}\sum_{\tilde{s}}q_{ij\tilde{s}}\sum_{\{s_{l*}\}_{l*\in\mathscr{L}(i)\setminus j}}\sum_{\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}}\left(\begin{array}{c}\ln\left[\begin{array}{c}\pi\left(\tilde{s}_{ij}\right)p\left(\tilde{a}_{ij1},\tilde{\phi}_{ij1}\mid\tilde{s}_{ij}\right)F(\tilde{x}_{il(i,t-1),t},\tilde{x}_{r(j,t-1)j,t}\mid\tilde{x}_{il(i,t),t},\tilde{x}_{r(j,t)j,t},\tilde{a}_{il(i,t-1)t-1},\tilde{a}_{r(j,t-1)j,t-1})\times\\ \prod_{l*\in\mathscr{L}(i)\setminus j}\prod_{t\in\mathscr{T}(i,l*)}P\left(s_{l*},\phi_{l*,t},\tilde{a}_{il*,t}\mid\phi_{it}\right)\prod_{r*\in\mathscr{R}(j)\setminus i}\prod_{t\in\mathscr{T}(j,r*)}P\left(s_{r*},\phi_{r*,t},\tilde{a}_{r*j,t}\mid\phi_{j,t}\right)\times\\ \prod_{t\in\mathscr{T}(i,j)\setminus\{t=1\}}P\left(\phi_{r(j,t),t}=\phi_{i,t},\phi_{l(i,t),t}=\phi_{j,t},\tilde{a}_{ij,t}\mid\phi_{j,t},\phi_{i,t}\right)\end{array}\right]\\ \times P\left(\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\mid\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t},s_i,s_j\right)\end{array}\right)$$

$$=\max_{p,\psi}\sum_{ij}\sum_{\tilde{s}}q_{ij\tilde{s}}\sum_{\{s_{l*}\}_{l*\in\mathscr{L}(i)\setminus j}}\sum_{\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}}\left(\begin{array}{c}\ln\left[\begin{array}{c}\pi\left(\tilde{s}_{ij}\right)p\left(\tilde{a}_{ij1},\tilde{\phi}_{ij1}\mid\tilde{s}_{ij}\right)F(\tilde{x}_{il(i,t-1),t},\tilde{x}_{r(j,t-1)j,t}\mid\tilde{x}_{il(i,t),t},\tilde{x}_{r(j,t)j,t},\tilde{a}_{il(i,t-1)t-1},\tilde{a}_{r(j,t-1)j,t-1})\times\\ \prod_{l*\in\mathscr{L}(i)\setminus j}\prod_{t\in\mathscr{T}(i,l*)}P\left(s_{l*},\phi_{l*,t},\tilde{a}_{il*,t}\mid\phi_{it}\right)\prod_{r*\in\mathscr{R}(j)\setminus i}\prod_{t\in\mathscr{T}(j,r*)}P\left(s_{r*},\phi_{r*,t},\tilde{a}_{r*j,t}\mid\phi_{j,t}\right)\times\\ \prod_{t\in\mathscr{T}(i,j)\setminus\{t=1\}}P\left(\phi_{r(j,t),t}=\phi_{i,t},\phi_{l(i,t),t}=\phi_{j,t},\tilde{a}_{ij,t}\mid\phi_{j,t},\phi_{i,t}\right)\end{array}\right]\\ \times P\left(\{s_{l*}\}_{l*\in\mathscr{L}(i)}\{s_{r*}\}_{r*\in\mathscr{R}(j)\setminus i}\mid\left\{\tilde{\phi}_{il(i,t),t},\tilde{\phi}_{r(j,t)j,t},\tilde{a}_{il(i,t),t},\tilde{a}_{r(j,t)j,t}\right\}_{ij,t},s_i,s_j\right)\end{array}\right)$$

$$
= \max_{p,\psi} \sum_{(i,j)} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)\setminus j}} \sum_{\{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i}} \left(\begin{array}{c} \left(\begin{array}{c} \ln \pi\left(\tilde{s}_{ij}\right) + \ln p\left(\tilde{a}_{ij1}, \tilde{\phi}_{ij1} | \tilde{s}_{ij}\right) + \ln F(\tilde{x}_{il(i,t-1),t}, \tilde{x}_{r(j,t-1)j,t} | \tilde{x}_{il(i,t),t}, \tilde{x}_{r(j,t)j,t}, \tilde{a}_{il(i,t-1)t-1}, \tilde{a}_{r(j,t-1)j,t-1}) \\ + \sum_{l^* \in \mathscr{L}(i)\setminus j} \sum_{t \in \mathscr{T}(i,l^*)} \ln P\left(s_{l^*}, \phi_{l^*,t}, \tilde{a}_{il^*,t} | \phi_{it}\right) + \sum_{r^* \in \mathscr{R}(j)\setminus i} \sum_{t \in \mathscr{T}(j,r^*)} \ln P\left(s_{r^*}, \phi_{r^*,t}, \tilde{a}_{r^*j,t} | \phi_{j,t}\right) \\ + \sum_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} \ln P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \end{array} \right) \times \\ P\left(\{s_{l^*}\}_{l^* \in \mathscr{L}(i)} \{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i} | \left\{ \tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t} \right\}_{ij,t}, s_i, s_j \right) \end{array} \right)
$$

$$
= \max_{p,\psi} \sum_{(i,j)} \sum_{\tilde{s}} q_{ij\tilde{s}} \left(\begin{array}{c} \left(\ln p\left(\tilde{a}_{ij1}, \tilde{\phi}_{ij1} | \tilde{s}_{ij}\right) + \sum_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} \ln P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \right) \\ \\ \times \left(\sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)\setminus j}} \sum_{\{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i}} P\left(\{s_{l^*}\}_{l^* \in \mathscr{L}(i)} \{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i} | \left\{ \tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t} \right\}_{ij,t}, s_i, s_j \right) \right) \end{array} \right)
$$

$$
+ \sum_{(i,j)} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)\setminus j}} \sum_{\{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i}} \sum_{t \in \mathscr{T}(i,l^*)} \ln P\left(s_{l^*}, \phi_{l^*,t}, \tilde{a}_{il^*,t} | \phi_{it}\right) P\left(\{s_{l^*}\}_{l^* \in \mathscr{L}(i)} \{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i} | \left\{ \tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t} \right\}_{ij,t}, s_i, s_j \right)
$$

$$
+ \sum_{(i,j)} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{\{s_{l^*}\}_{l^* \in \mathscr{L}(i)\setminus j}} \sum_{\{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i}} \sum_{t \in \mathscr{T}(j,r^*)} \ln P\left(s_{r^*}, \phi_{r^*,t}, \tilde{a}_{r^*j,t} | \phi_{j,t}\right) P\left(\{s_{l^*}\}_{l^* \in \mathscr{L}(i)} \{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i} | \left\{ \tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t} \right\}_{ij,t}, s_i, s_j \right)
$$

$$
= \max_{p,\psi} \sum_{(i,j)} \sum_{\tilde{s}} q_{ij\tilde{s}} \left(\ln p\left(\tilde{a}_{ij1}, \tilde{\phi}_{ij1} | \tilde{s}_{ij}\right) + \sum_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} \ln P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \right)
$$

$$
+ \sum_{(i,j)} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{l^* \in \mathscr{L}(i)\setminus j} \sum_{s_{l^*}} \underbrace{\sum_{t \in \mathscr{T}(i,l^*)} \ln P\left(s_{l^*}, \phi_{l^*,t}, \tilde{a}_{il^*,t} | \phi_{it}\right)}_{\substack{\text{again, for any } l^* \text{you can factor this out of the summations} \\ \text{that are not relevant to } l^*}} \underbrace{\sum_{r^* \in \mathscr{R}(j)\setminus i} \sum_{s_{r^*}} P\left(\{s_{l^*}\}_{l^* \in \mathscr{L}(i)} \{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i} | \left\{ \tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t} \right\}_{ij,t}, s_i, s_j \right)}_{\text{the remaining sums only affect the posterior probability}}
$$

$$
+ \sum_{(i,j)} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{r^* \in \mathscr{R}(j)\setminus i} \sum_{s_{r^*}} \sum_{t \in \mathscr{T}(j,r^*)} \ln P\left(s_{r^*}, \phi_{r^*,t}, \tilde{a}_{r^*j,t} | \phi_{j,t}\right) \sum_{l^* \in \mathscr{L}(i)\setminus j} \sum_{s_{l^*}} P\left(\{s_{l^*}\}_{l^* \in \mathscr{L}(i)} \{s_{r^*}\}_{r^* \in \mathscr{R}(j)\setminus i} | \left\{ \tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t} \right\}_{ij,t}, s_i, s_j \right)
$$

$$
= \max_{p,\psi} \sum_{(i,j)} \sum_{\tilde{s}} q_{ij\tilde{s}} \left(\ln p\left(\tilde{a}_{ij1}, \tilde{\phi}_{ij1} | \tilde{s}_{ij}\right) + \sum_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} \ln P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \right)
$$

$$
+ \sum_{(i,j)} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{l^* \in \mathscr{L}(i)\setminus j} \sum_{s_{l^*}} \sum_{t \in \mathscr{T}(i,l^*)} \ln P\left(s_{l^*}, \phi_{l^*,t}, \tilde{a}_{il^*,t} | \phi_{it}\right) P\left(s_{l^*} | \left\{ \tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t} \right\}_{ij,t}, s_i, s_j \right)
$$

$$
+ \sum_{(i,j)} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{r^* \in \mathscr{R}(j)\setminus i} \sum_{s_{r^*}} \sum_{t \in \mathscr{T}(j,r^*)} \ln P\left(s_{r^*}, \phi_{r^*,t}, \tilde{a}_{r^*j,t} | \phi_{j,t}\right) P\left(s_{r^*} | \left\{ \tilde{\phi}_{il(i,t),t}, \tilde{\phi}_{r(j,t)j,t}, \tilde{a}_{il(i,t),t}, \tilde{a}_{r(j,t)j,t} \right\}_{ij,t}, s_i, s_j \right)
$$

$$
= \max_{p,\psi} \sum_{(i,j)} \sum_{\tilde{s}} q_{ij\tilde{s}} \left(\begin{array}{c} \ln p\left(\tilde{a}_{ij1}, \tilde{\phi}_{ij1} | \tilde{s}_{ij}\right) + \sum_{t \in \mathscr{T}(i,j)\setminus\{t=1\}} \ln P\left(\phi_{r(j,t),t} = \phi_{i,t}, \phi_{l(i,t),t} = \phi_{j,t}, \tilde{a}_{ij,t} | \phi_{j,t}, \phi_{i,t}\right) \\ + \sum_{l^* \in \mathscr{L}(i)\setminus j} \sum_{s_{l^*}} \sum_{t \in \mathscr{T}(i,l^*)} \ln P\left(s_{l^*}, \phi_{l^*,t}, \tilde{a}_{il^*,t} | \phi_{it}\right) q_{s_{l^*}|\tilde{s}_{ij}} \\ + \sum_{r^* \in \mathscr{R}(j)\setminus i} \sum_{s_{r^*}} \sum_{t \in \mathscr{T}(j,r^*)} \ln P\left(s_{r^*}, \phi_{r^*,t}, \tilde{a}_{r^*j,t} | \phi_{j,t}\right) q_{s_{r^*}|\tilde{s}_{ij}} \end{array} \right)
$$

C.4 Derivation of the solution to (22)

Take the first order conditions with respect to $p(\tilde{a}|x,x',s,s')$

$$\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{t} \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \frac{\begin{array}{c} \mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}, x_{it}=x, x_{l(i,t)}=x'\right)+ \\ \mathbb{1}\left(s_{r(j,t)}=s, s_j=s', \tilde{a}_{r(j,t)j,t}=\tilde{a}, x_{jt}=x', x_{l(i,t)}=x\right) \\ -\mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}, x_{it}=x, x_{l(i,t)}=x', l(i,t)=j\right) \end{array}}{p(\tilde{a}|x,x',s,s')}=0$$

so for any $\tilde{a}\neq\tilde{a}'$ we have

$$\frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{t} \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \left(\begin{array}{c} \mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}, x_{it}=x, x_{l(i,t)}=x'\right) \\ +\mathbb{1}\left(s_{r(j,t)}=s, s_j=s', \tilde{a}_{r(j,t)j,t}=\tilde{a}, x_{jt}=x', x_{l(i,t)}=x\right) \\ -\mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}, x_{it}=x, x_{l(i,t)}=x', l(i,t)=j\right) \end{array}\right)}{p(\tilde{a}|x,x',s,s')}$$
$$=\frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{t} \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \left(\begin{array}{c} \mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}', x_{it}=x, x_{l(i,t)}=x'\right) \\ +\mathbb{1}\left(s_{r(j,t)}=s, s_j=s', \tilde{a}_{r(j,t)j,t}=\tilde{a}', x_{jt}=x', x_{l(i,t)}=x\right) \\ -\mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}', x_{it}=x, x_{l(i,t)}=x', l(i,t)=j\right) \end{array}\right)}{p(\tilde{a}'|x,x',s,s')}$$

Multiply both sides by $p(\tilde{a}|x,x',s,s')p(\tilde{a}'|x,x',s,s')$

$$p(\tilde{a}'|x,x',s,s')\left[\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{t} \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \left(\begin{array}{c} \mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}, x_{it}=x, x_{l(i,t)}=x'\right)+ \\ \mathbb{1}\left(s_{r(j,t)}=s, s_j=s', \tilde{a}_{r(j,t)j,t}=\tilde{a}, x_{jt}=x', x_{l(i,t)}=x\right)- \\ \mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}, x_{it}=x, x_{l(i,t)}=x', l(i,t)=j\right) \end{array}\right)\right]$$
$$=p(\tilde{a}|x,x',s,s')\left[\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{t} \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \left(\begin{array}{c} \mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}', x_{it}=x, x_{l(i,t)}=x'\right)+ \\ \mathbb{1}\left(s_{r(j,t)}=s, s_j=s', \tilde{a}_{r(j,t)j,t}=\tilde{a}', x_{jt}=x', x_{l(i,t)}=x\right) \\ -\mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}', x_{it}=x, x_{l(i,t)}=x', l(i,t)=j\right) \end{array}\right)\right]$$

Then, sum across all $\tilde{a}'$, $p(\tilde{a}'|x,x',s,s')$ on the LHS sums up to 1 and

$$\sum_{\tilde{a}'} \begin{array}{c} \mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}', x_{it}=x, x_{l(i,t)}=x'\right)+ \\ \mathbb{1}\left(s_{r(j,t)}=s, s_j=s', \tilde{a}_{r(j,t)j,t}=\tilde{a}', x_{jt}=x', x_{l(i,t)}=x\right)- \\ \mathbb{1}\left(s_i=s, s_{l(i,t)}=s', \tilde{a}_{il(i,t),t}=\tilde{a}', x_{it}=x, x_{l(i,t)}=x', l(i,t)=j\right) \end{array}$$

on the RHS sums up to

$$\begin{array}{c} \mathbb{1}\left(s_i=s, s_{l(i,t)}=s', x_{it}=x, x_{l(i,t)}=x'\right)+ \\ \mathbb{1}\left(s_{r(j,t)}=s, s_j=s', x_{jt}=x', x_{l(i,t)}=x\right)- \\ \mathbb{1}\left(s_i=s, s_{l(i,t)}=s', x_{it}=x, x_{l(i,t)}=x', l(i,t)=j\right) \end{array}$$

Hence,

$$p(\tilde{a}|x_{it},x_{lt},s,s') = \frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \begin{pmatrix} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', \tilde{a}_{il(i,t),t} = \tilde{a}, x_{it} = x, x_{l(i,t)} = x'\right) + \\ \mathbb{1}\left(s_{r(j,t)} = s, s_j = s', \tilde{a}_{r(j,t)j,t} = \tilde{a}, x_{jt} = x', x_{l(i,t)} = x\right) \\ -\mathbb{1}\left(s_i = s, s_{l(i,t)} = s', \tilde{a}_{il(i,t),t} = \tilde{a}, x_{it} = x, x_{l(i,t)} = x', l(i,t) = j\right) \end{pmatrix}}{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \begin{pmatrix} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', x_{it} = x, x_{l(i,t)} = x'\right) + \\ \mathbb{1}\left(s_{r(j,t)} = s, s_j = s', x_{jt} = x', x_{l(i,t)} = x\right) - \\ \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', x_{it} = x, x_{l(i,t)} = x', l(i,t) = j\right) \end{pmatrix}}$$

Similarly, the foc for $p^m(a|x,s)$

$$\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \frac{\mathbb{1}\left(a_{it} = a, s_i = s, \phi_{it} = \phi_i\right)}{p^m(a|x,s)} = 0$$

Then, for any a' we have

$$\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \frac{\mathbb{1}\left(a_{it} = a, s_i = s, \phi_{it} = \phi_i\right)}{p^m(a|x,s)} \sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \frac{\mathbb{1}\left(a_{it} = a', s_i = s, \phi_{it} = \phi_i\right)}{p^m(a'|x,s)}$$

Multiply both sides by $p^m(a|x,s)p^m(a'|x,s)$ and sum across a' to obtain

$$p^m(a|x,s) = \frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \mathbb{1}\left(a_{it} = a, s_i = s, \phi_{it} = \phi_i\right)}{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \mathbb{1}\left(s_i = s, \phi_{it} = \phi_i\right)}$$

The same estimator can be constructed for $p^f(a|x,s)$

In a model where $p^m(a|x,s) = p^f(a|x,s) = p(a|x,s)$ we have

$$\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \frac{\mathbb{1}\left(s_i = s, a_{it} = a, x_{it} = x\right) + \mathbb{1}\left(s_j = s, a_{jt} = a, x_{jt} = x\right)}{p(a|x,s)} = 0$$

For any a' we have

$$\begin{aligned} &\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \frac{\mathbb{1}\left(s_i = s, a_{it} = a, x_{it} = x\right) + \mathbb{1}\left(s_j = s, a_{jt} = a, x_{jt} = x\right)}{p(a|x,s)} \\ &= \sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \frac{\mathbb{1}\left(s_i = s, a_{it} = a', x_{it} = x\right) + \mathbb{1}\left(s_j = s, a_{jt} = a', x_{jt} = x\right)}{p(a'|x,s)} \end{aligned}$$

Multiply both sides by $p(a|x,s)p(a'|x,s)$ and sum across a' gives

$$p(a|x,s) = \frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \left[\mathbb{1}\left(s_i = s, a_{it} = a, x_{it} = x\right) + \mathbb{1}\left(s_j = s, a_{jt} = a, x_{jt} = x\right)\right]}{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \left[\mathbb{1}\left(s_i = s, x_{it} = x\right) + \mathbb{1}\left(s_j = s, x_{jt} = x\right)\right]}$$

Similarly, the FOC for $\psi^m(\phi', s_l|\phi, s_i)$

$$\frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s_l, \phi_{it} = \phi, \phi_{l(i,t)} = \phi'\right)}{\psi^m(\phi', s_l|\phi, s_i)} = 0$$

Again, equate to any ϕ''

$$\frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s_l, \phi_{it} = \phi, \phi_{l(i,t)} = \phi'\right)}{\psi^m(\phi', s_l|\phi, s_i)}$$
$$= \frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s_l, \phi_{it} = \phi, \phi_{l(i,t)} = \phi''\right)}{\psi^m(\phi'', s_l|\phi, s_i)}$$

Multiply both sides by $\psi^m(\phi', s_l|\phi, s_i)\psi^m(\phi'', s_l|\phi, s_i)$ and sum across ϕ'' to obtain

$$\psi^m(\phi'', s_l|\phi, s_i)\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s_l, \phi_{it} = \phi, \phi_{l(i,t)} = \phi'\right)$$
$$= \psi^m(\phi', s_l|\phi, s_i)\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s_l, \phi_{it} = \phi, \phi_{l(i,t)} = \phi''\right)$$
$$\psi^m(\phi', s_l|\phi, s_i) = \frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s_l, \phi_{it} = \phi, \phi_{l(i,t)} = \phi'\right)}{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)},s_{r(j,t)}} q_{s_{l(i,t)}s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s_l, \phi_{l(i,t)} = \phi''\right)}$$

An analogous estimator can be constructed for $\psi^f(\phi_r, s_r|\phi_j, s_j)$.
The FOC with respect to $p(\tilde{x}|\tilde{s})$ is

$$\frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \mathbb{1}\left(\tilde{x}_{ij,1} = \tilde{x}, \tilde{s}_{ij} = \tilde{s}\right)}{p(\tilde{x}|\tilde{s})} = 0$$

Again, equate to any other $\tilde{x}'$, multiply both sides by $p(\tilde{x}|\tilde{s})p(\tilde{x}'|\tilde{s})$ and then sum across $\tilde{x}'$

$$\frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \mathbb{1}\left(\tilde{x}_{ij,1} = \tilde{x}, \tilde{s}_{ij} = \tilde{s}\right)}{p(\tilde{x}|\tilde{s})} = \frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \mathbb{1}\left(\tilde{x}_{ij,1} = \tilde{x}', \tilde{s}_{ij} = \tilde{s}\right)}{p(\tilde{x}'|\tilde{s})}$$
$$p(\tilde{x}'|\tilde{s})\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \mathbb{1}\left(\tilde{x}_{ij,1} = \tilde{x}, \tilde{s}_{ij} = \tilde{s}\right) = p(\tilde{x}|\tilde{s})\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \mathbb{1}\left(\tilde{x}_{ij,1} = \tilde{x}', \tilde{s}_{ij} = \tilde{s}\right)$$
$$\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \mathbb{1}\left(\tilde{x}_{ij,1} = \tilde{x}, \tilde{s}_{ij} = \tilde{s}\right) = p(\tilde{x}|\tilde{s})\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \mathbb{1}\left(\tilde{s}_{ij} = \tilde{s}\right)$$
$$p(\tilde{x}|\tilde{s}) = \frac{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \mathbb{1}\left(\tilde{x}_{ij,1} = \tilde{x}, \tilde{s}_{ij} = \tilde{s}\right)}{\sum_{(i,j)}\sum_{\tilde{s}} q_{ij\tilde{s}} \mathbb{1}\left(\tilde{s}_{ij} = \tilde{s}\right)}$$

C.5 Derivation of the solution to (23)

The estimator for $p(\tilde{a}|x, x', s, s')$ is derived as

$$\sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{t} \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \frac{\begin{pmatrix} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', \tilde{a}_{il(i,t),t} = \tilde{a}, x_{it} = x, x_{l(i,t)} = x'\right) + \\ \mathbb{1}\left(s_{r(j,t)} = s, s_j = s', \tilde{a}_{r(j,t)j,t} = \tilde{a}, x_{jt} = x', x_{l(i,t)} = x\right) - \\ \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', \tilde{a}_{il(i,t),t} = \tilde{a}, x_{it} = x, x_{l(i,t)} = x', l(i,t) = j\right) \end{pmatrix}}{p(\tilde{a}|x, x', s, s')} + \frac{\sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \sum_{s'} q_{l(i',t)s'} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', \tilde{a}_{i'l(i',t),t} = \tilde{a}, x_{i't} = x, x_{l(i',t)} = x'\right)}{p(\tilde{a}|x, x', s, s')} = 0$$

Equate the FOC for $\tilde{a}$ and any $\tilde{a}'$, the multiply both sides by $p(\tilde{a}|x, x', s, s')p(\tilde{a}'|x, x', s, s')$ and sum across $\tilde{a}'$ to obtain

$$p(\tilde{a}|x, x', s, s') = \frac{\sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \begin{bmatrix} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', \tilde{a}_{il(i,t),t} = \tilde{a}, x_{it} = x, x_{l(i,t)} = x'\right) + \\ \mathbb{1}\left(s_{r(j,t)} = s, s_j = s', \tilde{a}_{r(j,t)j,t} = \tilde{a}, x_{jt} = x', x_{l(i,t)} = x\right) - \\ \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', \tilde{a}_{il(i,t),t} = \tilde{a}, x_{it} = x, x_{l(i,t)} = x', l(i,t) = j\right) \end{bmatrix} + \sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \sum_{s'} q_{l(i',t)s'} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', \tilde{a}_{i'l(i',t),t} = \tilde{a}, x_{i't} = x, x_{l(i',t)} = x'\right)}{\sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \begin{bmatrix} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', x_{it} = x, x_{l(i,t)} = x'\right) + \\ \mathbb{1}\left(s_{r(j,t)} = s, s_j = s', x_{jt} = x', x_{l(i,t)} = x\right) - \\ \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', x_{it} = x, x_{l(i,t)} = x', l(i,t) = j\right) \end{bmatrix} + \sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \sum_{s'} q_{l(i',t)s'} \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', x_{i't} = x, x_{l(i',t)} = x'\right)}$$

The estimator for $p^m(a|x, s)$ is derived from the FOC

$$\sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_{t} \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \frac{\mathbb{1}\left(s_i = s, a_{it} = a, x_{it} = x, l(i,t) = 0\right)}{p^m(a|x, s)} + \frac{\sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \mathbb{1}\left(s_i = s, a_{i',t} = a, x_{i't} = x, l(i,t) = 0\right)}{p^m(a|x, s)} = 0$$

Then, for any a'

$$\begin{aligned} &\sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \frac{\mathbb{1}(s_i = s, a_{it} = a, x_{it} = x, l(i,t) = 0)}{p^m(a|x,s)} + \frac{\sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \mathbb{1}\left(s_i = s, a_{i',t} = a, x_{i't} = x, l(i',t) = 0\right)}{p^m(a|x,s)} \\ = &\sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \frac{\mathbb{1}\left(s_i = s, a_{it} = a', x_{it} = x, l(i,t) = 0\right)}{p^m(a'|x,s)} + \frac{\sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \mathbb{1}\left(s_i = s, a_{i',t} = a', x_{i't} = x, l(i',t) = 0\right)}{p^m(a'|x,s)} \end{aligned}$$

Again, multiply both sides by $p^m(a|x, s)p^m(a'|x, s)$ and sum across a' to obtain

$$p^m(a|x, s) = \frac{\sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i = s, a_{it} = a, x_{it} = x, l(i,t) = 0\right) + \sum_{\{i':l(i,1)=0\}} q_{i's} \mathbb{1}\left(s_{i'} = s, a_{i't} = a, x_{i't} = x, l(i',t) = 0\right)}{\sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i = s, x_{it} = x, l(i,t) = 0\right) + \sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \mathbb{1}\left(s_i = s, s_{l(i,t)} = s', x_{i't} = x, x_{l(i',t)} = x', l(i',t) = 0\right)}$$

A similar estimator can be constructed for $p^f(a|x,s)$. If $p^m(a|x,s) = p^f(a|x,s) = p(a|x,s)$, then the estimator for $p(a|x,s)$ becomes:

$$\begin{array}{c} \sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \left[\frac{\mathbb{1}(s_i=s,a_{it}=a,x_{it}=x,l(i,t)=0)}{p(a|x,s)} + \frac{\mathbb{1}(s_j=s,a_{jt}=a,x_{jt}=x,r(j,t)=0)}{p(a|x,s)} \right] + \\ \frac{\sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \mathbb{1}\left(s_i=s,a_{i',t}=a,x_{i't}=x,l(i',t)=0\right)}{p(a|x,s)} + \frac{\sum_{\{j':r(j',1)=0\}} q_{j's} \sum_t \mathbb{1}\left(s_{j'}=s,a_{j',t}=a,x_{j't}=x,r(j',t)=0\right)}{p(a|x,s)} \end{array} = 0$$

For any a' we have

$$\begin{array}{c} \sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \left[\frac{\mathbb{1}(s_i=s,a_{it}=a,x_{it}=x,l(i,t)=0)}{p(a|x,s)} + \frac{\mathbb{1}(s_j=s,a_{jt}=a,x_{jt}=x,r(j,t)=0)}{p(a|x,s)} \right] + \\ \frac{\sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \mathbb{1}\left(s_i=s,a_{i',t}=a,x_{i't}=x,l(i',t)=0\right)}{p(a|x,s)} + \frac{\sum_{\{j':r(j',1)=0\}} q_{j's} \sum_t \mathbb{1}\left(s_{j'}=s,a_{j',t}=a,x_{j't}=x,r(j',t)=0\right)}{p(a|x,s)} \end{array}$$

$$= \begin{array}{c} \sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \left[\frac{\mathbb{1}\left(s_i=s,a_{it}=a',x_{it}=x,l(i,t)=0\right)}{p(a'|x,s)} + \frac{\mathbb{1}\left(s_j=s,a_{jt}=a',x_{jt}=x,r(j,t)=0\right)}{p(a'|x,s)} \right] + \\ \frac{\sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \mathbb{1}\left(s_i=s,a_{i',t}=a',x_{i't}=x,l(i',t)=0\right)}{p(a'|x,s)} + \frac{\sum_{\{j':r(j',1)=0\}} q_{j's} \sum_t \mathbb{1}\left(s_{j'}=s,a_{j',t}=a',x_{j't}=x,r(j',t)=0\right)}{p(a'|x,s)} \end{array}$$

Multiply both sides by $p(a|x,s)p(a'|x,s)$ and sum across a' to obtain:

$$p(a|x,s) = \frac{\begin{array}{c} \sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \left[\begin{array}{c} \mathbb{1}\left(s_i = s, a_{it} = a, x_{it} = x, l(i,t) = 0\right) + \\ \mathbb{1}\left(s_j = s, a_{jt} = a, x_{jt} = x, r(j,t) = 0\right) \end{array} \right] + \\ \sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \mathbb{1}\left(s_i = s, a_{i',t} = a, x_{i't} = x, l(i',t) = 0\right) \\ + \sum_{\{j':r(j',1)=0\}} q_{j's} \sum_t \mathbb{1}\left(s_{j'} = s, a_{j',t} = a, x_{j't} = x, r(j',t) = 0\right) \end{array}}{\begin{array}{c} \sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \left[\begin{array}{c} \mathbb{1}\left(s_i = s, x_{it} = x, l(i,t) = 0\right) + \\ \mathbb{1}\left(s_j = s, x_{jt} = x, r(j,t) = 0\right) \end{array} \right] + \\ \sum_{\{i':l(i,1)=0\}} q_{i's} \sum_t \mathbb{1}\left(s_i = s, x_{i't} = x, l(i',t) = 0\right) + \\ \sum_{\{j':r(j',1)=0\}} q_{j's} \sum_t \mathbb{1}\left(s_{j'} = s, x_{j't} = x, r(j',t) = 0\right) \end{array}}$$

An analogous estimator can be constructed for $\psi^m(\phi', s'|\phi, s)$.

$$\begin{array}{c} \frac{\sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i=s,\phi_{it}=\phi,s_{l(i,t)}=s',\phi_{l(i,t)}=\phi'\right)}{\psi^m(\phi',s'|\phi,s)} + \\ \frac{\sum_{\{i':l(i,1)=0\}} \sum_s q_{i's} \sum_t \sum_{s'} q_{s_{l(i,t)}|s_i} \mathbb{1}\left(s_i=s,\phi_{it}=\phi,s_{l(i,t)}=s',\phi_{l(i,t)}=\phi'\right)}{\psi^m(\phi',s'|\phi,s)} \end{array} = 0$$

For any ϕ'' we have

$$\begin{array}{c} \frac{\sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i=s,\phi_{it}=\phi,s_{l(i,t)}=s',\phi_{l(i,t)}=\phi'\right)}{\psi^m(\phi',s'|\phi,s)} + \\ \frac{\sum_{\{i':l(i,1)=0\}} \sum_s q_{i's} \sum_t \sum_{s'} q_{s_{l(i,t)}|s_i} \mathbb{1}\left(s_i=s,\phi_{it}=\phi,s_{l(i,t)}=s',\phi_{l(i,t)}=\phi'\right)}{\psi^m(\phi',s'|\phi,s)} \end{array}$$

$$= \begin{array}{c} \frac{\sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i=s,\phi_{it}=\phi,s_{l(i,t)}=s',\phi_{l(i,t)}=\phi''\right)}{\psi^m(\phi'',s'|\phi,s)} + \\ \frac{\sum_{\{i':l(i,1)=0\}} \sum_s q_{i's} \sum_t \sum_{s'} q_{s_{l(i,t)}|s_i} \mathbb{1}\left(s_i=s,\phi_{it}=\phi,s_{l(i,t)}=s',\phi_{l(i,t)}=\phi''\right)}{\psi^m(\phi'',s'|\phi,s)} \end{array}$$

Multiply both sides by $\psi^m(\phi'', s'|\phi, s)\psi^m(\phi', s'|\phi, s)$ and sum across ϕ'' to obtain

$$\psi^m(\phi'', s'|\phi, s) = \frac{\begin{array}{c} \sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i = s, \phi_{it} = \phi, s_{l(i,t)} = s', \phi_{l(i,t)} = \phi'\right) + \\ \sum_{\{i':l(i,1)=0\}} \sum_s q_{i's} \sum_t \sum_{s'} q_{s_{l(i,t)}|s_i} \mathbb{1}\left(s_i = s, \phi_{it} = \phi, s_{l(i,t)} = s', \phi_{l(i,t)} = \phi'\right) \end{array}}{\begin{array}{c} \sum_{\{i:l(i,1)=j\}} \sum_{\tilde{s}} q_{ij\tilde{s}} \sum_t \sum_{s_{l(i,t)}, s_{r(j,t)}} q_{s_{l(i,t)} s_{r(j,t)}|\tilde{s}} \mathbb{1}\left(s_i = s, \phi_{it} = \phi, s_{l(i,t)} = s'\right) + \\ \sum_{\{i':l(i,1)=0\}} \sum_s q_{i's} \sum_t \sum_{s'} q_{s_{l(i,t)}|s_i} \mathbb{1}\left(s_i = s, \phi_{it} = \phi, s_{l(i,t)} = s'\right) \end{array}}$$

The FOCs for $p(\tilde{x}_{ij,1}|\tilde{s}_{ij})$ and $p(x_{i'1}|s_{i'})$ are similarly derived.

D Finite dependence

Stage 2, single agent. The choice sequences $\{a_1, \emptyset, a_1\}$(replace engine, then operate individually, then replace engine again) and $\{a, \emptyset, a_1\}$ $a = a_2, a_3$ (no maintenance or partial maintenance, then operate individually, then replace engine) establish one-period finite dependence.

$$
\begin{aligned}
& v_t^m\left(a|\phi_{it}\right) - v_t^m\left(a_1|\phi_{it}\right) \\
=& u^m\left(a|\phi_{it}\right) - \cancel{u^m\left(a_1|\phi_{it}\right)} + \beta E_{\phi_{it+1}|a,\phi_{it}} U^m\left(\phi_{it+1}\right) - \beta E_{\phi_{it+1}|a_1,\phi_{it}} U^m\left(\phi_{it+1}\right) \\
=& u^m\left(a|\phi_{it}\right) + \beta E_{\phi_{it+1}|a,\phi_{it}}\left[V^m(\emptyset|\phi_{it+1}) + \cancel{\kappa^m(\emptyset|\phi_{it+1})} - \nu\ln\psi^m(\emptyset|\phi_{it+1})\cancel{+\mu+\nu\gamma}\right] \\
& - \beta E_{\phi_{it+1}|a_1,\phi_{it}}\left[V^m(\emptyset|\phi_{it+1}) + \cancel{\kappa^m(\emptyset|\phi_{it+1})} - \nu\ln\psi^m(\emptyset|\phi_{it+1})\cancel{+\mu+\nu\gamma}\right] \\
=& u^m\left(a|\phi_{it}\right) + \beta E_{\phi_{it+1}|a,\phi_{it}}\left[v^m(a_1|\phi_{it+1}) - \nu\ln p^m(a_1|\phi_{t+1})\cancel{+\mu+\nu\gamma} - \nu\ln\psi^m(\emptyset|\phi_{it+1})\right] \\
& - \beta E_{\phi_{it+1}|a_1,\phi_{it}}\left[v^m(a_1|\phi_{it+1}) - \nu\ln p^m(a_1|\phi_{t+1})\cancel{+\mu+\nu\gamma} - \nu\ln\psi^m(\emptyset\phi_{it+1})\right] \\
=& u^m\left(a|\phi_{it}\right) + \beta E_{\phi_{it+1}|a,\phi_{it}}\left[\cancel{u^m\left(a_1|\phi_{it+1}\right)} + \beta E_{\phi_{t+2}|a_1,\phi_{it+1}} U^m(\phi_{t+2}) - \nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset|\phi_{it+1})\right] \\
& - \beta E_{\phi_{it+1}|a_1,\phi_{it}}\left[\cancel{u^m\left(a_1|\phi_{it+1}\right)} + \beta E_{\phi_{t+2}|a_1,\phi_{it+1}} U^m(\phi_{t+2}) - \nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset|\phi_{it+1})\right] \\
=& u^m\left(a|\phi_{it}\right) + \cancel{\beta E_{\phi_{t+2}|a_1} U^m(\phi_{t+2})} + \beta E_{\phi_{it+1}|a,\phi_{it}}\left[-\nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset|\phi_{it+1})\right] \\
& - \cancel{\beta E_{\phi_{t+2}|a_1} U^m(\phi_{t+2})} - \beta E_{\phi_{it+1}|a_1,\phi_{it}}\left[-\nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset|\phi_{it+1})\right] \\
=& u^m\left(a|\phi_{it}\right) + \beta E_{\phi_{it+1}|a,\phi_{it}}\left[-\nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset|\phi_{it+1})\right] - \beta E_{\phi_{it+1}|a_1,\phi_{it}}\left[-\nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset|\phi_{it+1})\right]
\end{aligned}
$$

Where $E_{\phi_{it+1}|a,\phi_{it}}\left(E_{\phi_{t+2}|a_1,\phi_{it+1}} U^m(\phi_{t+2})\right) = E_{\phi_{it+1}|a_1,\phi_{it}}\left(E_{\phi_{t+2}|a_1,\phi_{it+1}} U^m(\phi_{t+2})\right) = E_{\phi_{t+2}|a_1} U^m(\phi_{t+2})$ holds because action a_1 is a renewal action.

Stage 2, couple: The choice sequences $\{\tilde{a}, \emptyset, a_1\}$(any action in $\mathscr{A}^2$ such that $a_i \neq a_1$, then operate individually, then replace engine) and $\{(a_1, a_1), \emptyset, a_1\}$ (replace engines in both buses in the couple, then operate individually, then replace engine) establish one-period finite dependence.

$$
\begin{aligned}
& v_t^m\left(\tilde{a}|\tilde{\phi}_{il(i,t),t}\right) - v_t^m\left((a_1,a_1)|\tilde{\phi}_{il(i,t),t}\right) \\
=& u^m\left(\tilde{a}|\tilde{\phi}_{il(i,t),t}\right) - \cancel{u^m\left(a_1|\tilde{\phi}_{il(i,t),t}\right)} + \beta E_{\tilde{\phi}_{il(i,t),t+1}|\tilde{a},\tilde{\phi}_{il(i,t),t}} U^m\left(\phi_{it+1}\right) - \beta E_{\tilde{\phi}_{il(i,t),t+1}|a_1,\tilde{\phi}_{il(i,t),t}} U^m\left(\phi_{it+1}\right) \\
=& u^m\left(\tilde{a}|\tilde{\phi}_{il(i,t),t}\right) + \beta E_{\tilde{\phi}_{il(i,t),t+1}|\tilde{a},\tilde{\phi}_{il(i,t),t}}\left[V_\emptyset^m(\phi_{it+1}) + \cancel{\kappa^m(\emptyset|\phi_{it+1},\phi_{l(i,t)t+1})} - \ln\psi^m(\emptyset|\tilde{\phi}_{il(i,t),t+1})\cancel{+\mu+\nu\gamma}\right] \\
& - \beta E_{\tilde{\phi}_{il(i,t),t+1}|a_1,\tilde{\phi}_{il(i,t),t}}\left[V_\emptyset^m(\phi_{it+1}) + \cancel{\kappa^m(\emptyset|\phi_{it+1},\phi_{l(i,t)t+1})} - \nu\ln\psi^m(\emptyset|\tilde{\phi}_{il(i,t),t+1})\cancel{+\mu+\nu\gamma}\right] \\
=& u^m\left(\tilde{a}|\tilde{\phi}_{il(i,t),t}\right) + \beta E_{\tilde{\phi}_{il(i,t),t+1}|\tilde{a},\tilde{\phi}_{il(i,t),t}}\left[v_\emptyset^m(a_1|\phi_{it+1}) - \nu\ln p^m(a_1|\phi_{t+1})\cancel{+\mu+\nu\gamma} - \nu\ln\psi^m(\emptyset|\tilde{\phi}_{il(i,t),t+1})\right] \\
& - \beta E_{\tilde{\phi}_{il(i,t),t+1}|a_1,\tilde{\phi}_{il(i,t),t}}\left[v_\emptyset^m(a_1|\phi_{it+1}) - \nu\ln p^m(a_1|\phi_{t+1})\cancel{+\mu+\nu\gamma} - \nu\ln\psi^m(\emptyset|\tilde{\phi}_{il(i,t),t+1})\right] \\
=& u^m\left(\tilde{a}|\tilde{\phi}_{il(i,t),t}\right) + \beta E_{\tilde{\phi}_{il(i,t),t+1}|\tilde{a},\tilde{\phi}_{il(i,t),t}}\left[\cancel{u^m\left(a_1|\phi_{it+1}\right)} + \beta E_{\tilde{\phi}_{t+2}|a_1,\tilde{\phi}_{il(i,t),t+1}} U^m(\tilde{\phi}_{t+2}) - \nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset\tilde{\phi}_{il(i,t),t+1})\right] \\
& - \beta E_{\tilde{\phi}_{il(i,t),t+1}|a_1,\tilde{\phi}_{il(i,t),t}}\left[\cancel{u^m\left(a_1\phi_{it+1}\right)} + \beta E_{\tilde{\phi}_{t+2}|a_1,\tilde{\phi}_{il(i,t),t+1}} U^m(\tilde{\phi}_{t+2}) - \nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset|\tilde{\phi}_{il(i,t),t+1})\right] \\
=& u^m\left(\tilde{a}|\tilde{\phi}_{il(i,t),t}\right) + \cancel{\beta E_{\tilde{\phi}_{t+2}|a_1} U^m(\tilde{\phi}_{t+2})} + \beta E_{\tilde{\phi}_{il(i,t),t+1}|\tilde{a},\tilde{\phi}_{il(i,t),t}}\left[-\nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset|\tilde{\phi}_{il(i,t),t+1})\right] \\
& - \cancel{\beta E_{\tilde{\phi}_{t+2}|a_1} U^m(\tilde{\phi}_{t+2})} - \beta E_{\tilde{\phi}_{il(i,t),t+1}|a_1,\tilde{\phi}_{il(i,t),t}}\left[-\nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset|\tilde{\phi}_{il(i,t),t+1})\right] \\
=& u^m\left(\tilde{a}|\tilde{\phi}_{il(i,t),t}\right) + \beta E_{\tilde{\phi}_{il(i,t),t+1}|\tilde{a},\tilde{\phi}_{il(i,t),t}}\left[-\nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset|\tilde{\phi}_{il(i,t),t+1})\right] \\
& - \beta E_{\tilde{\phi}_{il(i,t),t+1}|a_1,\tilde{\phi}_{il(i,t),t}}\left[-\nu\ln p^m(a_1|\phi_{t+1}) - \nu\ln\psi^m(\emptyset|\tilde{\phi}_{il(i,t),t+1})\right]
\end{aligned}
$$

Stage 1: The choice sequences $\left\{\phi_{r(j,t),t}, a_1, \emptyset, a_1\right\}$(match with any bus $\phi_{r(j,t),t}$, then replace engine, then operate individually, then replace engine) and $\{\emptyset, a_1, \emptyset, a_1\}$ (operate individually, replace engine, then operate individually, then replace engine) establish one-period finite dependence.

Notice that $V^m(\phi_{it}, \phi_{j,t})$ can be written as

$$
\begin{aligned}
& V^m(\phi_{r(j,t),t}, \phi_{j,t}) \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) \left(v^m\left(\tilde{a}|\tilde{\phi}\right) - \nu \ln p(a|\tilde{\phi}_{r(j,t)j,t}) \right) \cancel{+\mu + \nu\gamma} \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) \left(u^m\left(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}\right) + \beta \mathbb{E}_{\tilde{\phi}_{r(j,t)j,t+1}|\tilde{a},\tilde{\phi}_{r(j,t)j,t}} U^m\left(\tilde{\phi}_{r(j,t)j,t+1}\right) - \nu \ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) \right) \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) \left(\begin{array}{c} u^m\left(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}\right) - \nu \ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) \\ +\beta \mathbb{E}_{\tilde{\phi}_{r(j,t)j,t+1}|\tilde{a},\tilde{\phi}_{r(j,t)j,t}} \left[V^m_{\emptyset}\left(\phi^m_{j,t+1}\right) + \cancel{\kappa^m\left(\emptyset|\tilde{\phi}_{r(j,t)j,t+1}\right)} - \ln \psi^m\left(\emptyset|\tilde{\phi}_{r(j,t)j,t+1}\right) \right] \end{array} \right)
\end{aligned}
$$

Then,

$$
\begin{aligned}
& V^m_t(\phi_{r(j,t)t}, \phi_{j,t}) - V^m_{\emptyset t}(\phi_{j,t}) \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) \left(v^m\left(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}\right) - \nu \ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) \right) \cancel{+\mu + \nu\gamma} \\
&- v^m_{\emptyset t}\left(a^*, \phi_{j,t}\right) + \nu \ln p^m(a^*|\phi_{j,t}) \cancel{-\mu - \nu\gamma} \forall a^* \in \mathscr{A} \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) \left(v^m\left(\tilde{a}|\phi_{r(j,t),t}, \phi_{j,t}\right) - \nu \ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) - v^m_{\emptyset}\left(a^*|\phi_{j,t}\right) + \nu \ln p^m(a^*|\phi_{j,t}) \right) \ \forall a^* \in \mathscr{A}\left(\phi_{j,t}\right) \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) \left(v^m\left(\tilde{a}|\phi_{r(j,t),t}, \phi_{j,t}\right) - \nu \ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) - v^m_{\emptyset}\left(a_1|\phi_{j,t}\right) + \nu \ln p^m(a_1|\phi_{j,t}) \right) \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) \left(u^m\left(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}\right) + \beta \mathbb{E}_{\tilde{\phi}_{r(j,t)j,t+1}|\tilde{a},\tilde{\phi}_{r(j,t)j,t}} U^m\left(\tilde{\phi}_{r(j,t)j,t+1}\right) \right. \\
&- \cancel{u^m_{\emptyset}\left(a_1|\phi_{j,t}\right)} - \beta \mathbb{E}_{\phi_{j,t+1}|a_1,\phi_{j,t}} U^m\left(\phi_{j,t+1}\right) \\
&\left. - \nu \ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) + \nu \ln p^m(a_1|\phi^m_{j,t}) \right) \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) \left(u^m\left(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}\right) + \right. \\
& \beta \mathbb{E}_{\tilde{\phi}_{r(j,t)j,t+1}|\tilde{a},\tilde{\phi}_{r(j,t)j,t}} \left[V^m_{\emptyset}\left(\phi_{j,t+1}\right) + \cancel{\kappa^m\left(\emptyset|\tilde{\phi}_{r(j,t)j,t+1}\right)} - \nu \ln \psi^m\left(\emptyset|\tilde{\phi}_{r(j,t)j,t+1}\right) \cancel{+\mu + \nu\gamma} \right] \\
&- \beta \mathbb{E}_{\phi_{j,t+1}|a_1,\phi_{j,t}} \left[V^m_{\emptyset}\left(\phi_{j,t+1}\right) + \cancel{\kappa^m\left(\emptyset|\phi_{j,t+1}\right)} - \nu \ln \psi^m\left(\emptyset|\phi_{j,t+1}\right) \cancel{+\mu + \nu\gamma} \right] \\
&\left. - \nu \ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) + \nu \ln p(a_1|\phi_{j,t}) \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t})\Bigg(u_t^m\left(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}\right) + \\
&\beta \mathbb{E}_{\tilde{\phi}_{r(j,t)j,t+1}|\tilde{a},\tilde{\phi}_{r(j,t)j,t}}\left[v_\emptyset^m\left(a_1|\phi_{j,t+1}\right) - \nu \ln p(a_1|\phi_{j,t+1}) \cancel{+\mu+\nu\gamma} - \nu\ln\psi^m\left(\emptyset|\tilde{\phi}_{r(j,t)j,t+1}\right)\right] \\
&- \beta \mathbb{E}_{\phi_{jt+1}|a_1,\phi_{jt}}\left[v_\emptyset^m\left(a_1|\phi_{j,t+1}\right) - \nu\ln p(a_1|\phi_{j,t+1}) \cancel{+\mu+\nu\gamma} - \nu\ln\psi^m\left(\emptyset|\phi_{j,t+1}\right)\right] \\
&- \nu\ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) + \nu\ln p^m(a_1|\phi_{j,t})\Bigg) \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t})\Bigg(u^m\left(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}\right) + \\
&\beta \mathbb{E}_{\tilde{\phi}_{r(j,t)j,t+1}|\tilde{a},\tilde{\phi}_{r(j,t)j,t}}\left[\cancel{u_\emptyset^m\left(a_1|\phi_{j,t+1}\right)} + \beta\mathbb{E}_{\phi_{j,t+2}|a_1,\phi_{j,t+1}} U^m\left(\phi_{j,t+2}\right) - \nu\ln p^m(a_1|\phi_{j,t+1}) - \nu\ln\psi^m\left(\emptyset|\tilde{\phi}_{r(j,t)j,t+1}\right)\right] \\
&- \beta \mathbb{E}_{\phi_{j,t+1}|a_1,\phi_{j,t}}\left[\cancel{u_\emptyset^m\left(a_1|\phi_{j,t+1}\right)} + \beta\mathbb{E}_{\phi_{j,t+2}|a_1,\phi_{j,t+1}} U_{t+2}^m\left(\phi_{j,t+2}\right) - \nu\ln p^m(a_1|\phi_{j,t+1}) - \nu\ln\psi^m\left(\emptyset|\phi_{j,t+1}\right)\right] \\
&- \nu\ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) + \nu\ln p^m(a_1|\phi_{j,t})\Bigg) \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t})\Bigg(u^m\left(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}\right) + \\
&+ \cancel{\beta^2\mathbb{E}_{\phi_{j,t+2}|a_1} U_{t+2}^m\left(\phi_{j,t+2}\right)} + \beta \mathbb{E}_{\tilde{\phi}_{r(j,t)j,t+1}|\tilde{a},\tilde{\phi}_{r(j,t)j,t}}\left[-\nu\ln p^m(a_1|\phi_{j,t+1}) - \nu\ln\psi^m\left(\emptyset|\tilde{\phi}_{r(j,t)j,t+1}\right)\right] \\
&- \cancel{\beta^2\mathbb{E}_{\phi_{j,t+2}|a_1} U_{t+2}^m\left(\phi_{j,t+2}^m\right)} - \beta \mathbb{E}_{\phi_{j,t+1}|a_1,\phi_{j,t}}\left[-\nu\ln p^m(a_1|\phi_{j,t+1}) - \nu\ln\psi^m\left(\emptyset|\phi_{j,t+1}\right)\right] \\
&- \nu\ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) + \nu\ln p^m(a_1|\phi_{j,t})\Bigg) \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t})\Bigg(u^m\left(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}\right) + \\
&\beta \mathbb{E}_{\tilde{\phi}_{r(j,t)j,t+1}|\tilde{a},\tilde{\phi}_{r(j,t)j,t}}\left[-\nu\ln p^m(a_1|\phi_{j,t+1}) - \nu\ln\psi^m\left(\emptyset|\tilde{\phi}_{r(j,t)j,t+1}\right)\right] \\
&- \beta \mathbb{E}_{\phi_{j,t+1}^m|a_1,\phi_{j,t}^m}\left[-\nu\ln p^m(a_1|\phi_{j,t+1}) - \nu\ln\psi^m\left(\emptyset|\phi_{j,t+1}\right)\right] \\
&- \nu\ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) + \nu\ln p^m(a_1|\phi_{j,t})\Bigg) \\
&= \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) u^m\left(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}\right) \\
&+ \sum_{\tilde{a}\in\mathscr{A}^2} p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t})\Bigg(\beta \mathbb{E}_{\tilde{\phi}_{r(j,t)j,t+1}|a,\tilde{\phi}_{r(j,t)j,t}}\left[-\nu\ln p^m(a_1|\phi_{j,t+1}) - \nu\ln\psi^m\left(\emptyset|\tilde{\phi}_{r(j,t)j,t+1}\right)\right] \\
&- \beta \mathbb{E}_{\phi_{j,t+1}|0,\phi_{j,t}}\left[-\nu\ln p^m(a_1|\phi_{j,t+1}) - \nu\ln\psi^m\left(\emptyset|\phi_{j,t+1}\right)\right] - \nu\ln p(\tilde{a}|\tilde{\phi}_{r(j,t)j,t}) + \nu\ln p^m(a_1|\phi_{j,t})\Bigg)
\end{aligned}
$$

The last rearrangement conveniently separates nonparametrically identified conditional choice probabilities and functions of the structural parameters that need to be evaluated for every parameter candidate during estimation.

E Proofs

E.1 Proof of Lemma 1

For convenience, we define

$$\begin{aligned}
\mathring{x}_{l,t} &= \mathbb{1}\left(x_{l(i,t),t} = x_0\right) \text{ for some } x_0 \in \mathscr{X} \cup \emptyset \\
\mathring{a}_{il,t} &= \begin{cases} \mathbb{1}\left(\tilde{a}_{il(i,t),t} = \left(a_0, a_0^{'}\right)\right) \text{ for some } \left(a_0, a_0^{'}\right) \in \mathscr{A}^2 & if\, x_0 \in \mathscr{X} \\ \mathbb{1}\left(a_{i,t} = a_0\right) \text{ for some } a_0 \in \mathscr{A} & if\, x_0 = \emptyset \end{cases} \\
\mathring{d}_{il,t} &= \mathring{a}_{il,t} \times \mathring{x}_{l,t} \\
\lambda_{\varphi}^{\mathring{d}}(s) &= P^m\left(\mathring{d}_{il,t} = 1 | \varphi, s, h_{t_0}^m\right) \\
\lambda_{\varphi}^{a}(s) &= P^m\left(a_{i1} = a_0^{''}, x_{i1} = \varphi | s, h_{t_0}^m\right) = p^m(a_{i1} = a_0^{''}, x_{i1} = \varphi | s, h_{t_0}^m)\; for\, some\, a_0^{''} \in \mathscr{A}
\end{aligned}$$

for any observable state $\varphi \in \mathscr{X}$. This reduces agents' choices to a binary set, without loss of generality.

In the first part of this proof, we apply Kasahara and Shimotsu (2009)directly.

Consider a set of points in the state space $\{\varphi^*\} \cup \{\varphi_o\}_{o=1}^{S-1} \subset \Phi$. Define

$$\underset{(S\times S)}{L} = \begin{bmatrix} 1 & \lambda_{\varphi_1}^{\mathring{d}}(s_1) & \cdots & \lambda_{\varphi_{S-1}}^{\mathring{d}}(s_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{\varphi_1}^{\mathring{d}}(s_S) & \cdots & \lambda_{\varphi_{S-1}}^{\mathring{d}}(s_S) \end{bmatrix}$$

$$D_{\varphi^*} = diag\left(\lambda_{\varphi^*}^{a}(s_1), ..., \lambda_{\varphi^*}^{a}(s_S)\right)$$

$$V = diag\left(\pi(s_1|h_{t_0}^m), ..., \pi(s_S|h_{t_0}^m)\right)$$

The elements of L, V, and $D_{\varphi^*} \forall \varphi^* \in \Phi$ are the parameters to be identified.

Now, collect notation for matrices of observables. Only three periods of history $\{t_1, t_2, t_3\}$ per individual are necessary for identification. It is required that $t_1 = 1$ and that t_2 and t_3 be larger than 1 but they need not be consecutive. Then, define $(\varphi^*, \varphi_{t_2}, \varphi_{t_3}) \in \Phi^3$ to be a set of points in the state space, one per selected period of data. Then, we define some matrices of joint probabilities that can be consistently estimated from the data, where the asymptotic convergence occurs as the number of individual histories in the sample approaches infinity.

$$F_{\varphi^*,\varphi_{t_2},\varphi_{t_3}}^{*} \equiv \mathring{P}\left(a_{il(i,1),1} = a_0^{''}, \varphi^*, \left\{\mathring{d}_{il,t} = 1, \varphi_t\right\}_{t=t_2,t_3} | h_{t_0}^m\right) \equiv \sum_s \pi(s|h_{t_0}^m)\lambda_{\varphi^*}^{a}(s)\,\lambda_{\varphi_{t_2}}^{\mathring{d}}(s)\,\lambda_{\varphi_{t_3}}^{\mathring{d}}(s)$$

Sum across values of $\left(a_{il(i,1)}, \varphi^*\right)$. Only the term $\lambda_{\varphi^*}^{a}(s)$ depends on them and will sum up to one, yielding:

$$F_{\varphi_{t_2},\varphi_{t_3}} \equiv \mathring{P}\left(\left\{\mathring{d}_{il,t} = 1, \varphi_t\right\}_{t=t_2,t_3}\right) \equiv \sum_s \pi(s|h_{t_0}^m)\lambda_{\varphi_{t_2}}^{\mathring{d}}(s)\,\lambda_{\varphi_{t_3}}^{\mathring{d}}(s)$$

Similarly, we define the following "marginals" by summing across values of $\left\{\mathring{d}_{il,t}\right\}_{t=t_2,t_3}$ or $\left(a_{il(i,1)}, \varphi^*\right)$

$$\begin{aligned}
F_{\varphi^*,\varphi_{t_2}}^{*} &\equiv \mathring{P}\left(a_{il(i,1)} = a_0^{''}, \varphi^*, \mathring{d}_{il,t_2} = 1, \varphi_{t_2}\right) = \sum_s \pi(s|h_{t_0}^m)\lambda_{\varphi^*}^{a}(s)\,\lambda_{\varphi_{t_2}}^{\mathring{d}_{il,2}}(s) \\
F_{\varphi^*,\varphi_{t_3}}^{*} &\equiv \mathring{P}\left(a_{il(i,1)} = a_0^{''}, \varphi^*, \mathring{d}_{il,t_3} = 1, \varphi_{t_3}\right) = \sum_s \pi(s|h_{t_0}^m)\lambda_{\varphi^*}^{a}(s)\,\lambda_{\varphi_{t_3}}^{\mathring{d}_{il,3}}(s) \\
F_{\varphi_{t_2}} &\equiv \mathring{P}\left(\mathring{d}_{il,t_2} = 1, \varphi_{t_2}\right) = \sum_s \pi(s|h_{t_0}^m)\lambda_{\varphi_{t_2}}^{\mathring{d}_{il,3}}(s) \\
F_{\varphi_{t_3}} &\equiv \mathring{P}\left(\mathring{d}_{il,t_3} = 1, \varphi_{t_3}\right) = \sum_s \pi(s|h_{t_0}^m)\lambda_{\varphi_{t_3}}^{\mathring{d}_{il,3}}(s) \\
F_{\varphi^*}^{*} &\equiv \mathring{P}\left(a_{il(i,1)} = a_0^{''}, \varphi^*\right) = \sum_s \pi(s|h_{t_0}^m)\lambda_{\varphi^*}^{a}(s)
\end{aligned}$$

Using these definition create the matrices

$$P=\begin{bmatrix}1 & F_{\varphi_1} & \cdots & F_{\varphi_{S-1}}\\ F_{\varphi_1} & F_{\varphi_1,\varphi_1} & \cdots & F_{\varphi_1,\varphi_{S-1}}\\ \vdots & \vdots & \ddots & \vdots\\ F_{\varphi_{S-1}} & F_{\varphi_{S-1},\varphi_1} & \cdots & F_{\varphi_{S-1},\varphi_{S-1}}\end{bmatrix}$$

$$P^*_{\varphi^*}=\begin{bmatrix}F^*_{\varphi^*} & F^*_{\varphi^*,\varphi_1} & \cdots & F^*_{\varphi^*,\varphi_{S-1}}\\ F^*_{\varphi^*,\varphi_1} & F^*_{\varphi^*,\varphi_1,\varphi_1} & \cdots & F^*_{\varphi^*,\varphi_1,\varphi_{S-1}}\\ \vdots & \vdots & \ddots & \vdots\\ F^*_{\varphi^*,\varphi_{S-1}} & F^*_{\varphi^*,\varphi_{S-1},\varphi_1} & \cdots & F^*_{\varphi^*,\varphi_{S-1},\varphi_{S-1}}\end{bmatrix}$$

Now we can write

$$P=L'VL$$
$$P_{\varphi^*}=L'VD_{\varphi^*}L$$

If L is invertible, so is P, then we can define $A_{\varphi^*}=P^{-1}P_{\varphi^*}$.

$$A_{\varphi^*}=L^{-1}D_{\varphi^*}L$$
$$\implies A_{\varphi^*}L^{-1}=L^{-1}D_{\varphi^*}$$

Since D_{φ^*} is diagonal, we know that the columns of $L^{-1}$are the right eigenvectors of A_{φ^*} up to constant scaling, and the elements of D_{φ^*} are the corresponding eigenvalues. Denote $L^{-1}\Xi$ the right eigenvectors of A_{φ^*} for some diagonal matrix Ξ. Notice that

$$PL^{-1}\Xi=L'V\Xi$$

Since the first row of L' is a vector of ones and $V\Xi$ is a diagonal matrix, the first row of $PL^{-1}\Xi$ will contain the elements of $V\Xi$. Now that $V\Xi$ and $PL^{-1}\Xi$ are both known, we can uniquely determine L':

$$L'=\left(PL^{-1}\Xi\right)(V\Xi)^{-1}$$

Knowing L' and hence L, we can uniquely determine V by using the relation $P=L'VL$, either by inverting both L' and L or by noticing that $(L')^{-1}P=VL$ and the first column of L is a vector of ones, so the first row of $(L')^{-1}P$ will contain the elements of V.

From the values of $V$and L, we can construct the matrix P_{ζ^*} for any $\zeta^*\in\Phi$ the same way P_{φ^*} is constructed and use it to determine $D_{\zeta^*}=diag\left(\lambda^a_{\zeta^*}(s_1),...,\lambda^a_{\zeta^*}(s_S)\right)$ explointing the relation $P_{\zeta^*}=L'VD_{\zeta^*}L$.

We still have to identify $\lambda^{\mathring{d}}_{\xi}(s)$ at values of ξ that are not in $\{\varphi_o\}_{o=1}^{S-1}$. For any arbitrary $\xi\in\Phi$ we can define

$$\underset{(S\times 2)}{L_\xi}=\begin{bmatrix}1 & \lambda^{\mathring{d}}_{\xi}(s_1)\\ \vdots & \vdots\\ 1 & \lambda^{\mathring{d}}_{\xi}(s_S)\end{bmatrix}$$

$$\underset{(2\times M)}{P_\xi}=\begin{bmatrix}1 & F_{\varphi_1} & \cdots & F_{\varphi_{S-1}}\\ F_\xi & F_{\xi,\varphi_1} & \cdots & F_{\xi,\varphi_{S-1}}\end{bmatrix}$$

Since P_ξ can be consistently estimated from the data and $P_\xi=(L_\xi)'VL$,we can uniquely determine $(L_\xi)'=P_\xi(VL)^{-1}$.

Up until now, we have applied Kasahara and Shimotsu (2009) directly and we have identified $\lambda^{\mathring{d}}_{\varphi}(s)$, $\lambda^a_\varphi(s)$, and $\pi(s|h^m_{t_0})$.

One can use the same argument to show the identification of $p^m(a_{i1}=a_0^{''},x_{i1}=\varphi|s,h^m_{t_0})$ for any value of $a_0^{''}\in\mathscr{A}$. Next, determining $p^m(a|x,s,h^m_{t_0})$ is trivial.

$$p^m(a|x,s,h^m_{t_0})=\sum_z\lambda^a_x(s)/P(x|s,h^m_{t_0})$$

Where the distribution $P(x|s,h^m_{t_0})$ can be obtained from $\lambda^a_x(s)$ via

$$P(x|s,h^m_{t_0})=\sum_a p^m(a,x|s,h^m_{t_0})=\sum_a\lambda^a_x(s).$$

Notice that $\pi(s,\emptyset)=\pi(s|h^m_{t_0})P(h^m_{t_0})$ where $P(h^m_{t_0})$ is nonparametrically identified from the data because $h^m_{t_0}$ is observable.

E.2 Proof of Lemma 2

For convenience, we define

$$\mathring{x}_{lr,t} = \mathbb{1}\left(x_{l(i,t),t} = x_0, x_{r(j,t),t} = x_0^{'}\right) \text{ for some } x_0 \in \mathscr{X} \cup \emptyset, x_0^{'} \in \mathscr{X} \cup \emptyset$$
$$\mathring{a}_{ilrj,t} = \mathbb{1}\left(\tilde{a}_{il(i,t),t} = a_0, \tilde{a}_{r(j,t)j,t} = a_0^{'}\right) \; a_0, a_0^{'} \in \mathscr{A}$$
$$\mathring{d}_{ilrj,t} = \mathring{a}_{ilrj,t} \times \mathring{x}_{lr,t}$$
$$\lambda_{\tilde{\varphi}}^{\mathring{d}}(\tilde{s}) = P\left(\mathring{d}_{ilrj,t} = 1 | \tilde{\varphi}, \tilde{s}, \tilde{h}_{t_0}\right) \quad t = 2, ..., T$$
$$\lambda_{\tilde{\varphi}}^{\tilde{a}}(\tilde{s}) = P\left(\tilde{a}_{ij1} = \tilde{a}_0, \tilde{x}_{ij1} = \tilde{\varphi} | \tilde{s}, \tilde{h}_{t_0}\right) = p(\tilde{a}_{ij1}, \tilde{x}_{ij,t} | \tilde{s_{ij}}, \tilde{h}_{t_0}) \; for\ some\ \tilde{a}_0 \in \mathscr{A}(\tilde{\varphi})$$

for any observable state $\tilde{\varphi} \in \Phi^2$. This reduces agents' choices to a binary set, without loss of generality.

First part. Apply Kasahara Shimotsu (2009) The Kasahara and Shimotsu (2009) argument flows exactly like in the one-person household case, with the difference that now there are S^2 unobserved types of couples and an expanded choice set equal to $\Phi^2 \times A^2$. Consider a set of points in the joint state space of the couple$\{\tilde{\varphi}_o\}_{o=1}^{S^2-1} \subset \Phi^2$. Define

$$\underset{(S^2 \times S^2)}{L} = \begin{bmatrix} 1 & \lambda_{\tilde{\varphi}_1}^{\mathring{d}}(\tilde{s}_1) & \cdots & \lambda_{\tilde{\varphi}_{S^2-1}}^{\mathring{d}}(\tilde{s}_1) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{\tilde{\varphi}_1}^{\mathring{d}}(\tilde{s}_{S^2}) & \cdots & \lambda_{\tilde{\varphi}_{S^2-1}}^{\mathring{d}}(\tilde{s}_{S^2}) \end{bmatrix}$$
$$D_{\tilde{\varphi}^*} = diag\left(\lambda_{\tilde{\varphi}^*}^{\tilde{a}}(\tilde{s}_1), ..., \lambda_{\tilde{\varphi}^*}^{\tilde{a}}(\tilde{s}_{S^2})\right)$$
$$V = diag\left(\pi(s_1, s_1|\tilde{h}_{t_0}), \pi(s_1, s_2|\tilde{h}_{t_0}), ..., \pi(s_S, s_S|\tilde{h}_{t_0})\right)$$

Now, collect notation for matrices of observables. For $(\tilde{\varphi}^*, \tilde{\varphi}_{t_2}, \tilde{\varphi}_{t_3}) \in (\Phi)^{2\times 3}$:

$$F^*_{\tilde{\varphi}^*, \tilde{\varphi}_{t_2}, \tilde{\varphi}_{t_3}} = \mathring{P}\left(\tilde{a}_{ij1} = \tilde{a}_0, \tilde{\varphi}^*, \left\{\mathring{d}_{ilrj,t} = 1, \tilde{\varphi}_t\right\}_{t=t_2,t_3} | \tilde{h}_{t_0}\right) = \sum_{\tilde{s}} \pi(\tilde{s}) \lambda_{\tilde{\varphi}^*}^{\tilde{a}}(\tilde{s}) \lambda_{\tilde{\varphi}_{t_2}}^{\mathring{d}}(\tilde{s}) \lambda_{\tilde{\varphi}_{t_3}}^{\mathring{d}}(\tilde{s})$$

Sum across values of $(\tilde{a}_{ilrj,t}, \tilde{\varphi}^*)$. Only the term $\lambda_{\tilde{\varphi}^*}^{\tilde{a}}(\tilde{s})$ depends on them and will sum up to one, yielding:

$$F_{\tilde{\varphi}_{t_2}, \tilde{\varphi}_{t_3}} = \mathring{P}\left(\left\{\mathring{d}_{ilrj,t} = 1, \tilde{\varphi}_t\right\}_{t=t_2,t_3} | \tilde{h}_{t_0}\right) = \sum_{\tilde{s}} \pi(\tilde{s}) \lambda_{\tilde{\varphi}_{t_2}}^{\mathring{d}} \lambda_{\tilde{\varphi}_{t_3}}^{\mathring{d}}$$

Similarly, we define the following "marginals" by summing across elements of $\left\{\mathring{d}_{ilrj,t} = 1\right\}_{t=t_2,t_3}$ and $(\tilde{a}_{ilrj,t}, \tilde{\varphi}^*)$.

$$F^*_{\tilde{\varphi}^*, \tilde{\varphi}_{t_2}} = \mathring{P}\left(\tilde{a}_{ij,1} = \tilde{a}_0, \tilde{\varphi}^*, \mathring{d}_{ilrj,t_2} = 1, \tilde{\varphi}_{t_2} | \tilde{h}_{t_0}\right) = \sum_{\tilde{s}} \pi(\tilde{s}) \lambda_{\tilde{\varphi}^*}^{\tilde{a}}(\tilde{s}) \lambda_{\tilde{\varphi}_{t_2}}^{\mathring{d}}(\tilde{s})$$
$$F^*_{\tilde{\varphi}^*, \tilde{\varphi}_{t_3}} = \mathring{P}\left(\tilde{a}_{ij1} = \tilde{a}_0, \tilde{\varphi}^*, \mathring{d}_{ilrj,t_3} = 1, \tilde{\varphi}_{t_3} | \tilde{h}_{t_0}\right) = \sum_{\tilde{s}} \pi(\tilde{s}) \lambda_{\tilde{\varphi}^*}^{\tilde{a}}(\tilde{s}) \lambda_{\tilde{\varphi}_{t_3}}^{\mathring{d}}(\tilde{s})$$
$$F_{\tilde{\varphi}_{t_2}} \equiv \mathring{P}\left(\mathring{d}_{ilrj,t_2} = 1, \tilde{\varphi}_{t_2} | \tilde{h}_{t_0}\right) = \sum_{s} \pi(s|h^m_{t_0}) \lambda_{\tilde{\varphi}_{t_2}}^{\mathring{d}_{il,3}}(s)$$
$$F_{\tilde{\varphi}_{t_3}} \equiv \mathring{P}\left(\mathring{d}_{ilrj,t_3} = 1, \tilde{\varphi}_{t_3} | \tilde{h}_{t_0}\right) = \sum_{s} \pi(s|h^m_{t_0}) \lambda_{\tilde{\varphi}_{t_3}}^{\mathring{d}_{il,3}}(s)$$
$$F^*_{\tilde{\varphi}^*} \equiv \mathring{P}\left(\tilde{a}_{ij,1} = \tilde{a}_0, \tilde{\varphi}^* | \tilde{h}_{t_0}\right) = \sum_{\tilde{s}} \pi(\tilde{s}) \lambda_{\tilde{\varphi}^*}^{\tilde{a}}(\tilde{s})$$

Using these definition create the matrices

$$P = \begin{bmatrix} 1 & F_{\tilde{\varphi}_1} & \dots & F_{\tilde{\varphi}_{S^2-1}} \\ F_{\tilde{\varphi}_1} & F_{\tilde{\varphi}_1,\tilde{\varphi}_1} & \dots & F_{\tilde{\varphi}_1,\tilde{\varphi}_{S^2-1}} \\ \vdots & \vdots & \ddots & \vdots \\ F_{\tilde{\varphi}_{S^2-1}} & F_{\tilde{\varphi}_{S^2-1},\tilde{\varphi}_1} & \dots & F_{\tilde{\varphi}_{S-1},\tilde{\varphi}_{S^2-1}} \end{bmatrix}$$

$$P_{\tilde{\varphi^*}} = \begin{bmatrix} F^*_{\tilde{\varphi^*}} & F^*_{\tilde{\varphi^*},\tilde{\varphi}_1} & \dots & F^*_{\tilde{\varphi^*},\tilde{\varphi}_{S^2-1}} \\ F^*_{\tilde{\varphi^*},\tilde{\varphi}_1} & F^*_{\tilde{\varphi^*},\tilde{\varphi}_1,\tilde{\varphi}_1} & \dots & F^*_{\tilde{\varphi^*},\tilde{\varphi}_1,\tilde{\varphi}_{S^2-1}} \\ \vdots & \vdots & \ddots & \vdots \\ F^*_{\tilde{\varphi^*},\tilde{\varphi}_{S^2-1}} & F^*_{\tilde{\varphi^*},\tilde{\varphi}_{S^2-1},\tilde{\varphi}_1} & \dots & F^*_{\tilde{\varphi^*},\tilde{\varphi}_{S^2-1},\tilde{\varphi}_{S^2-1}} \end{bmatrix}$$

Now we can write

$$P = L'VL$$
$$P_{\tilde{\varphi^*}} = L'VD_{\tilde{\varphi^*}}L$$

If L is invertible, so is P, then we can define $A_{\tilde{\varphi^*}} = P^{-1}P_{\tilde{\varphi^*}}$.

$$A_{\tilde{\varphi^*}} = L^{-1}D_{\tilde{\varphi^*}}L$$
$$\implies A_{\tilde{\varphi^*}}L^{-1} = L^{-1}D_{\tilde{\varphi^*}}$$

Since $D_{\tilde{\varphi^*}}$ is diagonal, we know that the columns of $L^{-1}$are the right eigenvectors of $A_{\tilde{\varphi^*}}$ up to constant scaling, and the elements of $D_{\tilde{\varphi^*}}$ are the corresponding eigenvalues. Denote $L^{-1}\Xi$ the right eigenvectors of $A_{\tilde{\varphi^*}}$ for some diagonal matrix Ξ. Notice that

$$PL^{-1}\Xi = L'V\Xi$$

Since the first row of L' is a vector of ones, the first row of $PL^{-1}\Xi$ will contain the elements of the diagonal matrix$V\Xi$. Now that $V\Xi$and $PL^{-1}\Xi$ are both known, we can uniquely determine L':

$$L' = \left(PL^{-1}\Xi\right)(V\Xi)^{-1}$$

Knowing L' and hence L, we can uniquely determine V by using the relation $P = L'VL$, either by inverting both L' and L or by noticing that $(L')^{-1}P = VL$ and the first column of L is a vector of ones, so the first row of $(L')^{-1}P$ will contain the elements of V.

From the values of $V$and L, we can construct the matrix $P_{\tilde{\zeta}}$for any $\tilde{\zeta}^* \in \Phi^2$ the same way $P_{\tilde{\varphi^*}}$ is constructed and use it to determine $D_{\tilde{\zeta^*}} = diag\left(\lambda^{*1}_{\tilde{\zeta^*}}, ..., \lambda^{*S^2}_{\tilde{\zeta^*}}\right)$ exploiting the relation $P_{\tilde{\zeta^*}} = L'VD_{\tilde{\zeta^*}}L$.

We still have to identify $\lambda^{\tilde{s}}_{\tilde{\xi}}$ at values of $\tilde{\xi}$ that are not in $\{\varphi_g\}_{g=1}^{S-1}$. For any arbitrary $\tilde{\xi} \in \Phi^2$ we can define

$$\underset{(S\times 2)}{L_{\tilde{\xi}}} = \begin{bmatrix} 1 & \lambda^1_{\tilde{\xi}} \\ \vdots & \vdots \\ 1 & \lambda^{S^2}_{\tilde{\xi}} \end{bmatrix}$$

$$\underset{(2\times M)}{P_{\tilde{\xi}}} = \begin{bmatrix} 1 & F_{\tilde{\varphi}_1} & \dots & F_{\tilde{\varphi}_{S^2-1}} \\ F_{\tilde{\xi}} & F_{\tilde{\xi},\tilde{\varphi}_1} & \dots & F_{\tilde{\xi},\tilde{\varphi}_{S^2-1}} \end{bmatrix}$$

Since $P_{\tilde{\xi}} = \left(L_{\tilde{\xi}}\right)'VL$,we can uniquely determine $\left(L_{\tilde{\xi}}\right)' = P_{\tilde{\xi}}(VL)^{-1}$.

Now we have identified $\lambda^{\mathring{d}}_{\tilde{\varphi}}(\tilde{s})$, $\lambda^{\tilde{a}}_{\tilde{\varphi}}(\tilde{s})$, and $\pi(\tilde{s}|\tilde{h}_{t_0})$

One can use the same argument to show the identification of $p(\tilde{a}_{ij1} = \tilde{a}_0, \tilde{x}_{ij,1}|\tilde{s_{ij}}, \tilde{h}_{t_0})$ for all values of $(\tilde{a}_0, \tilde{x}_{ij1}, \tilde{s_{ij}})$. $p(\tilde{a}|\tilde{\phi}, \tilde{h}_{t_0})$ is identified from

$$p(\tilde{a}|\tilde{\phi}, \tilde{h}_{t_0}) = p(\tilde{a}, \tilde{x}|\tilde{s}, \tilde{h}_{t_0})/P(\tilde{x}|\tilde{s}, \tilde{h}_{t_0})$$

Where the distribution $P(\tilde{x}|\tilde{s}, \tilde{h}_{t_0})$ is obtained from

$$P(\tilde{x}|\tilde{s}, \tilde{h}_{t_0}) = \sum_{\tilde{a}} p(\tilde{a}, \tilde{x}|\tilde{s}, \tilde{h}_{t_0}) = \sum_{\tilde{a}} \lambda^{\tilde{a}}_{\tilde{x}}(\tilde{s})$$

Finally, notice that $\pi(s, s') = \pi(s, s'|\tilde{h}_{t_0})P(\tilde{h}_{t_0})$.

E.3 Proof of 1

Now that p's are identified, we need to identify the ψ's. We exploit the following relation concerning the one-person household

$$P\left(\tilde{a}_{il(i,t),t}, x_{l(i,t),t}|\phi_{it}, h^m_{t_0}\right) = \sum_{s_{l(i,t),t}} P\left(\tilde{a}_{il(i,t),t}, x_{l(i,t),t}, s_{l(i,t),t}|\phi_{it}, h^m_{t_0}\right) = \sum_{s_{l(i,t),t}} \psi^m(\phi_{l(i,t),t}|\phi_{it}) p(\tilde{a}_{il(i,t),t}|\tilde{\phi}_{il(i,t),t}) \tag{24}$$

Where $P\left(\tilde{a}_{il(i,t),t}, x_{l(i,t),t}|\phi_{it}, h^m_{t_0}\right)$ is identified in 1 and $p(\tilde{a}_{ij,t}|\tilde{\phi}_{ij,t})$ is identified in 2.
We define the matrices

$$\underset{(A^2-1)\times 1}{P_1(x_l, s_i, x_i)} = \left[\; P\left(\tilde{a} = \tilde{a}_1, x_l|x_i, s_i, h^m_{t_0}\right) \quad P\left(\tilde{a} = \tilde{a}_2, x_l|x_i, s_i, h^m_{t_0}\right) \quad \dots \quad P\left(\tilde{a} = \tilde{a}_{A^2-1}, x_l|x_i, s_i, h^m_{t_0}\right) \;\right]^T$$

$$\underset{(A^2-1)X\times 1}{P_1(s_i, x_i)} = \left[\; P_1(x_l = x_1, s_i, x_i, z_i) \quad P_1(x_l = x_2, s_i, x_i, z_i) \quad \dots \quad P_1(x_l = x_X, s_i, x_i, z_i) \;\right]^T$$

$$\underset{(A^2-1)XS\times 1}{P_1(x_i)} = \left[\; P_1(s_i = s_1, x_i) \quad P_1(s_i = s_2, x_i) \quad \dots \quad P_1(s_i = s_S, x_i) \;\right]^T$$

$$\underset{(A^2-1)X^2S\times 1}{P_1} = \left[\; P_1(x_i = x_1) \quad P_1(x_i = x_2) \quad \dots \quad P_1(x_i = x_X) \;\right]^T$$

$$\underset{(A^2-1)\times 1}{P_2(s_l, x_l, s_i, x_i)} = \left[\; p(\tilde{a}_{il} = \tilde{a}_1|s_l, x_l, s_i, x_i) \quad p(\tilde{a}_{il} = \tilde{a}_2|s_l, x_l, s_i, x_i) \quad \dots \quad p(\tilde{a}_{il} = \tilde{a}_{A^2-1}|s_l, x_l, s_i, x_i) \;\right]^T$$

$$\underset{(A^2-1)\times S}{P_2(x_l, s_i, x_i)} = \left[\; P_2(s_l = s_1, x_l, s_i, x_i) \quad P_2(s_l = s_2, x_l, s_i, x_i) \quad \dots \quad P_2(s_l = s_S, x_l, s_i, x_i) \;\right]$$

$$\underset{(A^2-1)X\times SX}{P_2(s_i, x_i)} = \begin{bmatrix} P_2(x_l = x_1, s_i, x_i) & 0 & \dots & 0 \\ 0 & P_2(x_l = x_2, s_i, x_i) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & P_2(x_l = x_X, s_i, x_i) \end{bmatrix}$$

$$\underset{(A^2-1)SX\times S^2X}{P_2(x_i)} = \begin{bmatrix} P_2(s_i = s_1, x_i) & 0 & \dots & 0 \\ 0 & P_2(s_i = s_2, x_i) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & P_2(s_i = s_S, x_i) \end{bmatrix}$$

$$\underset{(A^2-1)SX^2\times (SX)^2}{P_2} = \begin{bmatrix} P_2(x_i = x_1) & 0 & \dots & 0 \\ 0 & P_2(x_i = x_2) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & P_2(x_i = x_X) \end{bmatrix}$$

$$\underset{S\times 1}{P_3(x_l, s_i, x_i)} = \left[\; \psi(s_l = s_1, x_l|s_i, x_i) \quad \psi(s_l = s_2, x_l|s_i, x_i) \quad \dots \quad \psi(s_l = s_S, x_l|s_i, x_i) \;\right]^T$$

$$\underset{SX\times 1}{P_3(s_i, x_i)} = \left[\; P_3(x_l = x_1, s_i, x_i) \quad P_3(x_l = x_2, s_i, x_i) \quad \dots \quad P_3(x_l = x_X, s_i, x_i) \;\right]^T$$

$$\underset{S^2X\times 1}{P_3(x_i)} = \left[\; P_3(s_i = s_1, x_i) \quad P_3(s_i = s_2, x_i) \quad \dots \quad P_3(s_i = s_S, x_i) \;\right]^T$$

$$\underset{(SX)^2\times 1}{P_3} = \left[\; P_3(x_i = x_1) \quad P_3(x_i = x_2) \quad \dots \quad P_3(x_i = x_X) \;\right]^T$$

24 can be written as

$$P_1 = P_2 P_3$$

If $A^2 - 1 \geq S$, and there is sufficient variation in P_2 so that $P_2^{'} P_2$ is invertible, we can identify P_3.

$$P_3 = (P_2^{'} P_2)^{-1} P_2^{'} P_1$$

Notice that $P_3$is obtained as the result of an OLS regression of each column of P_1 onto P_2.