ON THE ASSET PRICING IMPLICATIONS OF INCOMPLETE INFORMATION IN SOVEREIGN DEBT MARKETS

BY

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Abstract

In the first chapter, we investigate the asset pricing implications of disagreement about the unobservable process that drives monetary policy. We construct a monetary economy with two investors who have heterogeneous priors about the unobservable expected growth of money, namely, the underlying monetary stance. The key friction is that information about this unobservable variable is symmetric but incomplete. Investors estimate it from observed realizations of money supply in a Bayesian framework. Investors value real money holdings intrinsically, which gives rise to a demand for money. In equilibrium, the demand for money and investors’ estimates endogenously determine the nominal short rates and expected inflation through a portfolio rebalancing channel. Disagreement about money growth is one driver of the conditional variation in all nominal interest rates, expected inflation, and inflation risk-premium. Through a parameterized example, we show that the volatility of nominal interest rates tend to rise with increases disagreement. We show that the relationship between disagreement about money growth and asset prices crucially depends on the persistence of the unobservable growth rate of money. We further use the model to investigate the consequences of allowing for heterogeneous interpretations of public information (e.g. central bank statements) on interest rates. We introduce a public signal on which investors may have different views. One "steadfast" investor believes the signal conveys relevant information about the unobservable process, while a "doubtful" one believes it contains only noise. Depending on the persistence of the unobservable process, interest rates may be more or less volatile in an economy populated by one steadfast investor and one doubtful investor that in an economy with two doubtful investors.\(^1\)

In the second chapter, we propose a method to measure the importance of private information about fundamentals as a determinant of market power in the primary market for sovereign debt. The method is based on the estimation of a divisible good uniform price auction model presented in Vives (2010, 2011). We establish conditions under which the model is identified from bidding data. Through an application to the Colombian debt market, we show that private information decreases and inventory costs rise after the rebalancing of two emerging markets debt indices. We use the model estimates to quantify the effect of these changes on bidders’ market power, and

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to evaluate the effectiveness of policies meant to reduce borrowing costs. We provide model-free evidence that supports the reduction in the importance of private information and the increase in inventory costs from our structural estimates. If bidders are privately informed in this market, then an auction’s cutoff price would reflect bidders’ private information about fundamentals that were not already captured by observable market prices. We show that auctions with high cutoff prices relative to a pre-auction benchmark predict high future secondary market prices. After the rebalancing, this effect either shrinks or vanishes altogether, which is consistent with a decline in the importance of private information. Furthermore, we find higher mean-reversion in inventories after the rebalancing. On average, it takes bidders 11.7 days to offload half of their inventories before the rebalancing, but only 7.5 days afterwards. Finally, we provide reduced-form evidence that is consistent with the assumptions of the model regarding primary market participation.²

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Chapter 1

Disagreeing about money growth in a monetary economy

1.1 Introduction

Blinder et al. (2008) argue that “[N]owadays, it is widely accepted that the ability of a central bank to affect the economy depends critically on its ability to influence market expectations about the future path of overnight interest rates, and not merely on their current level.” In a similar vein, King et al. (2008) assert that monetary policy is now mostly concerned about managing the expectations of the private sector. Moreover, over the last thirty years, and particularly since the financial crisis in 2008, central banks have increasingly relied on tools like forward guidance to explicitly guide market participants views on the future path of interest rate and the unobservable monetary policy stance ((Campbell et al., 2017)).

Nevertheless, these efforts to manage private sector’s views have not led to a concomitant decrease in disagreement among investors about the future path of interest rates. The empirical evidence indicates that interest rate expectations remain dispersed across market participants. Andrade et al. (2019) argue that there is substantial heterogeneity in the forecasts of the survey participants about inflation and the Federal Funds rate in the Blue Chip Financial Forecasts survey. Similarly, Caballero and Simsek (2020) document significant disagreement between Fed’s predictions and forward rates that are difficult to reconcile with traditional macroeconomic models. These levels of disagreement are likely to carry important asset pricing implications. For instance, Ehling et al. (2018) show empirically (and theoretically) that disagreement about inflation has a strong impact on the yield curve.

The monetary authorities track the private sector’s view on monetary policy stance and interest rate expectations. They are particularly interested in registering shifts in market views. For instance, the New York Fed performs the Survey of Primary Dealers (SPD) and the Survey of
Market Participants that reads “The objective of the Survey of Primary Dealers [...] is to gain insight into the expectations of primary dealer firms. [...] [R]espondents have been queried about their expectations for the future level of the federal funds rate. [...] Survey results [...] are used by Federal Reserve staff in their evaluations of market expectations for the economic outlook, monetary policy, and the financial markets. [...] [O]ccasionally, the Desk asks respondents to update their responses immediately following an FOMC meeting to gauge how expectations have changed in response to new information.”.\(^1\)

Given these considerations, we argue that it is desirable to develop a benchmark to evaluate the asset pricing implications of disagreement about the monetary policy and the future path of interest rates. In this paper, we construct a monetary model that investigates the relationship between interest rates, expected inflation, and the underlying monetary policy stance. Our monetary model is embedded into an exchange economy with exogenous paths for consumption and nominal money supply, both of which are observable. In contrast, we assume that the expected growth rate of money supply is an unobservable and stationary stochastic process. Investors use their heterogeneous priors and observations of changes in the money supply to estimate the expected growth rate of money. In the model’s equilibrium, their own estimates, on which they base their investment decisions, become the driver of the conditional variation in interest rates and expected inflation. Through their dependence on the investors' estimates, our model replicates the empirical relation between short nominal interest rate and inflation.\(^2\)

We relate the latent monetary policy stance to the unobservable expected growth rate of money in our model.\(^3\) The latter may capture in reduced-form smooth changes over time of the underlying monetary policy stance, as Campbell et al. (2014) and Song (2017) argue has been the case in the U.S. Intuitively, realizations of the nominal money supply contain relevant information from which investors can learn about the conduct of monetary policy. The monetarist literature has highlighted that changes in monetary aggregates indeed convey information about key unobservable variables that is not already contained in, for instance, interest-rate rules ((McCallum and Nelson, 2010; Ireland, 2017)).\(^4\)

We assume that investors derive utility from consumption and real money holdings. The intra-temporal rate of substitution between consumption and real balances gives rise to an endogenous demand for real money balances. In turn, this optimality condition determines the nominal short rate, namely, the nominally riskfree interest rate investors earn from investing in a bank account. The nominal short rate thus corresponds to the opportunity cost of holding money associated with

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\(^1\)See https://www.newyorkfed.org/markets/survey_market_participants.html for details.

\(^2\)The empirical relation between short nominal interest rates and inflation is what Carvalho and Nechio (2014) call one of the basic principles underlying the “Taylor rule”

\(^3\)King et al. (2008) assert that private agents do not know the underlying nature of the central banks’ policies.

\(^4\)See section (1.2.5) for a thorough discussion on the validity of relating the underlying stance of the monetary policy authority to the expected growth rate of money in our model.
standard models of the demand for money, which completes our monetary economy.\(^5\) The price level and interest rates in this economy endogenously adjust such that the markets for consumption and real balances clear, regardless of the exogenous paths for consumption and nominal balances. Therefore, a portfolio rebalancing channel arises endogenously in our model.

In a simple single-investor economy, the mechanism behind this channel is as follows: if the investor forms a high estimate of the expected growth rate of money, she would prefer to borrow to hold more real money balances today instead of the future when her marginal utility will be lower. For markets to clear, the nominal short rate needs to increase. The higher interest she will earn will afford her to increase future consumption, thus raising expected inflation. The level and conditional volatility of nominal interest rates across different maturities are determined in turn by her estimates through her inter-temporal decisions. In contrast, the real interest rates and the instantaneous inflation-risk premium are constant.

To study the asset pricing implications of disagreement about money growth, we introduce another investor into the economy. We assume that both investors observe the same public information, but they have heterogeneous priors about the volatility of the unobservable expected growth rate of money. Therefore, the model’s key friction in the model is that information about the monetary policy stance is symmetric but incomplete. Investors update their estimates of the expected growth rate of money based on realizations of the money supply. However, their heterogeneous priors lead to the formation of heterogeneous estimates. Disagreement about money growth is priced in the real and nominal state-price densities. In turn, it drives expected inflation, the inflation risk-premium, and nominal interest rates. In parameterized examples, we show that the volatility of nominal interest rates tend to rise with increases in the absolute value of disagreement, while the inflation risk-premium is an increasing function of its level. The effect of disagreement on the level of nominal interest rates is ambiguous. Crucially, we assert that the overall impact of disagreement on asset prices hinges on the persistence of the expected growth rate of money. In particular, the relationship between disagreement and equilibrium volatility and risk-premium is weakened when the unobservable process is less persistent. In such case, investors’ estimates are more heavily influenced by the long-run mean of the underlying process, on which both investors agree, than by their respective heterogeneous estimates.

Our model can be adapted to investigate the effects of heterogeneous interpretations of public information about the underlying monetary policy stance. For instance, central banks that attempt to manage expectations about future interest rates (e.g. forward guidance) rely on “unconventional” sources (statements, speeches, the Fed dots, among others) to communicate policy-relevant information to market participants. However, market participants may interpret the same information differently because they have different estimation models or priors ((Andrade et al., 2019); (Caballero and

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Simsek, 2020); (Sastry, 2021)). Motivated by this argument, we further examine the effect on our monetary model of introducing a public signal that conveys potentially useful and non-redundant information about the unobservable expected growth rate of money. We assume that a “steadfast” investor believes that the signal is informative. On the opposite side, a “doubtful” investor believes it conveys only noise. The steadfast investor’s estimate is then subject to two countervailing forces. Due to her beliefs, she believes she learns about the unobservable expected growth rate of money from observing the signal. Consequently, she reacts strongly to it when updating her estimate. In turn, she responds less strongly to the information about the unobservable variable contained in changes in money growth. In contrast, the doubtful investor bases revisions to her estimate solely on realizations of the latter. Hence, both investors have different estimation models and interpret the same public information heterogeneously. The signal itself becomes a risk factor that both investors are exposed to, despite that only the steadfast investor deems it informative. We show in two parameterized economies that the nominal interest rates are higher in a “mixed” economy with a steadfast and a doubtful investor than in a “homogeneous” economy, where both investors are doubtful. The higher the credibility the steadfast investor attaches to the signal, the higher the equilibrium nominal rates. However, the effects are economically minor. Instead, the informational tradeoff between money growth and signal stands out when comparing the volatility of interest rates in both economies. When the unobservable process is persistent, revisions to the steadfast investor’s estimate after observing the signal flow are less pronounced than when it is less persistent. Thus, the nominal rates volatility in the “mixed” economy are lower than those of the “homogeneous” economy when the state of the economy is more persistent. The opposite result obtains when it is less persistent.

We finish this section with a discussion of related literature. The remainder of this paper is as follows: in section (1.2) we introduce a single-investor economy to gain a better understanding of the portfolio rebalancing channel through which interest rates and expected inflation are determined in the model. We proceed to describe the investor’s filtering problem and characterize the model’s equilibrium. We conclude the section with a thorough discussion on whether our monetary economy is an appropriate model to study monetary policy. In section (1.3) we introduce another investor with different priors into the economy, thus giving rise to an endogenous disagreement variable. We discuss some general properties of the equilibrium of the disagreement economy and some differences with respect to the single-investor economy. In section (1.4) we calibrate the model to study the single-investor and disagreement economies in more detail. We conclude with a short summary of our main results and a brief discussion on directions for future research.

Related literature Our paper is closely related to the literature on asset prices and heterogeneous beliefs. Ehling et al. (2018) and Xiong and Yan (2010) study the asset pricing impact of disagreement about inflation. Ehling et al. (2018) focus on disagreement about the expected growth rate of
inflation, whereas Xiong and Yan (2010) study disagreement about the long-run inflation target. Buraschi and Whelan (2012) study the implications of disagreement about long-run expected growth rate of consumption and its implications on the term-structure of interest rates. Contrary to these works, our focus is on the nominal sector of the economy and, specifically, on the underlying monetary policy stance. Specifically, by endogenizing the demand for real money holdings, nominal interest rates in our model are jointly determined through a portfolio rebalancing channel. Thus, nominal interest rates are not subordinated by real interest rates in our model.

Croitoru and Lu (2015) also explore the asset pricing implications of disagreement in a monetary economy similar to ours. However, we allow for the expected growth rate of money to be governed by a stationary and stochastic process. This feature makes the the level and volatility of nominal interest rates and expected inflation time-varying, regardless of whether investors have heterogeneous beliefs. Importantly, we relate the expected growth rate of money to the underlying monetary policy stance, which as argued by Campbell et al. (2014) and Song (2017), changes over time. Moreover, our model can naturally accommodate the signaling information channel of monetary policy documented by Melosi (2017). The heterogeneous confidence about the informativeness of the public signal in our model is also related to the analysis of Sastry (2021). Other important works in this literature include Scheinkman and Xiong (2003), Gallmeyer and Hollifield (2008), Dumas et al. (2009), and Baker et al. (2016), although their focus is on the stock market and the real sector.

In the macro-finance literature, Caballero and Simsek (2020) study disagreement about the path of interest rates when markets are “opinionated”. Andrade et al. (2019) study monetary models where agents have heterogeneous priors. Their focus is on forward guidance announcements about the future macroeconomic outlook and how they can potentially be interpreted differently by the private sector. Agents disagreement in their models stem from aggregate demand not from the monetary policy stance, as in our model.

Our monetary model draws from a resurgent literature that emphasizes the interactions between the demand for real money holdings and monetary policy. Lucas Jr and Nicolini (2015); Belongia and Ireland (2016, 2017, 2019); Benati et al. (2021); Belongia and Ireland (2022) and Benati et al. (2021) bring the demand for money and modified monetary aggregates back to the academic debate about the conduct of monetary policy. They construct new monetary aggregates that restore the stability of the empirical relations between inflation, interest rates, and monetary aggregates. Belongia and Ireland (2022) argue that an explicit money growth rule delivers similar results to those of standard interest-rate rules in terms of its ability to stabilize inflation and output. Furthermore, money growth rules are particularly relevant when the zero-lower bound becomes binding. The focus of our paper is on asset prices as opposed to macroeconomic considerations.

Finally, the equilibrium of the single-investor economy shares some properties with some popular single-factor interest rate models such as that of Cox et al. (1985). In particular, there is a state
variable that drives the conditional variation of all variables (the investor’s estimate), and the nominal interest rates are non-negative, time-varying and heteroskedastic in the state variable. Our model has more parameters than Cox et al. (1985), so it can potentially allow for a rich array of dynamics. An important difference with that literature is that the underlying state variable is unobservable, so it is the estimate of the unobservable state the driver in the conditional variation of the interest rates.

1.2 A single-investor monetary economy

This section introduces a monetary economy where the price level, inflation and interest rates are all determined endogenously. The framework throughout the paper is a continuous-time pure exchange economy model. Uncertainty in the model is summarized by the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\). We first study a monetary economy with a single-representative investor (“investor \(i\)”) who derives utility from a perishable consumption good and real cash balances. Assuming that money enters directly the utility function of the investor can be rationalized by the fact that cash holdings facilitate consumption transactions using money. Models with money-in-the-utility-function have been adopted by Bakshi and Chen (1996); Lucas (2001); Basak and Gallmeyer (2001); Buraschi and Jiltsov (2005, 2007), and Balduzzi (2007), among others.

Before introducing disagreement, we introduce a the single-investor economy to gain a better understanding on the properties of the equilibrium of our monetary economy. We investigate the relationship between the consumption, the demand for money, interest rates and prices with the unobservable expected rate of money growth (i.e. monetary policy stance). Thus, we can isolate the effects that disagreement itself introduces into the equilibrium of the model.

1.2.1 Model setup

The model has two sectors: real and nominal. The real sector is represented by a consumption process denoted by \(C = (C_t)\), and the nominal sector consists of a nominal money process denoted by \(M = (M_t)\), where \(M_t\) is the dollar amount of total monetary aggregates held at time \(t\). We assume that aggregate consumption follows an exogenous processes of the form

\[
dC_t = C_t [\mu_c dt + \sigma_c dz_{c,t}] \tag{1.2.1}
\]

All the uncertainty in the real sector of this economy is driven by the Brownian motion process \(z_{c,t}\) which proxies for real shocks that affect consumption.

As in Woodford (1995), \(M_t\) corresponds to investor \(i\)’s direct or indirect holdings of the monetary base, namely, the sum of her currency holdings and deposits. We indistinctly refer to \(M_t\) as the
nominal money supply. Furthermore, we assume that $M_t$ is governed by the following stochastic process

$$dM_t = M_t [x_t dt + \sigma_m dz_{m,t}]$$

(1.2.2)

where $x_t$ denotes the expected growth rate of money. We assume that $x_t$ follows an Ornstein-Uhlenbeck process given by

$$dx_t = \kappa (\bar{x} - x_t) dt + \sigma_x dz_{x,t}$$

(1.2.3)

Thus, $x_t$ drives the variation in the conditional expectation of the money supply growth. The Brownian motions $z_{c,t}$, $z_{m,t}$ and $z_{x,t}$ are uncorrelated by assumption. By assumption, the investor does not observe $x_t$, although she knows it follows an Ornstein-Uhlenbeck process. Moreover, she observes the persistence parameter $\kappa$ and long-run mean $\bar{x}$, but does not observe the true volatility parameter $\sigma_x$. As we argue later on, $\kappa$ plays a crucial role in the relationship between disagreement about money growth and the asset prices.

Throughout this paper we refer to the system of Eqs. (1.2.2)-(1.2.3) as “monetary policy”. Furthermore, we indistinctly refer to Eq.(1.2.2) as the “unobservable”, “underlying”, or “latent” “state of the economy”. The expected growth rate of nominal money supply is stochastic and stationary. The exogeneity assumption is related to the monetarist literature, whereby traditionally the government exogenously specifies the path of money supply (see Woodford (1995) and references therein). We remit the reader to section (1.2.5) for a more thorough discussion on the appropriateness of our notion of a monetary policy stance based on the model above.

### 1.2.2 Public signal and money growth

We introduce a public source of information similar to the one in Dumas et al. (2009); Xiong and Yan (2010), but adapted to a monetary economy where the unobservable variable is the expected growth rate of money. Concretely, the investor observes a public signal, $s_t$, that may convey information about $x_t$. The investor believes that the signal obeys the following stochastic differential equation

$$ds_t = \alpha_i dz_{x,t} + \sqrt{1 - \alpha_i^2} dz_{s,t}$$

(1.2.4)

where $z_{s,t}$ is a Brownian motion uncorrelated with $z_{c,t}$, $z_{m,t}$ and $z_{x,t}$. Investor $i$ believes that the signal $s_t$ has a correlation parameter $\alpha_i \in [0, 1]$ with the unobservable process $x_t$ through their common dependence on the Brownian motion $z_{x,t}$. Thus, $\alpha_i$ is a subjective belief about the informativeness of the signal. When $\alpha_i \to 1$, investor $i$ believes that the signal is perfectly correlated with the innovation $z_{x,t}$, and, thereby, she can infer the unobservable state of the economy by
observing a realization of the signal, $s_t$. Conversely, when $\alpha_i \to 0$, investor $i$ believes that $s_t$ is pure noise. Therefore, she believes it conveys no relevant information.

Intuitively, this signal captures that, besides observed changes in monetary aggregates, the investor believes there are other sources from which she can learn about the unobservable process that drives money growth. The alternative sources could be related to statements released by central banks, speeches from board members, explicit forward guidance, among others. Different investors may have heterogeneous subjective beliefs about the degree of informativeness of the public signal. Accordingly, they may react differently to the flow of information conveyed by the signal (see section (1.3) for more details). Importantly, given that $\alpha_i$ is a subjective belief by assumption, the monetary authority does not have control over the informativeness of the signal. Rather, it is taken for granted as an input.

**Optimal filtering**

Following standard filtering theory (see theorem 12.7, Liptser and Shiryaev, 2001), as new information arrives, the money supply dynamics under investor’s $i$ beliefs are

$$dM_t = M_t \left[ x_i^i dt + \sigma_m dz_{m,t}^i \right] \quad (1.2.5)$$

Investor $i$’s optimal estimator of the expected growth rate $x_i^i$ has dynamics

$$dx_i^i = \kappa (\bar{x} - x_i^i) dt + \hat{\nu}_m^i dz_{m,t}^i + \hat{\nu}_s^i ds_t \quad (1.2.6)$$

where $\hat{\nu}_m^i$ and $\hat{\nu}_s^i$ are defined below. From Eqs. (1.2.5) and (1.2.6)

$$dz_{m,t}^i = \frac{1}{\sigma_m} \left( \frac{dM_t}{M_t} - x_i^i dt \right)$$

Therefore, $z_{m,t}^i$ is a “surprise” component of the change in money supply growth relative to the estimate $x_i^i$ adjusted by $\sigma_m$. As new information about $M_t$ becomes available, investor $i$ updates her estimate $x_i^i$ according to Eq. (1.2.6). We assume that the investor observes the money supply process for a sufficiently long time such that the variance of her estimate, $(\hat{\nu}_m^i)^2$ has reached its steady state level. $^6$ Therefore

$$\hat{\nu}_m^i = \sigma_m \left( \sqrt{\kappa^2 + \left( \frac{\sigma_{x,i}}{\sigma_m} \right)^2 (1 - \alpha_i^2)} - \kappa \right) \quad (1.2.7)$$

$(\hat{\nu}_m^i)^2$ is investor $i$’s posterior variance, that is, a measure of the uncertainty she attaches to her estimate $x_i^i$. The parameter $\sigma_{x,i}$ corresponds to the investor’s prior estimate of the unobservable

$^6$See Scheinkman and Xiong (2003); Dumas et al. (2009); Xiong and Yan (2010); Ehling et al. (2018).
parameter $\sigma_x$. In turn, $\hat{\nu}_s^i$ measures investor $i$’s uncertainty about the informativeness of $s_t$ about $x_t$. It is given by

$$\hat{\nu}_s^i = \sigma_{x,i} \alpha_i \quad (1.2.8)$$

We let investor $i$’s total uncertainty about $x^i_t$ be denoted by $\hat{\nu}^i$ such that $\hat{\nu}^i \equiv \sqrt{(\hat{\nu}_m^i)^2 + (\hat{\nu}_s^i)^2}$. From this definition and Eqs. (1.2.7) and (1.2.8), it is readily seen that an increase in the correlation coefficient between the signal $s_t$ and $z_{x,t}$ has two opposite effects on $\hat{\nu}^i$. On the one hand, a higher $\alpha_i$ leads to an increase in $\hat{\nu}_s^i$ and total perceived uncertainty $\hat{\nu}^i$. On the other hand, $\hat{\nu}_m^i$ and $\hat{\nu}^i$ fall as $\alpha_i$ increases. Intuitively, higher values of $\alpha_i$ imply that the investor assigns a larger weight to the signal as a source of information about $x_t$ relative to her own updating model. Thus, optimally she decides to react less strongly to her own surprise $z_{m,t}^i$ and more to the $s_t$. In the extreme case when the investor believes that $\alpha_i = 1$, then $\hat{\nu}_m^i = 0$ and she would only update her estimate $x^i_t$ based on the public signal. On the other extreme, when $\alpha_i = 0$, the investor believes that the signal is pure noise and she optimally chooses to ignore her such that $\hat{\nu}_s^i = 0$. Finally, in contrast to $\alpha_i$, a higher estimate of $\sigma_{x,i}$ raise $\hat{\nu}^i$ through both, $\hat{\nu}_m^i$ and $\hat{\nu}_s^i$.

### 1.2.3 Single investor problem

We describe investor $i$’s optimization problem. She solves an infinite time horizon problem and derives utility from consumption and the real value of the monetary holdings, i.e., $m_t \equiv M_t / P_t$, where $P_t$ is the unit dollar price of the consumption good. At time $t$ the investor has the opportunity to invest in a money account with nominal rate of return of $R_t$. Her optimization problem can be then written as

$$\sup_{(C_t, m_t)} \mathbb{E} \left[ \int_0^\infty e^{-\rho t} U(C_t, M_t / P_t) \right]$$

s.t. $\mathbb{E} \left[ \int_0^\infty \xi_t \left( C_t + \frac{M_t}{P_t} R_t \right) dt \right] \leq W_0 \quad (1.2.9)$

where $\xi_t$ denote the state-price density.

The FOC of the problem with respect to consumption and money holdings are, respectively

$$e^{-\rho t} U_c(C_t, M_t / P_t) = y \xi_t$$
$$e^{-\rho t} U_m(C_t, M_t / P_t) = y \xi_t R_t$$

where $y$ is the Lagrange multiplier of the optimization problem. Combining both FOC, we obtain the optimality condition for the nominal interest rate

$$R_t = \frac{U_m(C_t, M_t / P_t)}{U_c(C_t, M_t / P_t)} \quad (1.2.10)$$

Andrade et al. (2019); Sastry (2021) also introduce monetary models where agents have heterogeneous priors.
The intuition for this result is straightforward: if an investor has an extra dollar, she can invest it in the money account for which she will earn a nominal dollar interest of $R_t$. She may use these earnings to purchase $R_t P_t$ units of the consumption good. This action would raise her marginal utility by $U_c(C_t, M_t) R_t P_t$. Otherwise, she can keep the same dollar and have her utility increased by $U_m(C_t, M_t) 1/P_t$. At the optimum, these increments must equate one another.

1.2.4 Equilibrium

An investment of one dollar at time $t$ in the money account generates a continuous payment stream at the rate of $R_t$ dollars over time. The corresponding real investment at time $t$ is $1/P_t$ and the real dividend at time $u$ is $R_u P_u$ for $u \geq t$. Using the state-price density $\xi_t$ to price this investment, the following relationship obtains

$$\frac{1}{P_t} = \mathbb{E}_t \left[ \int_t^\infty \frac{\xi_u R_u}{\xi_t P_u} du \right]$$

Equation (1.2.11) may be interpreted as the familiar asset pricing equation, where the real price of a unit of money $(1/P_t)$ equals the expected present value of its implicit future “dividends” $R_u P_u$. In this context, $\frac{R_u}{P_u}$ represents the real value of the transaction services provided by holding money.

By the definition of the state-price density $\xi_t$ and the optimality condition for $R_t$ (1.2.10), the previous expression becomes

$$\frac{1}{P_t} = \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(u-t)} \frac{U_m(C_u, m_u)}{U_c(C_t, m_t)} \frac{1}{P_u} du \right]$$

As pointed out by Basak and Gallmeyer (2001), Eq. (1.2.12) is a backward stochastic differential equation. Based on tractability considerations, we assume that the investor has logarithmic utility given by

$$U(C_t, m_t) = \varphi \ln(C_t) + (1 - \varphi) \ln(m_t), \quad 0 \leq \varphi \leq 1$$

where $\varphi$ is the share of expenditure on consumption. With this functional form, Eq. (1.2.12) becomes

$$\frac{1}{P_t} = \frac{1 - \varphi}{\varphi} \mathbb{E}_t \left[ \int_t^\infty e^{-\rho(u-t)} \frac{C_t}{M_u} du \right]$$

Re-arranging terms in the equation above

$$\frac{1}{P_t} = \frac{1 - \varphi}{\varphi} \frac{C_t}{M_t} \int_t^\infty \mathbb{E}_t \left[ e^{-\rho(u-t)} \left( \frac{M_u}{M_t} \right)^{-1} \right] du$$

Using the previous expression and the dynamics for $M_t$ and $x_t$ in Eqs. (1.2.2) and (1.2.3), we calculate the equilibrium price level and nominal short rate of this economy in the following proposition.

---

*McCallum and Nelson (2010) argue that utility can be approximately separable in money and other arguments.*
Proposition 1.2.1. The equilibrium price level and short nominal rate in the monetary economy without disagreement are given, respectively, by

\[
\frac{1}{P_t} = \frac{1 - \varphi}{\varphi} C_t \tag{1.2.14}
\]

\[
R_t = Q(x_t^i)^{-1} \tag{1.2.15}
\]

where \(Q(x_t^i) \equiv \int_t^\infty M(x_t^i, u - t) \, du\) and

\[
M(x_t^i, u - t) = \mathbb{E}_t \left[ e^{-\rho(u-t)} \left( \frac{M_t}{M_t} \right)^{-1} \right] = \exp \left\{ -A(u - t) - B(u - t) x_t^i \right\} \tag{1.2.16}
\]

\[
B(u - t) = \frac{1}{\kappa} \left( 1 - e^{-\kappa(u-t)} \right) \tag{1.2.17}
\]

\[
A(u - t) = \left( \bar{\sigma}_m^2 - \frac{(\hat{\nu})^2}{2\kappa^2} - \frac{\hat{x}^i}{\kappa} \right) [(u - t) - B(u - t)] + \frac{(\rho - \sigma_m^2)(u - t)}{4\kappa} B(u - t)^2 \tag{1.2.18}
\]

Finally, the nominal short rate \(R_t\) is non-negative.

From Eq. (1.2.16), it is immediately seen that \(M(x_t^i, u - t) \geq 0\) for any \(x_t^i\), which implies that \(Q(x_t^i)\) is non-negative.\(^9\) Re-arranging terms in Eq. (1.2.14) and combining with Eq. (1.2.15) we obtain the following expression

\[
\ln \left( \frac{M_t}{P_t} \right) = \ln (m_t) = \text{constant} + \ln (C_t) - \ln (R_t) \tag{1.2.19}
\]

Eq. (1.2.19) has the natural interpretation of a demand for real money holdings: higher levels of income (consumption in equilibrium) increase the demand for money, while high nominal interest rates decrease it. The relationship between \(R_t\) and \(m_t\) is pinned down by \(Q(x_t^i)\), which in turn is a function of \(x_t^i\). From Eq. (1.2.19), it is readily seen \(Q(x_t^i)\) is proportional to the demand of real money holdings per unit of income (normalized demand for money hereafter). Therefore, the normalized demand for money is in itself determined by the estimate \(x_t^i\). Suppose that investor \(i\) infers that \(x_t\) will be high, either due to a large realization of the nominal money supply, \(dM_t\), or signal flow \(ds_t\). According to Eq. (1.2.16), \(M(x_t^i, u - t)\) must be lower. Therefore, the investor optimally decides to reduce her real money holdings in period \(t\), such that \(Q(x_t^i)\) falls. For markets to clear, the nominal short rate \(R_t\) must rise to induce the investor to substitute cash balances today for (nominal) interest-bearing assets in the future. Thus, the nominal short rate in this economy bears the common interpretation of the “opportunity cost” of holding cash at hand.

Eq. (1.2.1) shows that, in equilibrium, the nominal short rate is endogenously determined from the optimality condition (1.2.10) and the processes (1.2.1), (1.2.2) and (1.2.3). From Eq. (1.2.15),

\(^9\)Observe that \(M(x_t^i, u - t)\) is similar to the equilibrium bond price in the seminal interest rate model of Vasicek (1977).
we observe that the estimate of the expected growth rate of money, $x_i^t$, drives the time-variation in $R_t$.\footnote{For the sake of conciseness, we choose not to state the explicit dependence of $R_t$ on $x_i^t$ unless necessary.} Moreover, given that $\frac{\partial Q(x_i^t)}{\partial x_i^t} < 0$, $R_t$ is an increasing function of $x_i^t$. Indeed, we later show that $x_i^t$ fully describes the dynamics of all nominal interest rates and expected inflation in this economy.

Next, we compute the dynamics of the price level, the short nominal rate, and the real and nominal state-price densities to expand our understanding of the relationship between money, inflation and short interest rates.

**Proposition 1.2.2.** The dynamics of the price level, nominal short rate, and the real and nominal state-price densities in the monetary economy without disagreement are given, respectively, by

$$
\frac{dP_t}{P_t} = \pi_t dt + \phi^P_c dz_{c,t} + \phi^P_{m,t} dz^{i}_{m,t} + \phi^P_{s,t} ds_t \tag{1.2.20}
$$

$$
\frac{dR_t}{R_t} = \mu^R_t dt + \phi^R_{m,t} dz^{i}_{m,t} + \phi^R_{s,t} ds_t \tag{1.2.21}
$$

$$
\frac{d\xi_t}{\xi_t} = -r dt - \theta_c dz_{c,t} \tag{1.2.22}
$$

$$
\frac{d\xi^S_t}{\xi^S_t} = -R_t dt - \theta^S_{m,t} dz^{i}_{m,t} - \theta^S_{s,t} ds_t \tag{1.2.23}
$$

where $\xi^S_t = \frac{\xi_t}{P_t}$ is the nominal state-price density. The expected inflation rate, $\pi_t$, and the diffusion terms $\phi^P_c$, $\phi^P_{m,t}$, and $\phi^P_{z,t}$ are given by Eqs. (1.6.5), (1.6.6), (1.6.7), and (1.6.8), respectively. The drift and diffusion terms $\mu^R_t$ and $\phi^R_{z,t}$ are given by Eqs. (1.6.9), and (1.6.10), respectively. The real short rate, $r$, is given by $r = \rho + \mu_c - \sigma^2_c$ and $\theta_c = \sigma_c$. Finally, the nominal short rate, $R_t$, and the nominal prices of risk $\theta^S_{m,t}$ and $\theta^S_{s,t}$ are given, respectively, by Eqs. (1.6.16), (1.6.14), and (1.6.15) in the Appendix.

The first noticeable result from Proposition 1.2.2 is that there are three risk factors in this economy: a real risk factor associated with consumption, $z_{c,t}$, and two nominal risk factors, $z^{i}_{m,t}$ and $s_t$, which are associated with investor $i$’s adapted shock and the public signal, respectively. Therefore, whenever $\alpha_i > 0$, the signal becomes another risk factor for which investor $i$ requires compensation. In turn, the real short rate, $r$, is constant and independent of the nominal parameter $\sigma_m$ and of any realization of $x_i^t$. Moreover, the economy’s single real price of risk, $\theta_c$, is constant, while the nominal risk factors do not affect the real state-price density, $\xi_t$. Therefore, the monetary sector does not affect consumption and other real variables the economy. This result is a consequence of the log-utility assumption, which equalizes the substitution and income effects. In turn, the real risk factor is not priced in by the nominal state-price density, $\xi^S_t$. Therefore, in the model’s equilibrium, the nominal and real sectors of the economy are determined separately.
Furthermore, the drift of the nominal state-price density must be equal to the short rate defined in Eq. (1.2.15). Accordingly, as we show in the appendix

\[ R_t = \rho - \sigma_m^2 + x_t^i + \frac{G(x_t^i)}{Q(x_t^i)} (\kappa (x_t^i) - \hat{\nu}_x^i) - \frac{(\hat{\nu}_x^i)^2}{2} \frac{H(x_t^i)}{Q(x_t^i)} \]

where \( G(x_t^i) \) and \( H(x_t^i) \) are defined when proving proposition (1.2.2) in the Appendix.\(^{11}\) The expression for \( R_t \) above is perhaps more intuitive than then one in proposition (1.2.1). For once, it is immediately seen that a higher time discount increases the nominal short rate, whereas a higher value of \( \sigma_m \) lowers it due to risk-aversion, as in standard models. Furthermore, \( x_t^i \) first impacts \( R_t \) in a linear fashion: when nominal money supply is expected to grow at a faster rate, it takes a larger \( R_t \) to induce the investor to invest in the money account. This effect is similar to the linear impact of an increase in the expected growth rate of consumption, \( \mu_c \), on the short real rate, \( r \), arising from the Euler equation.

The remaining terms are adjustments accounting for the stochastic and unobservable nature of \( x_t \). The first one of them is due to the mean-reversion of the estimate \( x_t^i \). For instance, a high value of \( x_t^i \) brings the process farther away from its long-run mean, \( \bar{x} \). As a result, \( x_t^i \) is pulled back towards \( \bar{x} \). The speed of mean-reversion is modulated by \( \frac{G(x_t^i)}{Q(x_t^i)} \), the negative of the elasticity of the demand for real cash balances to changes in \( x_t^i \). \( \frac{G(x_t^i)}{Q(x_t^i)} \) is bounded below by zero and above by \( \kappa^{-1} \). The mean-reversion of \( R_t \) is large enough for \( R_t \) to never become negative, even for very low values of \( x_t^i \).\(^{12}\) Second, the steady-state volatility \( \hat{\nu}_x^i \) reduces \( R_t \) due to the filtering problem that investor \( i \) faces when estimating \( x_t \). The last term comes from Jensen’s inequality. It is determined by with the steady-state posterior variance \( (\hat{\nu}_x^i)^2 \) and \( \frac{H(x_t^i)}{Q(x_t^i)} \). Therefore, \( R_t \) is bounded above by \( x_t^i \) plus a constant. Finally, due to the independent determination of the real and nominal sectors, \( \mu_c \) and \( \sigma_c \) do not influence the nominal short rate.

We also observe from the dynamics of the nominal interest rate and state-price density that their conditional volatility is a function of \( x_t^i \), and therefore, stochastic. Thus, \( x_t \) is not only a driver of the conditional expectation of the nominal sector, but also of its conditional variances. We leave a more thorough discussion about the expected inflation rate, \( \pi_t \), for the comparative analysis of the model in section (1.4). This result is not present in Croitoru and Lu (2015) where time-variation of the system arises only after the introduction of disagreement.

We calculate the instantaneous inflation risk-premium and the real and nominal yield curve in proposition (1.2.3).

\(^{11}\)It can be shown that \( \frac{G(x_t^i)}{Q(x_t^i)} \) and \( \frac{H(x_t^i)}{Q(x_t^i)} \) are bounded below by zero and above by \( \frac{1}{\kappa} \) and \( \frac{1}{\kappa^2} \), respectively. Moreover, they are inverted sigmoid functions of \( x_t^i \).

\(^{12}\)This mechanism is similar to the pull in the widely used interest rate model from Cox et al. (1985), where the interest rate can never become negative either. As can be seen from \( \phi^R_{m,t} \) and \( \phi^R_{s,t} \) in proposition (1.2.2), the stochastic nature of the volatility of the nominal interest rate is another trait that our model shares with Cox et al. (1985).
Proposition 1.2.3. The instantaneous inflation risk-premium in period \( t \), \( \text{IRP}_t \), and the real and nominal yield curves between periods \( t \) and \( \tau \), \( y_{t,\tau} \) and \( y^\$_{t,\tau} \), are given, respectively, by

\[
\text{IRP}_t = \sigma_c^2 \tag{1.2.24}
\]

\[
y_{t,\tau} = \rho + \mu_c - \sigma_c^2 \tag{1.2.25}
\]

\[
y^\$_{t,\tau} = -\log Q(x^i_t, \tau) + \log Q(x^i_t) \tag{1.2.26}
\]

where \( Q(x^i_t, \tau) \equiv \int_\tau^\infty \mathcal{M}(x^i_t, u - t) \, du \). Furthermore, the dynamics of the nominal yield curve are

\[
dy^\$_{t,\tau} = \mu y_{t,\tau} dt + \phi^y_{m,t,\tau} dz_{m,t} + \phi^y_{s,t,\tau} ds_t \tag{1.2.27}
\]

The diffusion terms \( \phi^y_{m,t,\tau} \) and \( \phi^y_{s,t,\tau} \) are equal to

\[
\phi^y_{m,t,\tau} = \frac{\hat{\nu}i_m}{\tau - t} \left( \frac{G(x^i_t, \tau)}{Q(x^i_t, \tau)} - \frac{G(x^i_t)}{Q(x^i_t)} \right) \tag{1.2.28}
\]

\[
\phi^y_{s,t,\tau} = \frac{\hat{\nu}i_s}{\tau - t} \left( \frac{G(x^i_t, \tau)}{Q(x^i_t, \tau)} - \frac{G(x^i_t)}{Q(x^i_t)} \right) \tag{1.2.29}
\]

where \( G(x^i_t, \tau) \equiv \int_\tau^\infty B_n(u - \tau) \, \mathcal{M}(x^i_t, u - t) \, du \).

It is readily seen from Eq. (1.2.24) that the instantaneous inflation risk-premium is positive. Hence, investing in the nominally risk-free money account is risky in real terms because the inflation rate is itself stochastic, as can be seen from Eq. (1.2.20). Nevertheless, the instantaneous inflation risk-premium is constant and independent of \( x^i_t \), as is the real yield curve, \( y_{t,\tau} \), for any \( \tau \geq t \). These results are a further consequence of the log-utility assumption, which implies that the cross-derivative of the utility function with respect to consumption and real money holdings is zero (Bakshi and Chen (1996)).

In contrast, the nominal yield curve is time-varying and fully characterized by realizations of \( x^i_t \), just like the nominal short rate. For \( \tau > t \), it follows from Eq. (1.2.26) and the definition of \( Q(x^i_t, \tau) \) in proposition (1.2.3) that \( \int_\tau^\infty \mathcal{M}(x^i_t, u - t) \, du < \int_\tau^\infty \mathcal{M}(x^i_t, u - t) \, du \). Therefore, \( y^\$_{t,\tau} > 0 \) for any realization of \( x^i_t \). Additionally, the diffusion terms of the nominal yield curve \( \phi^y_{m,t,\tau} \) and \( \phi^y_{s,t,\tau} \) are in turn functions of \( x^i_t \), such that the instantaneous volatility of the yield curve is heteroskedastic. It is difficult to state more general properties about the shape of the nominal yield curve and its dynamics. Instead, we rely on a comparative numerical analysis of the level and volatility of the nominal in the single-investor economy in section (1.4).

\[\text{The stochastic nature of the price level also implies that the Fisher relation between the short term nominal and real interest rates does not hold in this economy.}\]
1.2.5 Discussion on monetary policy in the model

So far we have abstained from discussing whether our model conforms to an appropriate framework for the study of monetary policy. Our implicit assumption is that the monetary system, given by Eqs. (1.2.2)-(1.2.3), constitutes a good proxy for the unobservable monetary policy stance. In a seminal paper, Taylor (1993) shows that the Federal Reserve conducts monetary policy by gradually adjusting the short-term nominal interest rate to changes in inflation and some measure of output gap. Since then, most works in the macroeconomics and macro-finance tradition develop models with a production sector that study monetary policy under a “Taylor rule”. In contrast to these works, our model is framed as an exchange economy where prices are determined endogenously, while consumption (output in equilibrium) and nominal money supply are exogenous. This assumption rules out any meaningful measure of output gap derived directly from the model.14

Instead, our model pertains to the monetarist and quantitative-theoretic tradition, according to which aggregate measures of money play an important role in the conduct of monetary policy. This literature emphasizes the importance of money as a monetary policy instrument over the nominal short rate.15 Works within this tradition argue that, to the extent that the relationship between money and prices or interest rates (i.e. the demand for real money holdings) is stable, monetary policy should be tied to money growth targets.16

As shown in Eq. (1.2.19), a demand for real money holdings arises naturally in our model: for a given level of consumption, the demand for money adjusts to changes in the nominal short rate. In turn, the nominal short rate responds to variations in the expected growth rate of money supply, $x_t$. Even though investor $i$ cannot directly observe $x_t$, she knows it follows an Ornstein-Uhlenbeck process, with stationary dynamics. Investor $i$’s estimate, $\hat{x}_t^i$, inherits the stationarity of $x_t$. Moreover, as argued in the previous section, we know that $R_t$ is bounded below by zero and above by $x_t^i$ plus a constant. Therefore, the demand for real money holdings is stable. Given that the conditional variation in expected inflation $\pi_t$ is in turn fully characterized by $x_t^i$, a stable and positive relationship between $R_t$ and $\pi_t$ also follows. Thus, our model is able to replicate at least

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14 In addition, it is unfeasible to introduce a Taylor rule in our model due to the estimation problem faced by the investors. Standard filtering theory from Liptser and Shiryaev (2001) requires a state-space specification where measure and state variables are already solved for. On the contrary, a Taylor rule establishes an explicit relationship between inflation and the nominal short rate, which are endogenous variables.

15 Lucas Jr and Nicolini (2015) argue that one driver of the shift towards the nominal short rate as a policy instrument was the deterioration of the stability of the empirical relations between monetary aggregates, prices, and interest rates. They construct an updated monetary aggregate that accounts for changes in regulation that affected the substitutability between cash and bank deposits. Their updated measure restores the stability of these empirical relations. Their work and Benati et al. (2021); Belongia and Ireland (2016, 2017, 2019, 2022) and Benati et al. (2021), among other recent works, bring the demand for money and modified monetary aggregates back to the academic debate about the conduct of monetary policy. Belongia and Ireland (2022) argue that an explicit money growth rule delivers similar results to those of standard interest-rate rules in terms of the stability of inflation and output.

16 See Friedman (1988), McCallum (1989), Ireland (2017), among many others
the first part of what Carvalho and Nechio (2014) call the basic principles underlying the Taylor rule: short nominal interest rates tend to increase with inflation.

The second part of the monetarist prescription on the conduct of monetary policy is that money aggregates should grow at a nearly constant rate (e.g. Milton Friedman’s k-percent rule). Nevertheless, as McCallum (1989) points out, in reality “money supply is not itself a controllable instrument but is instead, [...] a variable that the monetary authority can manipulate only indirectly and with some significant [...] errors.” (italics in original text). We argue that our assumption regarding the stochastic nature of money supply and the shock $z_{m,t}$ in Eq. (1.2.2) is then justified on these grounds. In turn, Woodford (1995) argues that, in the quantity-theoretic tradition’s models, the path of nominal money supply is exogenously determined by the government.

We also assume that the expected growth rate of money, $x_t$ is a stochastic and unobservable process. As discussed in previous sections, the assumption that $x_t$ varies over time makes the nominal interest rates and expected inflation time-varying and stochastic regardless of whether there is disagreement about money growth. This result is important because, as we argue in upcoming sections, the persistence of $x_t$ exerts substantial influence on the impact of disagreement on interest rates and inflation. As for the unobservable nature of $x_t$, McCallum and Nelson (2010) assert that, under some circumstances, money growth might produce more accurate signals about the unobservable monetary policy stance than interest-rate rules. For instance, money growth may contain valuable information about the unobservable natural rate of interest rates. They argue that monetary aggregates reveal relevant information for future aggregate demand that is not conveyed by current output and interest rates. While we only speculate that $x_t$ might be related to the natural rate of interest rates or other latent variables containing pertinent information for future macroeconomic outcomes, the filtering problem in section (1.2.2) is consistent with the mechanism described by McCallum and Nelson (2010). In particular, investors in our model learn about the unobservable process $x_t$, the sole driver of conditional variation in the economy, from the information revealed by money growth over time. When producing her estimate, the investor accounts for the mean-reversion of $x_t$, which influences her expectations about the future path of

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17 In reduced-form, the time-varying nature of $x_t$ may accommodate smooth changes in the policy stance of the monetary authority. Campbell et al. (2014); Song (2017) document significant shifts in the aggressiveness of monetary policy and inflation in the U.S. Malmendier et al. (2021) argue that decisions made by new appointees to the board of the directors of a Central Bank may be substantially influenced by their previous experiences about inflation.

18 Belongia and Ireland (2019) argue that interest-rate rules are of limited value to learn about the monetary policy stance when the zero lower bound becomes a binding constrain on nominal short rates, as has been the case since 2008.

19 Moreover, the asset pricing results in our model follow from the adjustment of the relative prices of real money holdings, consumption, and bonds to the estimate $x_t$. This result is consistent with the view of Allan Meltzer on the monetary transmission mechanism, which relies on changes in relative prices and imperfect information. See Ireland (2017) and references therein for a more thorough discussion of the vast work of Allan Meltzer on monetary policy and its implications for the current academic and policy debate.
money growth and interest rates.

Overall, we believe that our model constitutes a reasonably sensible starting point to investigate the asset pricing implications of disagreement about monetary policy, to the extent that money growth is indeed able to capture the latter. Of course, we acknowledge the limitations of the answers our model can provide given its simple setup as an exchange economy.

1.3 A monetary economy with disagreement

In this section, we introduce disagreement about monetary policy among investors. To this end, we assume the existence of two investors who have heterogeneous beliefs about the dynamics of the expected growth rate of the money \( x_t \). As before, we assume that both investors observe \( C_t \), with dynamics given by Eq. (1.2.1) and \( M_t \), having the incomplete information filtration \( \mathcal{F}_t^M \subset \mathcal{F}_t \), \( t \in [0, \infty) \). Furthermore, we assume that investors have equivalent probability measures \( \mathbb{P}^i, i = A, B \). We assume that both investors deduce \( \sigma_m \) from the quadratic variation of \( M_t \), but can only draw inferences about \( x_t \) from realizations of \( M_t \). Specifically, we assume that investors know that \( x_t \) follows an Ornstein-Uhlenbeck process as the one in Eq. (1.2.3), but they may have different priors about the volatility of the expected growth rate of money, which yields different estimates about the process \( x_t \). Put differently, we assume that investors agree on the long-run mean \( \bar{x} \) and the speed of mean reversion \( \kappa \) in Eq. (1.2.3), but may have different beliefs about \( \sigma_x \), the volatility of the unobservable expected growth rate of money. Investor \( A \) knows investor \( B \)’s estimate \( \sigma_{x,B} \) and vice versa. However, they believe that their own estimate is correct. That is, they “agree to disagree”. We refer to this model as the “disagreement economy” for the remainder of this paper.

1.3.1 Investor’s problem

Here, we recast the optimization problem (1.2.9) for investor \( i \) where \( i = A, B \). Now, the information set on which every investor conditions their decisions are based on their own priors and their estimates. The investor’s problem is very similar to the one from the single-investor

\[
\sup_{(C_{i,t}, M_{i,t})} \mathbb{E}^i \left[ \int_0^\infty e^{-\rho t} U \left( C_{i,t}, \frac{M_{i,t}}{P_t} \right) \right] \\
\text{s.t. } \mathbb{E}^i \left[ \int_0^\infty \xi_{i,t} \left( C_{i,t} + \frac{M_{i,t} R_t}{P_t} \right) dt \right] \leq W_{i,0} 
\]

Assuming log-utility again, the optimality conditions of the problem for investor \( i \) are

\[
y_i \xi_{i,t} = \varphi e^{-\rho t} C_{i,t}^{-1} \\
y_i \xi_{i,t} = (R_t)^{-1} (1 - \varphi) e^{-\rho t} m_{i,t}^{-1}
\]
where \( y_i \) is the Lagrange multiplier from the static budget constraint in investor \( i \)'s optimization problem (1.3.1).

### 1.3.2 Sharing rules derivation

As in Basak (2005) and Ehling et al. (2013), the equilibrium sharing rules are derived using a state-dependent representative investor with utility given by

\[
U(C_t, m_t, \lambda_t) = \max_{\{C_A,t + C_B,t = C_t, m_A,t + m_B,t = m_t\}} \left[ \varphi \ln(C_A,t) + (1 - \varphi) \ln(m_A,t) \right] + \\
\lambda_t \left[ \varphi \ln(C_B,t) + (1 - \varphi) \ln(m_B,t) \right]
\]

where \( \lambda_t > 0 \) is a stochastic welfare weight that captures the impact of disagreement among investors on risk sharing.

The equilibrium allocation must solve the representative investor’s problem above implying that

\[
\lambda_t = \frac{C^{-1}_{A,t}}{C^{-1}_{B,t}} = \frac{y_1 \xi_{A,t}}{y_2 \xi_{B,t}} 
\]  

(1.3.4)

where the second equality comes from the optimality individual condition in Eq. (1.3.2). For \( u > t \), the random variable \( \lambda_u \), weighted by \( \frac{y_2}{y_1} \), is the Radon-Nikodym derivative of investor B’s beliefs with respect to investor A’s beliefs; i.e., \( \frac{dP_B}{dP_A} \).

From Girsanov’s theorem, innovations of investors A and B are linked by

\[
dz_{m,t}^B = dz_{m,t}^A - \psi_t dt
\]

(1.3.5)

where \( \psi_t = \frac{x_t^B - x_t^A}{\sigma_m} \). \( \psi_t \) captures the level of disagreement between \( x_t^B \) and \( x_t^A \) per unit of uncertainty of the nominal money supply. We also refer to \( \psi_t \) as the “beliefs spread”.

As pointed out by Basak (2005), since \( \psi_t \) follows directly from the exogenous endowment process and investors’ priors, we may treat it as an exogenous parameter if we do not impose any equilibrium restrictions on it. In proposition (1.3.1) we compute the dynamics of \( \lambda_t \), the sharing rules, and the disagreement process.

**Proposition 1.3.1.** The dynamics of the stochastic weight \( \lambda_t \) are

\[
\frac{d\lambda_t}{\lambda_t} = \psi_t dz_{m,t}^A
\]

(1.3.6)

Applying Ito’s lemma to \( \psi_t \) and re-arranging terms, we compute the dynamics of the disagreement process

\[
d\psi_t = -\beta \psi_t dt + \nu_{\psi,m} dz_{m,t}^A + \nu_{\psi,s} ds_t
\]

(1.3.7)
where \( \beta = \frac{\kappa_e + \hat{\nu}_m^B}{\sigma_m}, \nu_{\psi,m} = \frac{\hat{\nu}_m^B - \hat{\nu}_m^A}{\sigma_m} \) and \( \nu_{\psi,s} = \frac{\hat{\nu}_s^B - \hat{\nu}_s^A}{\sigma_m} \), where \( \hat{\nu}_m^i \) and \( \hat{\nu}_s^i \) are defined in Eqs. (1.2.7) and (1.2.8), respectively, for \( i = A, B \).

The equilibrium consumption allocations for investors A and B are given by

\[
C_{A,t} = f(\lambda_t) C_t \tag{1.3.8}
\]
\[
C_{B,t} = (1 - f(\lambda_t)) C_t \tag{1.3.9}
\]

where \( f_t = f(\lambda_t) = \frac{1}{1 + \lambda_t} \). The equilibrium consumption allocations for investors A and B are given by

\[
m_{A,t} = f(\lambda_t) m_t \tag{1.3.10}
\]
\[
m_{B,t} = (1 - f(\lambda_t)) m_t \tag{1.3.11}
\]

The real state-price densities for investors A and B are, respectively

\[
\xi_{A,t}^A = e^{-\rho t} \left( \frac{f(\lambda_t)}{f(\lambda_0)} \right)^{-1} \left( \frac{C_t}{C_0} \right)^{-1} \tag{1.3.12}
\]
\[
\xi_{B,t}^B = e^{-\rho t} \left( \frac{1 - f(\lambda_t)}{1 - f(\lambda_0)} \right)^{-1} \left( \frac{C_t}{C_0} \right)^{-1} \tag{1.3.13}
\]

The stochastic weighting process \( \lambda_t \), which is proportional to the likelihood ratio of the model, captures the impact of disagreement about the money growth process on the equilibrium quantities. The uncertainty of \( \lambda_t \) is driven by the adapted nominal shock \( z_{m,t}^A \), while Eq. (1.3.5) links \( z_{m,t}^A \) and \( z_{m,t}^B \). The disagreement process, \( \psi_t \), obeys an Ornstein-Uhlenbeck with long-run mean equal to zero. The diffusion terms \( \nu_{\psi,m} \) and \( \nu_{\psi,s} \) depend on the steady-state posterior variances \( (\hat{\nu}_m^i)^2 \) and \( (\hat{\nu}_s^i)^2 \) for \( i = A, B \) (see section (1.2.2) for details on the derivation of the optimal filtering problem faced by investor \( i \)). In general, for disagreement to persist over time, it must be the case that either \( \hat{\nu}_m^A \neq \hat{\nu}_m^B \) or \( \hat{\nu}_s^A \neq \hat{\nu}_s^B \), which only holds if either \( \sigma_{x,A} \neq \sigma_{x,B} \) or \( \alpha_A \neq \alpha_B \), respectively. Therefore, investors must have different estimates about the volatility of \( x_t \) \( (\sigma_{x,A} \neq \sigma_{x,B}) \) or about the informativeness of the public signal \( s_t \) \( (\alpha_A \neq \alpha_B) \). Otherwise, the disagreement process \( \psi_t \) becomes deterministic, as can be seen from Eq. (1.3.7). For the remainder of this paper, unless stated otherwise, we assume that at least one of these two conditions holds such that \( \psi_t \) is stochastic.

Suppose that \( \sigma_{x,A} > \sigma_{x,B} \), which means that investor A has a higher prior estimate of the volatility of \( x_t \) than B. Therefore, \( \hat{\nu}_m^A > \hat{\nu}_m^B \) and \( \nu_{\psi,m} < 0 \). If investor A observes a large and positive \( dM_t \) that is difficult to reconcile with her previous estimate \( x_t^B \), she would interpret it as a large money “surprise” \( d\psi_t \). Consequently, she would revise her estimate \( x_t^A \) upwards, which brings \( d\psi_t \) down. Thus, investor A becomes relatively more optimistic about \( x_t \).\(^{20}\) Alternatively, suppose that \( \alpha_A = 0, \alpha_B > 0 \). Thus, investor B believes that the signal conveys relevant information

\(^{20}\)We adopt the term “optimistic” for convenience to indicate that the expected growth rate of money is expected to rise. We employ the term “pessimistic” conversely.
about $x_t$, whereas $A$ believes it is pure noise, which implies that $\nu_{\psi,s} = \hat{\nu}_t^B > \hat{\nu}_t^A = 0$. A positive $d_{st}$ then means that investor $B$ infers good news about the unobservable expected growth rate of money from the signal flow. Consequently, she revises her estimate $x_t^B$ upwards such that $d_{\psi_t}$ rises in turn. Accordingly, investor $B$ becomes relatively more optimistic about $x_t$.

### 1.3.3 Equilibrium

The asset pricing equation that values an investment of one dollar in real terms under the beliefs of investor $i$ is given by

$$\frac{1}{P_t} = \mathbb{E}_t^i \left[ \int_t^{\infty} \frac{\xi_{i,u} R_u}{\xi_{i,t} P_u} du \right], \ i = A, B \quad (1.3.14)$$

Combining the FOCs (1.3.2) and (1.3.3), we obtain the optimality condition for the nominal short rate $R_t$

$$R_t = \frac{1 - \varphi}{\varphi} C_{i,t} \left( \frac{M_{i,t}}{P_t} \right)$$

Inserting the previous equation into Eq. (1.3.14), we obtain an analogous equation to (1.2.13) under the beliefs of investor $i$ where the price level is determined by

$$\frac{1}{P_t} = \frac{1 - \varphi}{\varphi} \mathbb{E}_t^i \left[ \int_t^{\infty} e^{-\rho(u-t)} \frac{C_{i,t}}{M_{i,u}} du \right], \ i = A, B \quad (1.3.15)$$

Specifically, under the beliefs of investor $A$ the equation above becomes

$$\frac{1}{P_t} = \frac{1 - \varphi}{\varphi} \mathbb{E}_t^A \left[ \int_t^{\infty} e^{-\rho(u-t)} \frac{C_{A,t}}{M_{1,u}} du \right]$$

From this expression we can derive the equilibrium price level and short nominal interest rate for the economy with disagreement.

**Proposition 1.3.2.** The equilibrium price level and nominal short rate are, respectively

$$P_t = \frac{\varphi}{1 - \varphi} Q \left( x_t^A, x_t^B, \lambda_t \right)^{-1} \left( \frac{C_t}{M_t} \right)^{-1} \quad (1.3.16)$$

$$R_t = Q \left( x_t^A, x_t^B, \lambda_t \right)^{-1} \quad (1.3.17)$$

where $Q \left( x_t^A, x_t^B, \lambda_t \right) \equiv \int_t^{\infty} \left( f_t M \left( x_t^A, u - t; \nu^A, \hat{\nu}_t^A \right) + (1 - f_t) \mathcal{M} \left( x_t^B, u - t; \nu^B, \hat{\nu}_t^B \right) \right) du$ and

$$\mathcal{M} \left( x_t^A, u - t; \nu^A, \hat{\nu}_t^A \right) \equiv \mathbb{E}_t^i \left[ e^{-\rho(u-t)} \left( \frac{M_u}{M_t} \right)^{-1} \right] \text{ is defined in Eq. (1.6.17) for } i = A, B.$$

From the proposition above, $Q \left( x_t^A, x_t^B, \lambda_t \right)$ can be decomposed into two components

$$Q \left( x_t^A, x_t^B, \lambda_t \right) = f_t Q \left( x_t^A, \nu^A, \hat{\nu}_t^A \right) + (1 - f_t) Q \left( x_t^B, \nu^B, \hat{\nu}_t^B \right) \quad (1.3.18)$$
where \( Q(x_i^t) \equiv \int_t^\infty \mathcal{M}(x_i^t, u - t; \hat{\nu}^i, \hat{\nu}_x^i) \, du \) for \( i = A, B \). Suppose that investor \( B \) estimates a high \( x_B^t \) in period \( t \). For the reasons discussed in the single-investor economy in section (1.2), this means that \( Q(x_B^t) \) would be low. Then, from Eqs. (1.3.17) and (1.3.18), the nominal short rate in the disagreement economy rises in turn. \( Q(x_i^t) \) is proportional to the normalized demand for real cash balances in an economy entirely populated by investor \( i \). Therefore, \( Q(x_A^t, x_B^t, \lambda_t) \) is proportional to the weighted-average of the demand for money that would prevail in the single-investor economies of \( A \) and \( B \), where the weights of both investors are given by the sharing rules \( f_t \) and \( 1 - f_t \), respectively. To facilitate the comparison between the single-investor economy in section (1.2) and the disagreement economy in this section, table (1.1) draws a mapping between some of the auxiliary variables in the single-investor economy and their corresponding counterparty in the disagreement economy. The latter are, in turn, averages of hypothetical single-investor economies weighted by the sharing rules.

We proceed to compute the dynamics of the price level, nominal short rate, and the real and nominal state-price densities of the disagreement economy in proposition (1.3.3). Then, we compare the results to those of the single-investor economy laid out in section (1.2).

**Proposition 1.3.3.** The dynamics of the price level, nominal short rate, and nominal and real state-price densities are given, respectively, by

\[
\frac{dP_t}{P_t} = \pi_t dt + \phi_{c,t}^P dz_{c,t} + \phi_{m,t}^P dz_{m,t} + \phi_s^P ds_t (1.3.19)
\]

\[
\frac{dR_t}{R_t} = \mu_t^R dt + \theta_{m,t}^R dz_{m,t} + \theta_{s,t}^R ds_t (1.3.20)
\]

\[
\frac{d\xi_{A,t}^A}{\xi_{A,t}^A} = -r dt - \theta_{c,t} dz_{c,t} - \theta_{m,t} dz_{m,t} (1.3.21)
\]

\[
\frac{d\xi_{A,t}^S}{\xi_{A,t}^S} = -R_t dt - \theta_{m,t}^S dz_{m,t} - \theta_{s,t}^S ds_t (1.3.22)
\]

where \( \xi_{A,t}^S = \frac{\xi_{A,t}^A}{P_t} \) is the nominal state-price density. The expected inflation rate, \( \pi_t \), and the diffusion terms \( \phi_{c,t}^P, \phi_{m,t}^P, \) and \( \phi_s^P \) are given by Eqs. (1.6.29), (1.6.30), (1.6.31), and (1.6.32), respectively. The drift and diffusion terms \( \mu_t^R, \phi_{m,t}^R, \) and \( \phi_{s,t}^R \) are given by Eqs. (1.6.34), (1.6.35), and (1.6.36), respectively. The real short rate, \( r \), and the real prices of risk, \( \theta_c \) and \( \theta_{m,t} \), are given respectively by \( r = \rho + \mu_c - \sigma_c^2 \), \( \theta_c = \sigma_c \), and \( \theta_{m,t} = -(1 - f_t) \psi_t \). The nominal short rate, \( R_t \), and the nominal prices of risk \( \theta_{m,t}^S \) and \( \theta_{s,t}^S \), a vector of nominal prices of risk are given, respectively, by Eqs. (1.6.40), (1.6.41), and (1.6.42) in the Appendix.

There are similarities and differences between the single-investor economy and the disagreement economy above. First, the same three risk factors are present in the disagreement economy. Furthermore, the real short interest rate and the single real risk factor are also constant. However,

\[21\] We omit the explicit dependence of \( Q(x_i^t) \) on the posterior variances \( \hat{\nu}^i, \hat{\nu}_x^i \) for the sake of conciseness.
the nominal risk factor $A_{m,t}$ now affects the real-state price density, with time-varying price of risk measured by $\theta_{m,t}$. Intuitively, disagreement about the money growth process leads to beliefs about the state of the economy and sharing rules that fluctuate over time. Given that the sharing rules determine real cash balances and consumption allocations, the prices of the risky real assets of this economy (e.g. stocks) are indeed impacted by the level of disagreement in the economy. This result is also present in Ehling et al. (2018) even with log-utility.

Before continuing with the analysis the impact of disagreement on the dynamics of the variables in proposition (1.3.3), we discuss the “disagreement adjustment term” $\Psi(x^A_t, x^B_t, \lambda_t)$ defined in Eq. (1.6.21). We call it a “disagreement adjustment” because it affects the level of the nominal short rate, expected inflation, and the instantaneous inflation risk-premium. In turn, it impacts the diffusion terms of the dynamics of the price level, the nominal short rate, state-price density and yield curve. Suppose that $x^B_t > x^A_t$, such that $\psi_t > 0$. Thus, $\mathcal{M}(x^A_t, u - t; \hat{\nu}^A, \hat{\nu}^A) < \mathcal{M}(x^B_t, u - t; \hat{\nu}^B, \hat{\nu}^B)$ when $\hat{\nu}^A$ and $\hat{\nu}^B$ are of similar magnitude. By construction, it then follows that $D(x^A_t, x^B_t) < 0$. When $x^B_t < x^A_t$ and $\psi_t < 0$, following an analogous reasoning it can be shown that $D(x^B_t, x^B_t) > 0$. Therefore, regardless of the sign of $\psi_t$, when $x^A_t \neq x^B_t$ it is the case that $\psi_t D(x^A_t, x^B_t) < 0$. Furthermore, the disagreement adjustment term depends on the product of the sharing rules $(1 - f_t)$ $f_t$, which is maximized at $f_t = 0.5$, that is, when both investors have an equal share of the endowment processes.

We continue with the analysis of proposition (1.3.3) with the disagreement term in mind. We begin with the drift of the dynamics of $\xi^A_{t+1}$, given by Eq. (1.6.13). This drift must coincide with the nominal short rate in Eq. (1.3.17). From table (1.1) and Eq. (1.6.40), we observe that all but the last two terms of the nominal short rate in the disagreement economy correspond to weighted-averages of the nominal short rates in hypothetical single-investor economies. The last two terms are $\theta_{m,t}$, whose sign depends on $\psi_t$, and $\Psi(x^A_t, x^B_t, \lambda_t)$, which tends to lower the nominal short rate. Both of them are due to disagreement. They imply that, in general, the equilibrium nominal short rate in the disagreement economy does not equal the weighted-average of the nominal short rates of the single-investor economies. Unless, $\psi_t = 0$, this result is at odds with that of Xiong and Yan (2010). In fact, the presence of these two terms induce additional volatility of the nominal interest rate not present in Xiong and Yan (2010). In sum, the overall effect of disagreement on the nominal short rate is ambiguous.

Finally, we compare the diffusion terms in proposition (1.3.3) to those from weighted-averages of single-investor economies in table (1.1). Given that investors agree on the consumption process and the public signal (although they may have heterogeneous subjective beliefs about its informativeness), the disagreement adjustment term $\Psi(x^A_t, x^B_t, \lambda_t)$ does not appear in the diffusion terms associated with them. Therefore, the diffusion terms associated with consumption and public signal correspond to weighted-averages of hypothetical single-investor economies, as can be seen in table (1.1). On
the contrary, it appears in the diffusion terms associated with investor A’s adapted shock \( z_{m,t} \), even in the case of the real state-price density. Therefore, any effect of the beliefs spread \( \psi_t \) on the volatility of the variables of interest arises from \( \Psi (x_t^A, x_t^B, \lambda_t) \), \( \theta_{m,t} \), and \( z_{m,t}^A \). We revisit the analysis of the volatility of the short rate in section (1.4).

We conclude the description of the disagreement economy with proposition (1.3.4), where we characterize the equilibrium instantaneous inflation risk-premium and the real and nominal yield curves.

**Proposition 1.3.4.** In the economy with disagreement, the instantaneous inflation risk-premium in period \( t \), \( IRP_t \), and the real and nominal yield curves between periods \( t \) and \( \tau \), \( y_t, \tau \) and \( y_t^s, \tau \), are given, respectively, by

\[
IRP_t = \sigma_c^2 + \theta_{m,t} \left[ -\sigma_m - \frac{G(x_t^A, x_t^B, \lambda_t; \hat{\psi}_m)}{Q(x_t^A, x_t^B, \lambda_t)} + \Psi (x_t^A, x_t^B, \lambda_t) \right] \quad (1.3.23)
\]

\[
y_t, \tau = \rho + \mu_c - \sigma_c^2 \quad (1.3.24)
\]

\[
y_t^s, \tau = -\frac{\log Q(x_t^A, x_t^B, \lambda_t)}{\tau - t} + \frac{\log Q(x_t^A, x_t^B, \lambda_t)}{\tau - t} \quad (1.3.25)
\]

where \( Q(x_t^A, x_t^B, \lambda_t, \tau) = \int_{\infty}^{\tau} \left( \frac{M(x, u-\tau; \hat{\psi}^A, \hat{\phi}^A) + \lambda_t M(x, u-\tau; \hat{\psi}^B, \hat{\phi}^B)}{1 + \lambda_t} \right) du \). Furthermore, the dynamics of the nominal yield curve are

\[
dy_t, \tau = \mu_{t, \tau} dt + \varphi_{m, t, \tau} dz_{m, t} + \varphi_{s, t, \tau} dz_{s, t} \quad (1.3.26)
\]

where

\[
\varphi_{m, t, \tau} = \left( \frac{1}{\tau - t} \right) \left[ \frac{G(x_t^A, x_t^B, \lambda_t, \tau; \hat{\psi}_m)}{Q(x_t^A, x_t^B, \lambda_t, \tau)} - \frac{G(x_t^A, x_t^B, \lambda_t; \hat{\psi}_m)}{Q(x_t^A, x_t^B, \lambda_t)} \right]
\]

\[
\varphi_{s, t, \tau} = \left( \frac{1}{\tau - t} \right) \left[ \frac{G(x_t^A, x_t^B, \lambda_t, \tau; \hat{\psi}_s)}{Q(x_t^A, x_t^B, \lambda_t, \tau)} - \frac{G(x_t^A, x_t^B, \lambda_t; \hat{\psi}_s)}{Q(x_t^A, x_t^B, \lambda_t)} \right] \quad (1.3.27)
\]

where \( G(x_t^A, x_t^B, \lambda_t, \tau; \hat{\psi}_l) \) for \( l \in \{m, s\} \), and \( D(x_t^A, x_t^B, \tau) \) are given by Eqs. (1.6.43) and (1.6.44), respectively.

Contrary to the single-investor economy, the instantaneous inflation risk-premium is time-varying in the disagreement economy. Therefore, despite the log-utility assumption, investors in this economy require a compensation to invest in nominal bonds over real bonds. Such compensation varies with their estimates of the unobservable money growth process \( x_t \). It is readily seen that, as expected, \( \Psi (x_t^A, x_t^B, \lambda_t) \) affects the inflation risk-premium. However, its overall effect is multiplied by \( \theta_{m,t} \), whose sign depends on sign of \( \psi_t \) so there is an ambiguous prediction about the impact of \( \Psi (x_t^A, x_t^B, \lambda_t) \) on \( IRP_t \). Instead, suppose that \( \psi_t > 0 \) such that investor B is more optimistic.
about $x_t$ than $A$. Thus, $\theta_{m,t} \Psi (x^A_t, x^P_t, \lambda_t)$ would tend to be positive, thus increasing the inflation risk-premium. Intuitively, from the point of view of investor $A$, who is relatively more pessimistic about $x_t$ than $B$, it takes a higher inflation risk-premium to induce her to invest in nominal bonds. If $\psi_t < 0$, then the opposite result obtains. Moreover, if the economy experiences a positive shock to money supply growth, then investor $B$ would confirm her more optimistic view on $x_t$, thereby increasing her weight $1 - f_t$, which in turn affects $\theta_{m,t}$. This feedback exacerbates the effect of the initial bet about $x_t$ between investors $A$ and $B$.

As in the single-investor economy, the real yield curve is constant and flat. In contrast, it is immediately seen from Eq. (1.3.25) that the nominal yield curve is a function of disagreement. We draw a similar conclusion regarding its conditional volatility from the diffusion terms in Eqs. (1.3.27) and (1.3.28).

## 1.4 Calibration and comparative analysis

We calibrate the model to gain a better understanding of the equilibrium of the economies in sections (1.2) and (1.3). The set of parameters is chosen to be roughly consistent to those in Ehling et al. (2013, 2018). The model’s calibration is summarized in table (1.2). Throughout this section, we focus our analysis on three key parameters: $\kappa$, $\sigma_{x,i}$, and $\alpha_i$. The persistence parameter $\kappa$ is observed, whereas $\sigma_{x,i}$ is an estimate of the volatility of the unobservable process $x_t$. In turn, $\alpha_i$ is the subjective belief about the correlation between the public and $x_t$ according to investor $i$. Given the importance of these three parameters in our model, we allow them to vary across exercises. Consequently, they are not always set at the values in table (1.2).

In section (1.4.1) we first evaluate the impact of the parameters of interest on the level of the nominal interest rates and expected inflation, and the volatility of nominal interest rates in the single-investor economy. When studying the disagreement economy, we split the analysis in two. First, in section (1.4.2), we assume that investors $A$ and $B$ disagree about their estimates $\sigma_{x,i}$ but have the same beliefs about $\alpha_i$. Then, in section (1.4.2) we assume that investors disagree about $\alpha_i$ but have the same estimate $\sigma_{x,i}$.

### 1.4.1 Single-investor economy

In this section, we analyze the equilibrium of the single-investor economy characterized in Propositions (1.2.2) and (1.2.3). We focus on the estimate of the expected growth rate of money, $x^i_t$, because it is the sole driver of conditional variation in this economy. We calculate the equilibrium of the model for the variables of interest as a function of $x^i_t$. The selected domain of $x^i_t$ is fixed within two standard deviations below and above its unconditional mean, $\bar{x}$ in Eq. (1.6.2), for $u \gg t$.

Figure (1.1) depicts the nominal short rate, $R_t$, and expected inflation rate, $\pi_t$ as a function of $x^i_t$ across multiple values of three key parameters of interest: $\kappa$, $\sigma_{x,i}$, and $\alpha_i$. Regardless of the
parameter of choice, $R_t$ and $\pi_t$ are both increasing in $x^i_t$. Intuitively, when investor $i$ infers that money supply will grow at a larger pace in the future, she will lower her demand for real money holdings today relative to tomorrow. For markets to clear, $R_t$ must increase. From the optimality condition in Eq. (1.2.10), the interest to be earned in the future from the reduction in her cash holdings today relative to tomorrow. For markets to clear, $R_t$ must grow at a larger pace in the future, she will lower her demand for real money holdings regardless of the shocks that hit the economy. Furthermore, increasing the estimate for real money holdings to $x^i_t$, approaches zero and $R_t$ becomes approximately linear in $x^i_t$.

In turn, increasing $\kappa$, which is inversely related to the persistence of $x_t$, flattens the responsiveness of $R_t$ and $\pi_t$ to the estimate $x^i_t$. For high levels of $\kappa$, future values of the expected growth rate will likely be close to $\bar{\pi}$ and, thus, will be less influenced by the current latent state of the economy, $x_t$. In the limit, as $\kappa$ increases, $R_t$ and $\pi_t$ become flat because $x^i_t$ would be set at $\bar{\pi}$ at all times regardless of the shocks that hit the economy. Furthermore, increasing the estimate $\sigma_{x,i}$ of the volatility parameter $\sigma_x$ shifts the nominal short rate down across all $x^i_t$. Thus, in an economy where the investor has more uncertainty about the volatility of $x_t$, she will prefer to invest more money in the bank account. This effect results in lower values of $R_t$. Subsequently, she substitutes real cash balances for consumption, whose expected growth rate is constant and observable, thus raising expected inflation. Finally, the higher the level of $\alpha_i$, the higher the nominal interest rate $R_t$ across all $x^i_t$. When investor $i$ believes that the public signal conveys valuable information about $x_t$, she is less willing to invest in the money account. Thus, for markets to clear, the nominal short rate increases and expected inflation falls. Intuitively, according to Eq. (1.10), higher values of $\alpha_i$ reduce $\hat{\nu}_s$, the risk adjustment due to the unobservable nature of $x_t$. Although, $R_t$ increases in equilibrium, this effect is, however, economically negligible.

From Figure (1.2), we observe that the instantaneous volatility of the nominal short rate is an increasing function of $x^i_t$. Furthermore, the nominal short rate becomes more volatile as $\kappa$ rises. As for the uncertainty parameter $\sigma_{x,i}$, we know that more uncertainty unambiguously leads to higher posterior variance $(\hat{\nu}_m^i)^2$, which in turn shifts upwards the volatility curve of $R_t$ for any $x^i_t$. In turn, increases in $\alpha_i$ reduce the investor’s posterior variance $(\hat{\nu}_m^i)^2$, but instead increase $(\hat{\nu}_s^i)^2$. Thus, the investor responds more strongly to any realization of the public signal. In the equilibrium, the increase in $(\hat{\nu}_s^i)^2$ more than offsets the reduction in $(\hat{\nu}_m^i)^2$, and volatility rises with $\alpha_i$ for any $x^i_t$.

We plot in Figure (1.3) the nominal yield curve between $t$ and $\tau$ for two levels of $x^i_t$: low and high, which correspond to values below and above $\bar{\pi}$, respectively. The first important result is that the slope of the yield curve is a function of the estimate $x^i_t$. In particular, when $x^i_t$ is low relative to $\bar{\pi}$, the yield curve is upward-sloping, while it is downward-sloping for high values of $x^i_t$ relative to $\bar{\pi}$. Intuitively, when $x^i_t$ is low, the nominal short rate is concomitantly low for the reasons discussed above. In addition, given that the expected growth rate of money is mean-reverting, the investor expects that $x^i_t$ will be pulled up towards the long-run mean in $\tau > t$. Thus, the nominal interest rate between $t$ and $\tau$ is expected to rise in the future, such that the yield curve slopes upwards.
Larger deviations of $x_i^t$ from $\bar{x}$ imply stronger pulls towards the long-run mean, thereby raising its slope even more. Conversely, when $x_i^t$ is high, the investor expects the growth rate of money to be pulled down to $\bar{x}$ and the opposite result for the slope of the yield curve follows. We also observe that larger values of $\kappa$ make the yield curve less responsive to $x_i^t$. Furthermore, more uncertainty $\sigma_{x,i}$ shifts downward the yield curve for all $\tau \geq t$, and a higher degree of confidence $\alpha_i$ has the opposite effect. These three effects are analogous to those above alluding to the nominal short rate, so we do not reiterate the same discussion here for the sake of brevity.

We conclude this section by analyzing the instantaneous volatility of the yield curve between $t$ and $\tau$ for low and high levels of $x_i^t$ in Figure (1.4). The main takeaway is that the volatility is decreasing in the horizon of the yield curve, regardless of the parameter configuration. Moreover, higher estimates $x_i^t$ increase the volatility of the interest rates between $\tau$ and $t$. The impact of varying the three parameters of interest on the instantaneous volatility of the yield curve is analogous to that of the volatility of the nominal short rate.

1.4.2 Disagreement economy

In this section, we first explore the asset pricing implications of disagreement when one investor is more confident than the other one regarding the uncertainty parameter ($\sigma_{x,B} < \sigma_{x,A}$; section (1.4.2)). Then, we study a scenario where one investor believes that the public signal is informative, while the other one does not ($\alpha_B > 0, \alpha_A = 0$, section (1.4.2)). In either case, we perform comparative analysis of the disagreement economy described in section (1.3).

Investors A and B must have different priors about the volatility of the expected growth rate of money supply or the informativeness of the signal. If they had the same priors, then they would converge to the same beliefs about $x_t$ for any initial level of disagreement, as can be seen in Eq. (1.3.7). For the remainder of this paper, without loss of generality, we adopt the convention that investor B is either more confident, in which case $\sigma_{x,B} < \sigma_{x,A}$, or assigns a larger credibility to the signal, such that $\alpha_B > \alpha_A$. Furthermore, as argued before, $\psi_t$ follows directly from the exogenous endowment process and investors’ priors and, hence, it can be treated as an exogenous variable. Additionally, conditional on information up to period $t$, the stochastic weighting $\lambda_t$ is itself a state variable that determines investors’ consumption and real money holdings allocations through the sharing-rule $f(\lambda_t)$, which endogenously fluctuates between zero and one. Thus, in the disagreement economy, the set of state variables expands to include $\psi_t$ and $f(\lambda_t)$. In the parameterized economy, we construct a grid for $\psi_t$ within two standard deviations below and above zero, the unconditional mean of $\psi_t$ (see Eq. (1.3.7)). In turn, we pick key values for the sharing rule such that $f(\lambda_t) \in \{0.1, 0.5, 0.9\}$. 

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No public signal

We first assume there is no public signal and $\sigma_{x,A} > \sigma_{x,B}$, that is, investors have different estimates about the volatility of the process $x_t$. Figure (1.5) depicts the impact of disagreement $\psi_t$ on the disagreement adjustment term $\Psi(x^A_t, x^B_t, \lambda_t)$ and the nominal short rate $R_t$ for three different sharing rules. The first result is that the adjustment term is an inverted parabola in $\psi_t$. When $\sigma_{x,B}$ is close to $\sigma_{x,A}$, its vertex is approximately zero, meaning that $\Psi(x^A_t, x^B_t, \lambda_t)$ is decreasing in the absolute value of disagreement, namely, $|\psi_t|$. Otherwise, $\Psi(x^A_t, x^B_t, \lambda_t)$ tends to fall as disagreement increases either because either investor $B$ is becoming more optimistic relative to investor $A$, or vice versa. Additionally, for a given level of disagreement, $\Psi(x^A_t, x^B_t, \lambda_t)$ is largest (in absolute value) when the sharing rules are equal, i.e. $f(\lambda_t) = 0.5$.

In contrast, the impact of the disagreement on the nominal short rate is ambiguous. The reason for this is that, as evidenced in Eq. (1.6.40), there are two components that determine the nominal short rate: the weighted-average of the nominal interest rates that would prevail in hypothetical single-investor economies, and the disagreement terms $\Psi(x^A_t, x^B_t, \lambda_t)$ and $\theta_{m,t}$. The impact of disagreement in the former is ambiguous. For instance, fixing $x^A_t$ and then increasing (decreasing) $x^B_t$ such that $\psi_t$ rises (falls) in turn, leads to a high (low) $R_t$. The decrease in $\Psi(x^A_t, x^B_t, \lambda_t)$ that tends to follow an increase in $\psi_t$ is then dominated.

In Figure (1.6) we study the effect of disagreement on instantaneous volatility of the nominal short rate and inflation risk-premium. First, we observe that the volatility of $R_t$ depicts a parabola in $\psi_t$, with its vertex slightly shifted to the left of $\psi_t$. Its shape resembles that of the disagreement adjustment $\Psi(x^A_t, x^B_t, \lambda_t)$, except that now it is (mostly) increasing in the absolute value of $\psi_t$. Therefore, a larger level of disagreement in either direction tends to make the nominal short rate more volatile. As before, the impact of disagreement on volatility is maximum when $f(\lambda_t) = 0.5$, for any given $\psi_t$. The mechanism behind this result is the usual one in the literature of heterogeneous beliefs. Upon the observation of a positive shock that raises the nominal money supply, the more optimistic investor (i.e. the one with the largest estimate $x^i_t$) would confirm her bullish view on $x_t$ and, hence, would bet on a rise of interest rates. The weight of the more optimistic investor in the economy would endogenously grow. This feedback effect would in turn raise the nominal interest rate in the economy. Furthermore, the larger the weight of one of the investors in the economy, the more the equilibrium asset prices would resemble those of a single-investor economy. Thus, the views of the dominant investor are proportionally more influential in the determination of asset prices than those of the other investor. The effect of the endogenous wealth fluctuation is then

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23 Alternatively, we could have assumed that $\alpha_A = \alpha_B = 0$ and the same results would follow because investors would optimally choose to ignore the signal. This assumption of different estimates about the volatility of the unobservable process is similar to that of Ehling et al. (2018).

24 The vertex is tilted towards the left of $\psi_t = 0$ because, by assumption, $\sigma_{x,A} > \sigma_{x,B}$.  

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more pronounced when the investors’ weight are the same.

As for the instantaneous inflation risk-premium, it is an increasing function of $\psi_t$, while it is constant in the single-investor economy. To understand this result, observe that when $\psi_t > 0$, investor $A$ is more optimistic about $x_t$ than $B$. In turn, consistent with her optimistic view on $x_t$, she expects higher interest rates in the future than investor $A$. Thus, investor $B$ expects to earn more interest from investing in the money market than investor $A$. For the money market to clear, investor $A$ requires an inflation-risk premium for investing in nominal short bonds. Otherwise, she would prefer to invest in riskfree real short bonds that remain unaffected by the beliefs spread. A higher value of $\psi_t$ would prompt investor $A$ to command a larger compensation in case investor $B$ turns out to be right. We also observe that the magnitude of the instantaneous inflation risk-premium decreases as $f(\lambda_t)$ rises. By assumption, investor $A$ has less certainty about the volatility parameter $\sigma_x$ than $B$, such that $\sigma_{x,A} > \sigma_{x,B}$. Thus, when her weight in the economy is low (e.g. $f(\lambda_t) = 0.1$), the instantaneous inflation-risk premium must be higher to induce her to invest in the bank account for markets to clear.

**Impact of disagreement under different levels of persistence of expected growth rate**

We investigate in this section the impact of the persistence of the expected growth rate of money, $\kappa$, on the relationship between disagreement and asset prices in the disagreement economy. In Figure (1.7) we depict the same four variables as before (disagreement adjustment $\Psi(x_t^A, x_t^B, \lambda_t)$, nominal short rate $R_t$, and instantaneous inflation risk-premium and nominal short rate volatility) as a function of $\psi_t$ across multiple values of $\kappa$. For the sake of space, we only report the plots when $f(\lambda_t)$ is fixed at 0.5, but the results remain unaffected across different sharing rules. We first observe that both, disagreement adjustment term and the instantaneous volatility of the nominal short rate, become less responsive to changes in $\psi_t$ as $\kappa$ increases. Furthermore, the response of the instantaneous inflation-risk premium to $\psi_t$ is also lower for high values of $\kappa$, while it is ambiguous in the case of the nominal short rate.

This argument can be stated formally by looking at Eq. (1.6.1). For $u > t$, low values of $\kappa$ imply that the term $e^{-\kappa(u-t)}$ in the conditional expectation of $x_u^i$ is large, thereby making $x_u^i$ highly dependent on the current estimate $x_t^i$. In contrast, when $\kappa$ is high $e^{-\kappa(u-t)}$, falls rapidly. Thus, the conditional expectation of $x_u^i$ becomes more heavily influenced by the long-run mean $\bar{x}$ than by $x_t^i$. Of course, investors would continue to “agree to disagree” in this scenario. They would still interpret realizations of the money supply growth through the lens of their own models and priors. However, their updated estimates would fluctuate over $\bar{x}$, a parameter they both agree on, as opposed to their own respective previous estimates, on which they have different views. Therefore, the feedback effect between investment decisions and endogenous fluctuations of wealth is lessened.

In Figure (1.8), we analyze the relation between money growth disagreement and the level and instantaneous volatility of the nominal yield curve across multiple values of $\kappa$. We do this for three
different levels of $\psi_t$: no disagreement, low and high belief spreads, where the last two correspond to two standard deviations below and above the unconditional mean of $\psi_t$ (i.e. zero disagreement), as can be seen from Eq. (1.3.7). Furthermore, we fix $x_t^A$ at 90% of its long-run mean. We observe that the slope of the yield curve depends on the level of disagreement and the persistence parameter. First, low levels of $\psi_t$ mean that $x_t^B$ is low. Thus, investor $B$ expects for $x_t$ to be pulled up towards $\bar{x}$ in the future, which implies that she expects high future interest rates. The result is an upward-sloping nominal yield curve between periods $\tau$ and $t$, regardless of the value of $\kappa$. Conversely, a downward-sloping nominal yield follows from a high value of $\psi_t$.

The intermediate case in Figure (1.8) illustrates an interesting situation. Concretely, in this scenario $x_t^B = x_t^A = 0.9\bar{x}$ and disagreement is equal to zero. In turn, both investors have estimates below their long-run mean $\bar{x}$. Given the mean-reversion of $x_t$, we might expected that nominal interest rates between $\tau$ and $t$ would rise. This is indeed the case for most values of $\kappa$ in the plot. However, for the lowest $\kappa$, the opposite result obtains. Thus, even though both investors believe that money growth will be higher in the future, the yield curve slopes downwards. We interpret this situation as follows: despite the bullish view on future monetary policy stance that both investors share, the high persistence of the unobservable monetary policy stance implies that the individual estimates $x_t^i$ will fail to be pushed up fast enough for long-term interest rates to rise. This situation is compounded by the fact that the estimates are close to their long-run mean.\footnote{This scenario bears some resemblance to the “Greenspan’s Conundrum”, whereby rising short-term interest rates are associated with falling long-term interest rates.}

**Heterogeneous interpretation of public signal**

In this section, we consider a situation where investor $B$ believes that the public signal, $s_t$, conveys relevant information about money growth, i.e. $\alpha_B > 0$, while investor $A$ believes it is not informative. Thus, investor $B$ is “steadfast” about her ability to partly infer $x_t$ from the public signal. From Eq. (1.2.6), investor $B$'s model then relies on the signal flow, realizations of the money supply, and her own priors to adjust her estimate of $x_t$. In contrast, investor $A$ is “doubtful” about the signal as she believes it conveys only noise and, consequently, decides to optimally ignore it. In this scenario, investors effectively employ different estimation models based on the same public information. We investigate to what extent the adoption of different estimation models, affect the equilibrium asset prices and volatility. To isolate the effect of having heterogeneous subjective beliefs about the informativeness of the signal, we assume for the remainder of this section that $\sigma_{x,A} = \sigma_{x,B}$.\footnote{This exercise is similar to that of Scheinkman and Xiong (2003); Dumas et al. (2009); Xiong and Yan (2010), among others. These papers assume that one investor is “overconfident” about the informativeness of a public signal relative to a “prudent” investor who, correctly, infers that the public signal is uninformative about the unobservable state of the economy. In the context of monetary policy, this exercise is related to the “signaling effects” documented}

Using table (1.2), we calculate the equilibrium in two economies: a “mixed economy” where
\( \alpha_B \in (0, 1), \alpha_B = 0, \) and a “doubtful economy” where \( \alpha_A = \alpha_B = 0. \) Then, for every case we calculate asset prices and volatility when the beliefs spread is zero, low, and high. We set the beliefs spread within two standard deviations below and above zero for the low and high cases, respectively.\(^{27}\) Afterwards, we compute the ratio of a variable of interest (e.g., the nominal short rate) in the mixed economy to its corresponding value in the doubtful economy. If the resulting ratio for \( R_t \) is equal to, say, 1.06, then \( R_t \) is 1.06 larger in the mixed-economy than in the benchmark homogeneous economy. We calculate these ratios for a grid of \( \alpha_B \) between zero and one across multiple values of \( \kappa. \) We plot the resulting ratios for the nominal short rate, \( R_t, \) and the 10-year nominal rate, \( y^{\$}_{t,t+10}, \) in Figure (1.9), and for the instantaneous volatility of the nominal short rate and 10-year nominal rate in Figure (1.10).

We first observe from Figure (1.9) that both the nominal short and 10-year rates are higher in the mixed economy than in the doubtful economy. Moreover, the ratio is increasing in \( \alpha_B, \) which means that the higher credibility investor \( B \) attaches to the signal, the larger the level of nominal interest rates relative to the benchmark economy. Given that investor \( B \) believes she can learn about \( x_t \) from \( s_t, \) she is less willing to invest in nominally riskfree assets. The nominal interest rates must rise in equilibrium for markets to clear. We also observe that the wedge between the economies is larger for low values of \( \kappa. \) However, the overall effect is economically small, regardless of the value of \( \kappa \) and the beliefs spread.

In Figure (1.4.2) we study the volatility of the nominal interest rates. The first noticeable result is that for a low \( \kappa, \) the instantaneous volatility of the nominal short and 10-year rates are smaller in the mixed-economy with \( \alpha_B > 0 \) than one where \( \alpha_B = 0. \) In fact, the wedge between both economies is larger than 10% for all \( 0 < \alpha_B < 1 \) and all three levels of the belief spread. In contrast, for a high level of \( \kappa, \) the opposite result obtains. As argued in section (1.2.2), two countervailing forces affect investor \( B \)’s filtering problem when she believes that the public signal conveys relevant information. On the one hand, from Eq. (1.2.6), higher \( \alpha_B \) will reduce her reliance on her own “surprise” \( z_{m,t}^B, \) thereby reducing the updated volatility estimates \( \hat{\nu}_x^B \) and \( \hat{\nu}_m^B. \) On the other hand, she will respond more strongly to any realization of \( s_t, \) thus increasing \( \hat{\nu}_s^B. \) Each one of these estimates increases volatility, so the overall impact of varying \( \alpha_B \) depends on which effect prevails in the end. Our results indicate that the answer hinges, once more, on the persistence of the underlying money growth process. Investor \( B \) updates \( dx_t^B \) when she receives the signal flow

\(^{27}\)To control for the beliefs spread, we assume that \( \psi_t \) is the same in the mixed economies and doubtful economies. Of course, a non-zero beliefs spread in the “\( \alpha \)-agreement economy” cannot be sustained indefinitely because \( \alpha_A = \alpha_B \) and \( \sigma_{x,A} = \sigma_{x,B}. \) Thus, both investors will converge to the same estimate eventually. Given that we perform comparative statics in this section, we temporarily ignore this property.
\( ds_t \) regardless of the value of \( \kappa \). However, when \( \kappa \) is low, the current state \( x_t^i \) carries significant influence on the magnitude of the revision. Therefore, \( dx_t^B \) does not respond as strongly to \( ds_t \). On the contrary, when the underlying money growth is less persistent, \( dx_t^B \) reacts strongly to any realization of the signal, thereby making the dynamics of \( x_t^B \) and, hence, the nominal interest rates more volatile. Finally, the effect of \( \alpha_B \) on the volatility wedge in either scenario is monotonic in \( \alpha_B \).

### 1.5 Final remarks and further discussion

We investigate the asset pricing implications of disagreement about the unobservable process that drives monetary policy. We present a monetary economy embedded into an exchange economy with exogenous paths for consumption and nominal money supply. The monetary economy is populated by two investors with heterogeneous priors about the unobservable expected growth of money, which we identify as the underlying monetary policy stance. The key model friction is that information about this unobservable variable is symmetric but incomplete. Investors estimate it in a Bayesian framework where they combine their priors and observed realizations of the money supply. Investors value holding real cash balances intrinsically, which gives rise to a demand for money. In equilibrium, the demand for money and investors’ estimates endogenously determine the nominal short rate and expected inflation for any given path of money supply. This is a portfolio rebalancing channel.

Disagreement about money growth drives the conditional variation in all nominal interest rates, expected inflation, and their volatility. Through a parameterized example, we show that the volatility of nominal interest rates tends to rise with increases in the absolute value of disagreement. In turn, the inflation risk-premium increases with the view of the more optimistic investor. The effect on the nominal interest rates is ambiguous. We show that the relationship between disagreement about money growth and asset prices crucially depends on the persistence of the unobservable growth rate of money. For instance, when the unobservable process is less persistent, disagreement about money growth has a weaker impact on asset prices.

Our model is also well-suited to address questions about the consequences of having heterogeneous interpretations about unconventional monetary policy on interest rates. To this end, we recast the model by introducing a public signal on which investors may have different views. Concretely, one “steadfast” investor believes the signal conveys relevant information about the unobservable process, while a “doubtful” one believes it is pure noise. The volatility of the nominal rates of an economy populated by one steadfast investor and one doubtful investor may be larger or smaller than the volatility of an economy with two doubtful investors. Whichever economy is more volatile depends, in turn, on the persistence of the unexpected growth rate of money.

Despite all reasonable caveats of our paper, we believe that our simple monetary model constitutes
a sensible starting point to investigate the asset pricing implications of disagreement about monetary policy in more general settings. In particular, the log-utility assumption makes the model tractable. We show that the impact of disagreement about money growth on nominal interest rates, expected inflation, and inflation risk-premium are independent of the log-utility assumption. However, due to this assumption the effects of disagreement on the real bond market are largely muted. As documented by Ehling et al. (2018), this is an economic relationship with strong empirical support. By considering more general risk-aversion configurations, the model may become a building block to accommodate a broader research question. For instance, it could include a properly defined real sector with investment on physical capital, which then could be used to investigate the impact of disagreement about money growth on the stock market. This is a topic that has received little attention in the literature.

Another avenue for future research is related to the study of disagreement about other type of policies such as “quantitative easing” (QE). As pointed out by Ireland (2017), the three rounds of QE were directed at raising the monetary base to prevent inflation from decreasing. Moreover, QE is supposed to work through a change in the relative price of the private sector’s portfolio, namely, a portfolio rebalancing channel (Bernanke (2013)). Both components, monetary growth and a portfolio rebalancing channel play a preponderant role in our model. Moreover, Beckworth (2017) argues that one crucial component of the effectiveness of QE is the ability of the Federal Reserve to commit to a permanent expansion of the monetary base. As we have argued throughout the paper, the persistence of the underlying monetary policy stance is a key determinant of our results. For all these reasons, we believe that our model can be extended to address this pertinent question.
1.6 Appendix

1.6.1 Tables and figures

Table 1.1: Variable correspondence between single-investor and disagreement economies

<table>
<thead>
<tr>
<th>Single-investor economy</th>
<th>Definition</th>
<th>Disagreement economy</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{M}(x_t^i, u - t)$</td>
<td>$E^t_1 \left[ e^{-\nu(u-t)} \left( \frac{M_t}{M_t} \right)^{-1} \right] !$</td>
<td>$\mathcal{M}(x_t^l, u - t; \hat{\nu}^l, \hat{\nu}^l_\lambda)$</td>
<td>$E^t_1 \left[ e^{-\nu(u-t)} \left( \frac{M_t}{M_t} \right)^{-1} \right] !$</td>
</tr>
<tr>
<td>$Q(x_t^i)$</td>
<td>$\int_x^\infty \mathcal{M}(x_t^i, u - t) du$</td>
<td>$Q(x_t^A, x_t^B, \lambda_t)$</td>
<td>$\frac{\int_x^\infty \mathcal{M}(x_t^A, u - t; \hat{\nu}^A, \hat{\nu}^A_\lambda)}{1 + \lambda_t} !$</td>
</tr>
<tr>
<td>$\mathcal{G}(x_t^i)$</td>
<td>$\int_x^\infty \mathcal{B}_\kappa (u - t) \mathcal{M}(x_t^i, u - t) du$</td>
<td>$G(x_t^i)$</td>
<td>$\int_x^\infty \mathcal{B}<em>\kappa (u - t) \mathcal{M}(x_t^i, u - t; \hat{\nu}^i, \hat{\nu}^i</em>\lambda) du$</td>
</tr>
<tr>
<td>$\mathcal{H}(x_t^i)$</td>
<td>$\int_x^\infty \mathcal{B}_\kappa (u - t)^2 \mathcal{M}(x_t^i, u - t) du$</td>
<td>$H(x_t^i)$</td>
<td>$\int_x^\infty \mathcal{B}<em>\kappa (u - t)^2 \mathcal{M}(x_t^i, u - t; \hat{\nu}^i, \hat{\nu}^i</em>\lambda) du$</td>
</tr>
<tr>
<td>$\mathcal{G}(x_t^i) (\kappa (\mathcal{F} - x_t^i) - \hat{\nu}^i_\lambda)$</td>
<td>$D(x_t^A, x_t^B)$</td>
<td>$\int_x^\infty \left[ \mathcal{M}(x_t^B, u - t; \hat{\nu}^B, \hat{\nu}^B_\lambda) - \mathcal{M}(x_t^A, u - t; \hat{\nu}^A, \hat{\nu}^A_\lambda) \right] du$</td>
<td></td>
</tr>
<tr>
<td>$\hat{\nu}^l_\lambda \mathcal{G}(x_t^i)$, for $l \in {m, s}$</td>
<td>$\Psi(x_t^A, x_t^B, \lambda_t)$</td>
<td>$(1 - f_t) \left( \frac{D(x_t^A, x_t^B)}{\hat{\mathcal{G}}(x_t^A, x_t^B, \lambda_t)} \right) \hat{\nu}^l_\lambda$</td>
<td></td>
</tr>
<tr>
<td>$(\hat{\nu}^l)^2 \mathcal{H}(x_t^i)$</td>
<td>$\mathcal{G}(x_t^A, x_t^B, \lambda_t; \hat{\nu}^l_\lambda)$</td>
<td>$f_t G(x_t^A) (\kappa (\mathcal{F} - x_t^i) - \hat{\nu}^A_\lambda) + (1 - f_t) \left( \kappa (\mathcal{F} - x_t^B) - \hat{\nu}^B_\lambda \right)$</td>
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Table 1.2: Parameter calibration

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>Time preference parameter</td>
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<tr>
<td>$f(\lambda_0)$</td>
<td>Initial cons/money hold. allocation</td>
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<tr>
<td>$\mu_c$</td>
<td>Expected cons. growth</td>
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<tr>
<td>$\sigma_c$</td>
<td>Volatility of cons.</td>
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<tr>
<td>$\sigma_m$</td>
<td>Money supply total vol.</td>
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</tr>
<tr>
<td>$\bar{x}$</td>
<td>Long-run mean</td>
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</tr>
<tr>
<td>$\kappa$</td>
<td>Mean reversion</td>
<td>0.5</td>
</tr>
<tr>
<td>$\sigma_x$</td>
<td>Vol. of expected money growth (true)</td>
<td>0.05</td>
</tr>
<tr>
<td>$\sigma_{x,A}$</td>
<td>Vol. of expected money growth for investor A</td>
<td>0.06</td>
</tr>
<tr>
<td>$\sigma_{x,B}$</td>
<td>Vol. of expected money growth for investor B</td>
<td>0.02</td>
</tr>
</tbody>
</table>
Figure 1.1: Nominal short rate and expected inflation as a function of $x_t^i$

Note: This figure plots the nominal short rate and expected inflation as a function of estimates the expected growth rate of money, $x_t^i$, in the single-investor economy described in section (1.2). The parameters come from table (1.2). The grid of $x_t^i$ is within two standard deviations below and above its unconditional mean, $\bar{x}$. See Eq. (1.6.1).
Figure 1.2: Nominal short rate instantaneous volatility as a function of estimates of expected growth rate of money, $x_i^t$

Note: This figure plots the instantaneous volatility of the nominal short rate, $R_t$, as a function of estimates of the expected growth rate of money, $x_i^t$, in the single-investor economy described in section (1.2). The parameters come from table (1.2). The grid of $x_i^t$ is within two standard deviations below and above its unconditional mean, $\bar{x}$. See Eq. (1.6.1).
Figure 1.3: Nominal yield curve as a function of estimates of expected growth rate of money, $x_t^i$

Note: This figure plots the level of the nominal yield, $y^i_{t,τ}$, across two values of the expected growth rate of money, $x_t^i$, in the single-investor economy described in section (1.2). The parameters come from table (1.2). Low and high values of $x_t^i$ are given by 0.5σ and 1.5σ, respectively.
Figure 1.4: Instant. volatility of nominal yield curve as a function of $x^i_t$

Note: This figure plots the volatility of the nominal yield, $y_{t,\tau}$, across two values of the expected growth rate of money, $x^i_t$, in the single-investor economy described in section (1.2). The parameters come from table (1.2). Low and high values of $x^i_t$ are given by $0.5\sigma$ and $1.5\sigma$, respectively.
Figure 1.5: Impact of disagreement on $\Psi (x^A_t, x^B_t, \lambda_t)$ and nominal short rate across multiple sharing rules

**Note**: This figure plots the disagreement adjustment, $\Psi (x^A_t, x^B_t, \lambda_t)$, and nominal short rate, $R_t$, across multiple beliefs spreads, $\psi_t$, and sharing rules, $f (\lambda_t)$, in the disagreement economy described in section (1.3). The parameters come from table (1.2). $x^A_t$ is fixed at 0.9. The grid of $\psi_t$ is within two standard deviations below and above its unconditional mean, zero. See Eq. (1.3.7).
Figure 1.6: Impact of disagreement on instant. volatil. and inflat. risk-premium across multiple sharing rules

(a) Instant. volatility of $R_t$: $f(\lambda_t) = 0.1$

(b) Instant. inflation risk-premium, IRP$_t$: $f(\lambda_t) = 0.1$

(c) Instant. volatility of $R_t$: $f(\lambda_t) = 0.5$

(d) Instant. inflation risk-premium, IRP$_t$: $f(\lambda_t) = 0.5$

(e) Instant. volatility of $R_t$: $f(\lambda_t) = 0.9$

(f) Instant. inflation risk-premium, IRP$_t$: $f(\lambda_t) = 0.9$

Note: This figure plots the instantaneous volatility of the nominal short rate and inflation risk-premium across multiple beliefs spreads, $\psi_t$, and sharing rules, $f(\lambda_t)$, in the disagreement economy described in section (1.3). The parameters come from table (1.2). $x\lambda^t$ is fixed at 0.9. The grid of $\psi_t$ is within two standard deviations below and above its unconditional mean, zero. See Eq. (1.3.7).
Figure 1.7: Impact of disagreement on $\Psi (x_t^A, x_t^B, \lambda_t)$ and inflation risk-premium across multiple $\kappa$

Note: This figure plots the disagreement adjustment, $\Psi (x_t^A, x_t^B, \lambda_t)$, and nominal short rate, $R_t$, across multiple beliefs spreads, $\psi_t$, and sharing rules, $f (\lambda_t)$, in the disagreement economy described in section (1.3). The parameters come from table (1.2). $x_t^A$ is fixed at 0.9. The grid of $\psi_t$ is within two standard deviations below and above its unconditional mean, zero. See Eq. (1.3.7).
Figure 1.8: Impact of disagreement on level and volatility of nominal yield curve across multiple $\kappa$

Note: This figure plots the level and instantaneous volatility of the nominal yield curve across multiple beliefs spreads, $\psi_t$, and sharing rules, $f(\lambda_t)$, in the disagreement economy described in section (1.3). The parameters come from table (1.2). $x^i_t$ is fixed at 0.97. Low and high $\psi_t$ correspond to, respectively, two standard deviations below and above its unconditional mean, zero. See Eq. (1.3.7).
Figure 1.9: Ratio of nominal short and 10-year rates in mixed to doubtful economies as $\alpha_B$ varies

Note: This figure plots the ratio of selected variables in two economies: a “mixed economy” where $\alpha_B \in (0, 1)$, $\alpha_B = 0$, and a “doubtful economy” where $\alpha_A = \alpha_B = 0$. A ratio is calculated for every variable across three values of $\psi_t$: low, null, and high. Low and high correspond to two standard deviations below and above zero, respectively. The ratios are calculated for a grid of $\alpha_B$ between zero and one. The selected variables are the nominal short rate and 10-year rate. The equilibrium corresponds to disagreement economy described in section (1.3). The parameters come from table (1.2).
Figure 1.10: Ratio of volatility of nominal short and 10-year rates in mixed to doubtful economies as $\alpha_B$ varies

Note: This figure plots the ratio of selected variables in two economies: a “mixed economy” where $\alpha_B \in (0,1)$, $\alpha_B = 0$, and a “doubtful economy” where $\alpha_A = \alpha_B = 0$. A ratio is calculated for every variable across three values of $\psi_t$: low, null, and high. Low and high correspond to two standard deviations below and above zero, respectively. The ratios are calculated for a grid of $\alpha_B$ between zero and one. The selected variables are the volatilities of the nominal short rate and 10-year rate. The equilibrium corresponds to disagreement economy described in section (1.3). The parameters come from table (1.2).
1.6.2 Proofs and auxiliary results

**Proof of Proposition 1.2.1**

**Price level:** The dynamics of investor $i$’s estimate of expected growth rate of money, $x^i_t$, follow a standard Ornstein–Uhlenbeck process. Thus, it has an explicit solution for $u \geq t$ of the form

$$x^i_u = e^{-\kappa(u-t)}x^i_t + \bar{\pi}(1 - e^{-\kappa(u-t)}) + \bar{\nu}_m \int_t^u e^{-\kappa(u-v)}dz^i_{m,v} + \bar{\nu}_s \int_t^u e^{-\kappa(u-v)}ds_v$$

The conditional moments of $x^i_u$ are given by

$$\mathbb{E}_t [x^i_u] = e^{-\kappa(u-t)}x^i_t + \bar{\pi}(1 - e^{-\kappa(u-t)}) \quad (1.6.1)$$

$$\text{var}_t [x^i_u] = \frac{1}{2\kappa}[(\bar{\nu}_m)^2 + (\bar{\nu}_s)^2] \left(1 - e^{-2\kappa(u-t)}\right) = \frac{(\bar{\nu}_m)^2}{2\kappa} \left(1 - e^{-2\kappa(u-t)}\right) \quad (1.6.2)$$

Furthermore, the dynamics of $M_t$ have an explicit solution for $u \geq t$ given by

$$\ln (M_u) = \ln (M_t) + \int_t^u x_vdv - \frac{\sigma^2_m}{2}(u-t) + \sigma_m (z_{m,u} - z_{m,t})$$

The conditional moments of $\ln (M_u)$ are computed using the Fubini’s theorem for stochastic integrals (see Munk (2013)), and Eqs. (1.6.1) and (1.6.2). Thus

$$\mathbb{E}_t [\ln (M_u)] = \ln (M_t) + \frac{1}{\kappa} \left(1 - e^{-\kappa(u-t)}\right) (x_t - \bar{\pi}) + \left(\bar{\pi} - \frac{\sigma^2_m}{2}\right)(u-t) \quad (1.6.3)$$

$$\text{var}_t [\ln (M_u)] = \frac{(\bar{\nu}_m)^2}{4\kappa^2} \left(e^{-2\kappa(u-t)} - 1\right) + \left(\frac{(\bar{\nu}_m)^2}{2\kappa} + \sigma^2_m\right)(u-t) \quad (1.6.4)$$

Let $\mathcal{M}(x^i_t, u-t) \equiv \mathbb{E}_t \left[e^{-\rho(u-t)} \left(\frac{M_u}{M_t}\right)^{-1}\right] = \exp \left\{-\rho(u-t) - \mathbb{E}_t \left[\ln \left(\frac{M_u}{M_t}\right)\right] + \frac{1}{2} \text{var}_t \left[\ln \left(\frac{M_u}{M_t}\right)\right]\right\}$ where the last equality follows from log-normality. Inserting the conditional moments (1.6.3) and (1.6.4) into this expression, we obtain $\mathcal{M}(x^i_t, u-t) = \exp \{ -\mathcal{A}(u-t) - \mathcal{B}_{\kappa}(u-t) x^i_t \}$ where $\mathcal{A}(u-t)$ and $\mathcal{B}_{\kappa}(u-t)$ are given by Eqs. (1.2.18) and (1.2.17), respectively. Inserting $\mathcal{M}(x^i_t, u-t)$ into Eq. (1.2.13), we conclude the proof of the first part of Proposition 1.2.1.

**Nominal short rate** The second part of the proposition follows from inserting the optimality condition (1.2.10) into the equilibrium price level in Eq. (1.2.13) and the definition of real money holdings, $m_t \equiv \frac{M_t}{P_t}$. Thus, we obtain that $R_t$ moves is inversely proportional to $Q(x^i_t)$.

**Proof of Proposition 1.2.2**
Dynamics of the price level: For convenience, let $q_t \equiv \frac{1}{q_t}$ such that Eq. (1.2.13) becomes

$$q_t = \frac{1 - \varphi}{\varphi} C_t Q(x_t)$$

or $q_t \equiv q(C_t, M_t, x_t^i)$. Before computing the dynamics of $q_t$, note that

$$\frac{\partial M(x_t^i, u - t)}{\partial x_t^i} = -B_{\kappa} (u - t) M(x_t^i, u - t) < 0; \quad \frac{\partial B M(x_t^i, u - t)}{\partial x_t^i} = B_{\kappa} (u - t)^2 M(x_t^i, u - t) > 0$$

Moreover

$$\frac{\partial Q(x_t^i)}{\partial x_t^i} = -G(x_t^i) < 0$$

where $G(x_t^i) \equiv \int_t^\infty B_{\kappa} (u - t) M(x_t^i, u - t) \, du$. Therefore

$$\frac{\partial q_t}{\partial x_t^i} = \left(1 - \varphi\right) C_t \frac{\partial Q(x_t^i)}{\partial x_t^i} = \frac{G(x_t^i)}{Q(x_t^i)} q_t$$

and using the dynamics for $q_t$ computed above

$$\frac{d q_t}{q_t} = \frac{dC_t}{C_t} - \frac{d M_t}{M_t} + \frac{\partial q_t}{\partial x_t^i} (dx_t^i) + \left(\frac{d M_t}{M_t}\right)^2 + \frac{1}{2} \frac{\partial^2 q_t}{\partial (x_t^i)^2} (dx_t^i)^2$$

$$= \mu_t^q dt + \phi_m^q dz_{c,t} + \phi_m^q dz_{m,t} + \phi_{s,t}^q ds_t$$

where $\mu_t^q \equiv \mu_c + \sigma_m^2 - x_t^i + \frac{G(x_t^i)}{Q(x_t^i)} \left(\hat{\nu}_t - \kappa (\tau - x_t^i)\right) + \frac{(\hat{\nu}_t)^2}{2} \frac{H(x_t^i)}{Q(x_t^i)}$, $\phi^q_c \equiv \sigma_c$, $\phi^q_{m,t} \equiv -\sigma_m - \hat{\nu}_m \frac{G(x_t^i)}{Q(x_t^i)}$, and $\phi_{s,t}^q \equiv -\hat{\nu}_s^i \frac{G(x_t^i)}{Q(x_t^i)}$.

Applying Ito’s lemma to $P_t = \frac{1}{q_t}$ and using the dynamics for $q_t$ computed above

$$\frac{d P_t}{P_t} = -\frac{d q_t}{q_t} + \left(\frac{d q_t}{q_t}\right)^2$$

$$= \pi_t dt + \phi^p_{c,t} dz_{c,t} + \phi^p_{m,t} dz_{m,t} + \phi_{s,t}^p ds_t$$

where

$$\pi_t \equiv \sigma_c^2 - \mu_c + x_t^i + \frac{G(x_t^i)}{Q(x_t^i)} (\kappa (\tau - x_t^i) + \hat{\nu}_t) - \frac{(\hat{\nu}_t)^2}{2} \frac{H(x_t^i)}{Q(x_t^i)} + \frac{(\hat{\nu}_t)^2}{2} \left(\frac{G(x_t^i)}{Q(x_t^i)}\right)^2$$

$$\phi^p_c \equiv -\sigma_c$$

$$\phi^p_{m,t} \equiv \sigma_m + \hat{\nu}_m \frac{G(x_t^i)}{Q(x_t^i)}$$

$$\phi_{s,t}^p \equiv \hat{\nu}_s \frac{G(x_t^i)}{Q(x_t^i)}$$

(1.6.5 - 1.6.8)
Dynamics of the short nominal rate: Applying Ito’s lemma to Eq. (1.2.15) and using the results from above when computing the dynamics of $q_t$

\[
dR_t = -R_t \left( \frac{\partial Q(x^t)}{\partial x^t} (dx^t) + \frac{1}{2} \frac{\partial^2 Q(x^t)}{\partial (x^t)^2} (dx^t)^2 \right) + R_t \left( \frac{\partial Q(x^t)}{\partial x^t} (dx^t) \right)^2
\]

\[
= \mu_t^R dt + \phi_{m,t}^R dz_{m,t} + \phi_{s,t}^R ds_t
\]

where

\[
\mu_t^R = \begin{bmatrix} \sigma_m^2 - \rho + R_t - x^t + \hat{\mu}_m^i \frac{G(x^t)}{Q(x^t)} + (\hat{\mu}_m^i)^2 \left( \frac{\sigma^2_m}{\sigma^2_m} \right) (dx^t)^2 \end{bmatrix}
\]

(1.6.9)

\[
\phi_{m,t}^R = \hat{\mu}_m \frac{G(x^t)}{Q(x^t)} R_t
\]

(1.6.10)

\[
\phi_{s,t}^R = \hat{\mu}_s \frac{G(x^t)}{Q(x^t)} R_t
\]

(1.6.11)

Dynamics of the state-price densities: We first compute the dynamics of the real state-price density. By definition

\[
\xi_t = e^{-\rho t} u_c(C_t, m_t)
\]

\[
= \phi e^{-\rho t} C_t^{-1}
\]

The dynamics are given by

\[
\frac{d\xi_t}{\xi_t} = -\rho dt - \left( \frac{dC_t}{C_t} \right) + \left( \frac{dC_t}{C_t} \right)^2
\]

\[
= -r_t dt - \theta_{c,t} dz_{c,t}
\]

where

\[
r_t = \rho + \mu_c - \sigma_c^2
\]

(1.6.12)

is the instantaneous real rate and $\theta_{c,t} \equiv \sigma_c$ is the real price of risk which only depends on “real variables”.

The dynamics of the nominal state-price density are computed from

\[
\xi_t^\$ = \xi_t q_t
\]

By Ito’s lemma

\[
\frac{d\xi_t^\$}{\xi_t^\$} = \left( \frac{d\xi_t}{\xi_t} \right) + \left( \frac{dq_t}{q_t} \right) + \left( \frac{d\xi_t}{\xi_t} \right) \left( \frac{dq_t}{q_t} \right)
\]

\[
= -R_t dt - \theta_{m,t}^\$ dz_{m,t} - \theta_{s,t}^\$ ds_t
\]
where the nominal short rate is given by

\[ R_t = r_t - \mu_{q,t} + \theta_{c,t} \phi_{q,t} \]  

(1.6.13)
and the vector of nominal prices of risk \( \Theta_t^s = [\theta_{m,t}^s, \theta_{s,t}^s]^t \) is given by

\[ \theta_{m,t}^s = \sigma_m + \hat{\nu}_m \frac{G(x_t^i)}{Q(x_t^i)} \]  

(1.6.14)

\[ \theta_{s,t}^s = \nu_s \frac{G(x_t^i)}{Q(x_t^i)} \]  

(1.6.15)

Inserting the explicit solutions we found above for \( r_t, \mu_{q,t}, \theta_{c,t}, \) and \( \phi_{q,t} \) into \( R_t \), we obtain an alternative expression for \( R_t \)

\[ R_t = \rho - \sigma_m^2 + x_t^i + \frac{G(x_t^i)}{Q(x_t^i)} (\kappa (\bar{x} - x_t^i) - \hat{\nu}_x^i) - \frac{(\hat{\nu}^i)^2}{Q(x_t^i)} \]  

(1.6.16)

**Proof of Proposition 1.2.3**

**Inflation risk-premium:** After noting that \((\phi_{c,t}^q)^2 + (\phi_{m,t}^q)^2 + (\phi_{s,t}^q)^2 = \text{var}_t \left( \frac{dP_t}{P_t} \right) = \text{var}_t \left( \frac{dP_t}{P_t} \right)\), we can re-arrange Eq. (1.6.5) as follows

\[ \mu_t^q = -\pi_t + \text{var}_t \left( \frac{dP_t}{P_t} \right) \]

By inserting this expression into Eq. (1.6.13) and re-arranging terms, we obtain

\[ \left[ R_t - \pi_t + \text{var}_t \left( \frac{dP_t}{P_t} \right) \right] - r_t = \theta_{c,t} \phi_{q,t} \]

The difference between the model implied real rate \( R_t - \pi_t + \text{var}_t \left( \frac{dP_t}{P_t} \right) \) and \( r_t \) is the inflation risk-premium. By letting IRP\(_t\) denote the instantaneous inflation risk-premium, it follows that \( \text{IRP}_t = \theta_{c,t} \phi_{q,t} = \sigma_c^2 \).

**Real yield curve:** The price of a real bond in \( t \) with maturity in \( \tau \) can be computed from the expression \( B_{t,\tau} = \mathbb{E}_t \left[ \frac{\xi_{\tau}}{\xi_t} \right] = \mathbb{E}_t \left[ e^{-\rho(\tau-t)} \left( \frac{C_t}{C_{\tau}} \right)^{-\gamma} \right] \). From log-normality and Eq. (1.2.1)

\[ B_{t,\tau} = e^{-(\rho + \mu_c - \sigma_c^2)(\tau-t)} \]

By letting the real yield curve between \( t \) and \( \tau \) be equal to \( y_{t,\tau} = -\frac{\log(B_{t,\tau})}{\tau-t} \) the result obtains.
Level of nominal yield curve: The price of a nominal bond in \( t \) with maturity \( \tau \) is given by \( B_{t,\tau} = \mathbb{E}_t \left[ \xi_t^\tau \right] = \mathbb{E}_t \left[ \frac{\xi_t^\tau B_t}{\xi_t} \right] \). We can manipulate the expression inside the conditional expectation using the price level from Eq. (1.2.13) and the real state-price density into this expression as follows

\[
\frac{\xi_t^\tau}{\xi_t} = \frac{\mathbb{E}_t \left[ \int_\tau^\infty e^{-\rho u} M_u^{-1} du \right]}{\mathbb{E}_t \left[ \int_\tau^\infty e^{-\rho u} M_u^{-1} du \right]} = \frac{\int_\tau^\infty \mathbb{E}_\tau \left[ e^{-\rho (u-t)} \left( \frac{M_u}{M_t} \right)^{-1} \right] du}{Q(x_t)}
\]

Inserting the expression above into the price of a nominal bond with maturity \( \tau \geq t \) at time \( t \)

\[
B_{t,\tau}^s = \mathbb{E}_t \left[ \int_\tau^\infty \mathbb{E}_\tau \left[ e^{-\rho (u-t)} \left( \frac{M_u}{M_t} \right)^{-1} \right] du \right] \left[ \frac{1}{Q(x_t)} \right] \int_\tau^\infty \mathbb{E}_t \left[ e^{-\rho (u-t)} \left( \frac{M_u}{M_t} \right)^{-1} \right] du
\]

where the last equality follows from the tower property of conditional expectations. Thus, the price of the nominal bond \( B_{t,\tau}^s \) has an explicit solution given by

\[
B_{t,\tau}^s = \frac{\int_\tau^\infty M(x_t^i, u-t) du}{\int_t^\infty M(x_t^i, u-t) du}
\]

The desired result obtains from the definition of the yield curve \( y_{t,\tau}^s = -\log \left( \frac{B_{t,\tau}^s}{\tau - t} \right) \) and by letting \( Q(x_t^i, \tau) \) be given by \( Q(x_t^i, \tau) \equiv \int_\tau^\infty M(x_t^i, u-t) du \) (with some abuse of notation).

Dynamics of nominal yield curve: From an application of Ito’s lemma to \( y_{t,\tau}^s \) and by letting \( G(x_t^i, \tau) \equiv \int_\tau^\infty B\kappa(u-\tau)M(x_t^i, u-t) du \), we obtain the diffusion terms. We omit the exact expression for the drift of the yield curve for the sake of briefness.

Proof of Proposition 1.3.1

Dynamics of stochastic weight: The Radon-Nikodym derivative of the model is the density function \( \lambda_u = \frac{dP^B}{dP^A} \). By Girsanov’s theorem, \( \lambda_u \) is given by the stochastic exponential

\[
\lambda_u = e^{w_u}
\]

where \( w_u \equiv \int_t^s \psi_u dz^A_{m,u} - \frac{1}{2} \int_t^s \psi_u^2 du \) is a standard Ito’s process adapted to \( z^A_{m,s} \). Equivalently

\[
dw_t = -\frac{1}{2} \psi_t^B dt + \psi_t dz^A_{m,t}
\]

Applying Ito’s lemma to \( \lambda_u \) and inserting \( dw_t \) into the resulting equation

\[
d\lambda_t = \frac{\partial \lambda_t}{\partial w_t} (dw_t) + \frac{1}{2} \frac{\partial^2 \lambda_t}{\partial w_t^2} (dw_t)^2
\]

\[
= \lambda_t \psi_t dz^A_{m,t}
\]
Equilibrium allocations: Inserting the optimality condition for investor $i$ in Eq. (1.3.2) into the feasibility constraint for total consumption, along with the stochastic weights $\lambda_t$ that we identified in Eq. (1.3.4) and re-arranging terms

$$C_t = C_{A,t} + C_{B,t}$$
$$\implies C_t = (\varphi^{-1}e^{rt} \xi_A y_1)^{-1} (1 + \lambda_t)$$

Using again Eq. (1.3.2) and re-arranging terms

$$C_{A,t} = f(\lambda_t) C_t$$

where $f(\lambda_t) \equiv \frac{1}{1+\lambda_t}$. Thus, we obtain Eqs. (1.3.8) and (1.3.9). Following a similar procedure for $m_t = \frac{M_t}{P_t}$ from the feasibility constraint (in real terms) $\frac{M_t}{P_t} = \frac{M_{A,t}}{P_t} + \frac{M_{B,t}}{P_t}$ and the optimality condition (1.3.3), we arrive at Eqs. (1.3.10) and (1.3.11). Finally, inserting the equilibrium consumption allocations into the individual optimality condition (1.3.2), we obtain the equilibrium real state-price densities

$$\xi_{A,t} = \varphi y_1 e^{-rt} \varphi f(\lambda_t)^{-1} C_t^{-1}$$

We set $y_1 = f(\lambda_0)^{-1} C_0^{-1}$ w.l.o.g. for $\xi_{A,0} = \varphi$ and $\xi_{B,0} = \varphi$. Thus, the previous expression becomes

$$\xi_{A,t} = \frac{\varphi y_1 e^{-rt} \varphi f(\lambda_t)^{-1} C_t^{-1}}{\lambda_0}$$

Furthermore, from the definition of the stochastic weight $\lambda_t$

$$y_2 = \frac{f(\lambda_0)^{-1} C_0^{-1}}{\lambda_0}$$

Inserting into the state-price density for investor B

$$\xi_{B,t} = \frac{e^{-rt} (1 - f(\lambda_t))^{-1} (C_t/C_0)^{-1}}{1 - f(\lambda_0)^{-1}}$$

Proof of Proposition 1.3.2

Price level: We start from the equation that determines the price level under the beliefs of investor A

$$\frac{1}{P_t} = \frac{1-\varphi}{\varphi} \mathbb{E}_t^A \left[ \int_t^{\infty} e^{-\rho(u-t)} \frac{C_{A,t}}{M_{1,u}} du \right]$$

Inserting the sharing rules we derived in Proposition 3.2 into the previous expression

$$\frac{1}{P_t} = \frac{1-\varphi}{\varphi} \mathbb{E}_t^A \left[ \int_t^{\infty} e^{-\rho(u-t)} \frac{f(\lambda_t) C_t}{f(\lambda_u) M_u} du \right]$$
Re-arranging terms and by Fubini’s theorem

\[
\frac{1}{P_t} = \frac{1 - \varphi f(\lambda_t) C_t}{\varphi M_t} \int_t^\infty e^{-\rho(u-t)} \left( E_t^A \left[ \left( \frac{M_u}{M_t} \right)^{-1} \right] + E_t^A \left[ \lambda_u \left( \frac{M_u}{M_t} \right)^{-1} \right] \right) du
\]

Changing the probability measure of the last term in the equation above to that of investor B by multiplying and dividing by \( \lambda_t \), and recalling that \( \lambda_u = \frac{\nu_u}{\nu_2} \, dB_t^{B} \) and \( \lambda_t = \frac{\nu_t}{\nu_2} \) for \( u > t \), it follows that

\[
\frac{1}{P_t} = \frac{1 - \varphi f(\lambda_t) C_t}{\varphi M_t} \int_t^\infty e^{-\rho(u-t)} \left( E_t^A \left[ \left( \frac{M_u}{M_t} \right)^{-1} \right] + \lambda_t E_t^A \left[ \left( \frac{\lambda_u}{\lambda_t} \right) \left( \frac{M_u}{M_t} \right)^{-1} \right] \right) du
\]

where \( M (x^i_t, u - t ; \hat{\nu}^i, \hat{\nu}^j_x) \equiv E_t^i \left[ e^{-(u-t)} \left( \frac{M_u}{M_t} \right)^{-1} \right] \) for \( i = A, B \). Following the same steps from the proof proposition (1.2.2), we can show that

\[
M (x^i_t, u - t ; \hat{\nu}^i, \hat{\nu}^j_x) = \exp \left\{ -A (u - t ; \hat{\nu}^i, \hat{\nu}^j_x) - B (u - t) x^i_t \right\}
\]

where \( A (u - t ; \hat{\nu}^i, \hat{\nu}^j_x) \) and \( B (u - t) \) are defined as in Eqs. (1.2.18) and (1.2.17) from Proposition 1.2.1, respectively for \( i = A, B \). Observe that, after introducing disagreement, we make the dependence of \( A (u - t ; \hat{\nu}^i, \hat{\nu}^j_x) \) and \( M (x^i_t, u - t ; \hat{\nu}^i, \hat{\nu}^j_x) \) on \( \hat{\nu}^i \) and \( \hat{\nu}^j_x \) explicit to emphasize that investors A and B have different confidence regarding their estimates of \( x_t \).

**Nominal short rate:** The proof of the nominal short rate is the same as the one in Proposition 1.2.1.

**Proof of Proposition 1.3.3**

**Dynamics of the price level:** Let \( q_t \equiv \frac{1}{P_t} \) and \( Q_t \equiv Q \left( x^A_t, x^B_t, \lambda_t \right) \) for convenience. Thus

\[
q_t = q \left( x^A_t, x^B_t, \lambda_t \right) = \frac{\varphi}{1 - \varphi} \left( \frac{C_t}{M_t} \right) Q \left( x^A_t, x^B_t, \lambda_t \right)
\]

Before calculating the dynamics of \( q_t \), we calculate the first and second derivatives of \( q_t \) with respect to its arguments. To this end, observe that

\[
\frac{\partial Q \left( x^A_t, x^B_t, \lambda_t \right)}{\partial x^A_t} = -\frac{1}{1 + \varpi \lambda_t} G \left( x^A_t \right), \quad \frac{\partial Q \left( x^A_t, x^B_t, \lambda_t \right)}{\partial x^B_t} = -\frac{\varpi \lambda_t}{1 + \varpi \lambda_t} G \left( x^B_t \right)
\]

where \( G \left( x^i_t \right) \equiv \int_t^\infty B \left( u - t \right) M \left( x^i_t, u - t ; \hat{\nu}^i, \hat{\nu}^j_x \right) du \). Additionally

\[
\frac{\partial^2 Q \left( x^A_t, x^B_t, \lambda_t \right)}{\left( \partial x^A_t \right)^2} = \frac{1}{1 + \varpi \lambda_t} H \left( x^A_t \right), \quad \frac{\partial^2 Q \left( x^A_t, x^B_t, \lambda_t \right)}{\left( \partial x^B_t \right)^2} = \frac{\varpi \lambda_t}{1 + \varpi \lambda_t} H \left( x^B_t \right)
\]
where \( H(x^1_t) \equiv \int_t^\infty B_\kappa (u-t)^2 \mathcal{M}(x^1_t, u-t; \hat{\nu}^2, \hat{\nu}^2) \, du \). Thus

\[
\begin{align*}
\frac{\partial q_t}{\partial x^1_t} &= -\frac{f_tG(x^1_t)}{Q(x^1_t, x^B_t, \lambda_t)} q_t, \quad \frac{\partial q_t}{\partial x^B_t} = -\left(1 - f_t \right) \frac{G(x^B_t)}{Q(x^A_t, x^B_t, \lambda_t)} q_t \\
\frac{\partial^2 q_t}{\partial (x^1_t)^2} &= \frac{f_tH(x^1_t)}{Q(x^1_t, x^B_t, \lambda_t)} q_t, \quad \frac{\partial^2 q_t}{\partial (x^B_t)^2} = \left(1 - f_t \right) \frac{H(x^B_t)}{Q(x^A_t, x^B_t, \lambda_t)} q_t
\end{align*}
\]

Furthermore, the derivatives of \( Q \) with respect to the likelihood ratio are

\[
\frac{\partial Q(x^A_t, x^B_t, \lambda_t)}{\partial \lambda_t} = \left(1 + \frac{1}{\lambda_t}\right)^2 D(x^A_t, x^B_t), \quad \frac{\partial^2 Q(x^A_t, x^B_t, \lambda_t)}{\partial \lambda^2} = -\frac{2}{(1 + \lambda)^3} D(x^A_t, x^B_t)
\]

where

\[
D(x^A_t, x^B_t) \equiv \int_t^\infty [\mathcal{M}(x^B_t, u-t; \hat{\nu}^2, \hat{\nu}^2) - \mathcal{M}(x^A_t, u-t; \hat{\nu}^A, \hat{\nu}^A)] \, du \quad (1.6.18)
\]

Similarly, the cross-derivatives are given by

\[
\begin{align*}
\frac{\partial^2 q_t}{\partial \lambda_t \partial x^1_t} &= \frac{f_t^B G(x^A_t)}{Q(x^A_t, x^B_t, \lambda_t)} q_t, \quad \frac{\partial^2 q_t}{\partial \lambda_t \partial x^B_t} = -\frac{f_t^B G(x^B_t)}{Q(x^A_t, x^B_t, \lambda_t)} q_t \\
\frac{\partial^2 q_t}{\partial \lambda_t \partial M_t} &= -\frac{f_t^B D(x^A_t, x^B_t)}{Q(x^A_t, x^B_t, \lambda_t)} M_t, \quad \frac{\partial^2 q_t}{\partial \lambda_t \partial C_t} = \frac{f_t^B D(x^A_t, x^B_t)}{Q(x^A_t, x^B_t, \lambda_t)} C_t \\
\frac{\partial^2 q_t}{\partial x^1_t \partial M_t} &= \frac{f_t G(x^A_t)}{Q(x^A_t, x^B_t, \lambda_t)} q_t, \quad \frac{\partial^2 q_t}{\partial x^B_t \partial M_t} = \frac{1 - f_t G(x^B_t)}{Q(x^A_t, x^B_t, \lambda_t)} M_t \\
\frac{\partial^2 q_t}{\partial x^1_t \partial C_t} &= -\frac{f_t G(x^A_t)}{Q(x^A_t, x^B_t, \lambda_t)} C_t, \quad \frac{\partial^2 q_t}{\partial x^B_t \partial C_t} = -\frac{1 - f_t G(x^B_t)}{Q(x^A_t, x^B_t, \lambda_t)} C_t
\end{align*}
\]

In turn, the instantaneous covariances of the stochastic processes that characterize the economy are summarized in the following table:

<table>
<thead>
<tr>
<th>( \frac{dC_t}{C_t} )</th>
<th>( \frac{dM_t}{M_t} )</th>
<th>( dx^A_t )</th>
<th>( dx^B_t )</th>
<th>( \frac{d\lambda_t}{\lambda_t} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{dC_t}{C_t} )</td>
<td>( \sigma^2_c )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{dM_t}{M_t} )</td>
<td>0</td>
<td>( \sigma^2_m )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( dx^A_t )</td>
<td>( \sigma_m \hat{\nu}_m^A )</td>
<td>( (\hat{\nu}_m^A)^2 + (\hat{\nu}_m^A)^2 = (\hat{\nu}_m^A)^2 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( dx^B_t )</td>
<td>( \sigma_m \hat{\nu}_m^B )</td>
<td>( \hat{\nu}_m^A \hat{\nu}_m^B + \hat{\nu}_s^A \hat{\nu}_s^B )</td>
<td>( (\hat{\nu}_m^B)^2 + (\hat{\nu}_s^B)^2 = (\hat{\nu}_s^B)^2 )</td>
<td></td>
</tr>
<tr>
<td>( d\lambda_t )</td>
<td>( x_t^B - x_t^A )</td>
<td>( \hat{\nu}_m^A \psi_t )</td>
<td>( \hat{\nu}_m^B \psi_t )</td>
<td>( \psi_t^B )</td>
</tr>
</tbody>
</table>
Armed with these results, we apply Ito’s lemma to $q_t$

$$dq_t = q_t \left( \frac{dC_t}{C_t} - \frac{dM_t}{M_t} \right) + \frac{1}{2} \frac{\partial^2 q_t}{\partial x_t^2} \left( dx_t^2 \right) + \frac{1}{2} \frac{\partial^2 q_t}{\partial x_t^2} \left( dx_t^2 \right) + \frac{1}{2} \frac{\partial^2 q_t}{\partial x_t^2} \left( dx_t^2 \right) - q_t \left( \frac{dC_t}{C_t} \right) \left( \frac{dM_t}{M_t} \right) +$$

After much algebra, we arrive at

$$dq_t = \mu_t^q dt + \phi_c^q dz_{c,t} + \phi_{m,t}^q ds_{m,t} + \phi_{s,t}^q ds_t \quad (1.6.19)$$

where the drift terms are given by

$$\mu_t^q \equiv -x_t^A + \mu_c + \sigma_m^2 - \frac{G \left( x_t^A, x_t^B, \lambda_t \right)}{Q \left( x_t^A, x_t^B, \lambda_t \right)} \left( x_t^A, x_t^B, \lambda_t \right) \frac{1}{2} \frac{H \left( x_t^A, x_t^B, \lambda_t \right)}{Q \left( x_t^A, x_t^B, \lambda_t \right)} +$$

$$\psi_t \left[ \frac{G \left( x_t^A, x_t^B, \lambda_t ; \hat{\nu}_m \right)}{Q \left( x_t^A, x_t^B, \lambda_t \right)} - \left( \sigma_m f_t \frac{D \left( x_t^A, x_t^B \right)}{Q \left( x_t^A, x_t^B, \lambda_t \right)} + \Psi \left( x_t^A, x_t^B, \lambda_t \right) \right) \right] \quad (1.6.20)$$

where

$$\Psi \left( x_t^A, x_t^B, \lambda_t \right) = \left( 1 - f_t \right) f_t \frac{D \left( x_t^A, x_t^B \right)}{Q \left( x_t^A, x_t^B, \lambda_t \right)} \psi_t \quad (1.6.21)$$

$$G \left( x_t^A, x_t^B, \lambda_t \right) \equiv f_t G \left( x_t^A \right) \left( \kappa \left( \bar{x} - x_t^A \right) - \hat{\nu}_t^A \right) + \left( 1 - f_t \right) \left( \kappa \left( \bar{x} - x_t^B \right) - \hat{\nu}_t^B \right) \quad (1.6.22)$$

$$G \left( x_t^A, x_t^B, \lambda_t ; \hat{\nu}_t \right) \equiv f_t G \left( x_t^A \right) \hat{\nu}_t^A + \left( 1 - f_t \right) G \left( x_t^B \right) \hat{\nu}_t^B \quad (1.6.23)$$

$$H \left( x_t^A, x_t^B, \lambda_t ; \hat{\nu}_t \right) \equiv f_t H \left( x_t^A \right) \left( \hat{\nu}_t^A \right)^2 + \left( 1 - f_t \right) H \left( x_t^B \right) \left( \hat{\nu}_t^B \right)^2 \quad (1.6.24)$$

Thus, $\Psi \left( x_t^A, x_t^B, \lambda_t \right)$ is a disagreement adjustment term, $G \left( x_t^A, x_t^B, \lambda_t \right)$ is an adjusted mean-reversion term, $G \left( x_t^A, x_t^B, \lambda_t ; \hat{\nu}_t \right)$ for $l = m, s$ is an uncertainty adjustment, and $H \left( x_t^A, x_t^B, \lambda_t ; \hat{\nu}_t \right)$ is a Jensen’s inequality term.

Furthermore, the diffusion terms are

$$\phi_c^q = \sigma_c \quad (1.6.25)$$

$$\phi_{m,t}^q = -\sigma_m - \frac{G \left( x_t^A, x_t^B, \lambda_t ; \hat{\nu}_m \right)}{Q \left( x_t^A, x_t^B, \lambda_t \right)} + \Psi \left( x_t^A, x_t^B, \lambda_t \right) \quad (1.6.26)$$

$$\phi_{s,t}^q = -\frac{G \left( x_t^A, x_t^B, \lambda_t ; \hat{\nu}_s \right)}{Q \left( x_t^A, x_t^B, \lambda_t \right)} \quad (1.6.27)$$

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Applying Ito’s lemma to \( P_t = \frac{1}{q_t} \), we obtain the dynamics of the price level
\[
\frac{dP_t}{P_t} = \pi_t dt + \phi^p_c dz_c,t + \phi^p_{m,t} dz_{m,t} + \phi^p_{s,t} ds_t
\]  
(1.6.28)
where the expected inflation (drift term) is given by
\[
\pi_t = -\mu_t^q + (\phi^q_{c,t})^2 + (\phi^q_{m,t})^2 + (\phi^q_{s,t})^2
\]  
(1.6.29)
and the diffusion terms by
\[
\begin{align*}
\phi^p_c &\equiv -\sigma_c \quad \text{(1.6.30)} \\
\phi^p_{m,t} &\equiv \sigma_m + \frac{G(x^A_t, x^B_t, \lambda_t; \hat{\nu}_m)}{Q(x^A_t, x^B_t, \lambda_t)} - \Psi(x^A_t, x^B_t, \lambda_t) \quad \text{(1.6.31)} \\
\phi^p_{s,t} &\equiv \frac{G(x^A_t, x^B_t, \lambda_t; \hat{\nu}_s)}{Q(x^A_t, x^B_t, \lambda_t)} \quad \text{(1.6.32)}
\end{align*}
\]

**Dynamics of the short nominal rate:** Making use of the relationship between \( R_t \) and \( Q_t = Q(x^A_t, x^B_t, \lambda_t) \) from proposition (1.3.2) and Ito’s lemma, we obtain
\[
dR_t = \mu^R_t dt + \theta^R_{m,t} dz_{m,t} + \theta^R_{s,t} ds_t
\]  
(1.6.33)
with drift terms given by
\[
\mu^R_t = R_t \left( \mu^Q_t - \left[ (\phi^Q_{m,t})^2 + (\phi^Q_{s,t})^2 \right] \right)
\]  
(1.6.34)
and diffusion terms by
\[
\begin{align*}
\theta^R_{m,t} &\equiv \frac{G(x^A_t, x^B_t, \lambda_t; \hat{\nu}_m)}{Q(x^A_t, x^B_t, \lambda_t)} R_t - \Psi(x^A_t, x^B_t, \lambda_t) R_t \quad \text{(1.6.35)} \\
\theta^R_{s,t} &\equiv \frac{G(x^A_t, x^B_t, \lambda_t; \hat{\nu}_s)}{Q(x^A_t, x^B_t, \lambda_t)} \quad \text{(1.6.36)}
\end{align*}
\]
To arrive at the previous results, we first calculated the dynamics of \( Q_t \) from Ito’s lemma
\[
dQ_t = \mu^Q_t dt + \phi^Q_{m,t} dz_{m,t} + \phi^Q_{s,t} ds_t
\]
where
\[
\begin{align*}
\mu^Q_t &\equiv -\left( f_t G(x^A_t) \kappa(x^A_t - x^B_t) + (1 - f_t) G(x^B_t) \kappa(x^A_t - x^B_t) \right) + \frac{1}{2} \frac{\mathcal{H}(x^A_t, x^B_t, \lambda_t; \hat{\nu})}{Q(x^A_t, x^B_t, \lambda_t)} + \\
&\quad \left( 1 - f_t \right) \psi_t \left[ \frac{G(x^A_t, x^B_t, \lambda_t; \hat{\nu}_m)}{Q(x^A_t, x^B_t, \lambda_t)} - \Psi(x^A_t, x^B_t, \lambda_t) \right] \\
\phi^Q_{m,t} &\equiv -\frac{G(x^A_t, x^B_t, \lambda_t; \hat{\nu}_m)}{Q(x^A_t, x^B_t, \lambda_t)} + \Psi(x^A_t, x^B_t, \lambda_t) \\
\phi^Q_{s,t} &\equiv -\frac{G(x^A_t, x^B_t, \lambda_t; \hat{\nu}_s)}{Q(x^A_t, x^B_t, \lambda_t)}
\end{align*}
\]
Dynamics of the real state-price density: The real state-price density of investor $A$ is given by

$$
\xi_{A,t} = y_1^{-1}\varphi e^{-\rho t} f(\lambda_t)^{-1} C_t^{-1} = y_1^{-1}\varphi e^{-\rho t} C_t^{-1} (1 + \lambda_t)
$$

By Ito’s lemma

$$
\frac{d\xi_{A,t}}{\xi_{A,t}} = -\rho dt - \left(\frac{dC_t}{C_t}\right) + \left(\frac{d\xi_{A,t}}{\xi_{A,t}}\right)\left(\frac{d\lambda_t}{\lambda_t}\right)
$$

where $r_t$ is the instantaneous real risk-free rate given by

$$
r_t \equiv \rho + \mu_c - \sigma_c^2
$$

and the prices of risk are

$$
\theta_c \equiv \sigma_c
$$

$$
\theta_{m,t} \equiv -(1 - f_t) \psi_t
$$

Dynamics of the nominal state-price density: The nominal state-price density is given by

$$
\xi^s_{A,t} = q_t \xi_{A,t}
$$

By Ito’s lemma

$$
\frac{d\xi^s_{A,t}}{\xi^s_{A,t}} = \left(\frac{d\xi_{A,t}}{\xi_{A,t}}\right) + \left(\frac{dq_t}{q_t}\right) + \left(\frac{d\xi_{A,t}}{\xi_{A,t}}\right)\left(\frac{ds_t}{s_t}\right)
$$

where the nominal short rate $R_t$ is

$$
R_t = \rho + x_1^1 - \sigma_m^2 + \frac{G(x^A_t, x^B_t, \lambda_t)}{Q(x^A_t, x^B_t, \lambda_t)} - \frac{1}{2} \frac{H(x^A_t, x^B_t, \lambda_t)}{Q(x^A_t, x^B_t, \lambda_t)} - \sigma_m \theta_{m,t} + \sigma_m \Psi(x^A_t, x^B_t, \lambda_t) \quad (1.6.40)
$$

In turn, the vector of nominal prices of risk, $\Theta^S_t \equiv [\theta^S_{m,t}, \theta^S_{s,t}]'$ is

$$
\theta^S_{m,t} \equiv \theta_{m,t} - \phi^q_{m,t} = \sigma_m + \frac{G(x^A_t, x^B_t, \lambda_t \nu_m)}{Q(x^A_t, x^B_t, \lambda_t)} - \Psi(x^A_t, x^B_t, \lambda_t) + \theta_{m,t} \quad (1.6.41)
$$

$$
\theta^S_{s,t} \equiv -\phi^q_{s,t} = \frac{G(x^A_t, x^B_t, \lambda_t \nu_s)}{Q(x^A_t, x^B_t, \lambda_t)} \quad (1.6.42)
$$

and $\mu^q_t, \theta_c, \theta_{m,t}, \phi^q_c$, and $\phi^q_{m,t}$ are given by Eqs. (1.6.20), (1.6.38), (1.6.39), (1.6.25), and (1.6.26), respectively.
Proof of Proposition 1.3.4

**Instantaneous inflation risk-premium:** This proof is very similar to the one of proposition (1.2.3). The only difference is that now we have to account for an additional term \( \theta_{m,t}\phi_{m,t}^q \) in Eq. (1.6.40). This additional term implies that the instantaneous inflation risk-premium in the model with disagreement is given by \( \text{IRP}_t \equiv R_t - \pi_t + \text{var}_t \left( \frac{\varphi_t}{\theta_t} \right) - r_t \). Thus

\[
\text{IRP}_t = \theta_{c,t}\phi_{c,t}^q + \theta_{m,t}\phi_{m,t}^q
= \sigma^2_c - (1 - f_t) \psi_t \left[ -\sigma_m - \frac{G(x_t^A, x_t^B, \lambda_t; \psi_t)}{Q(x_t^A, x_t^B, \lambda_t)} + \Psi(x_t^A, x_t^B, \lambda_t) \right]
\]

**Real yield curve:** We begin with the definition of the price in \( t \) of a real bond with maturity in period \( \tau \) under the beliefs of investor \( A \), that is \( B_{t,\tau} = \mathbb{E}_t^A \xi_{\Lambda,\tau} = \mathbb{E}_t^A \left[ e^{-\rho(\tau-t)} \left( \frac{f(\lambda_t)C_{t,\tau}}{f(\lambda_t)C_{t,t}} \right)^{-1} \right] \), where the last equality follows from Eq. (1.3.12). After inserting the sharing rule and re-arranging terms, the bond price becomes

\[
B_{t,\tau} = \frac{e^{-\rho(\tau-t)}}{1 + \lambda_t} \left( \mathbb{E}_t^A \left[ \left( \frac{C_{t,\tau}}{C_{t,t}} \right)^{-1} \right] + \lambda_t \mathbb{E}_t^A \left[ \frac{\lambda_t}{\lambda_t} \left( \frac{C_{t,\tau}}{C_{t,t}} \right)^{-1} \right] \right)
\]

Changing the probability measure of the last term in the equation above to that of investor \( B \) as we did when proving proposition (1.3.2), we obtain \( \mathbb{E}_t^A \left[ \frac{\Lambda_t}{\Lambda_t} \left( \frac{C_{t,\tau}}{C_{t,t}} \right)^{-1} \right] = \mathbb{E}_t^B \left[ \left( \frac{C_{t,\tau}}{C_{t,t}} \right)^{-1} \right] = \mathbb{E}_t^A \left[ \left( \frac{C_{t,\tau}}{C_{t,t}} \right)^{-1} \right] \). The last equality follows from the assumption that information about the consumption process is complete. Therefore, from log-normality

\[
B_{t,\tau} = e^{-\rho(\tau-t)} \mathbb{E}_t^A \left[ \left( \frac{C_{t,\tau}}{C_{t,t}} \right)^{-1} \right] = e^{-\rho(\mu_c - \sigma^2_c)(\tau-t)}
\]

The real yield curve follows directly from \( y_{t,\tau} = -\frac{\log(B_{t,\tau})}{\tau-t} \) and the expression above obtains.

**Nominal yield curve:** The price in \( t \) of a nominal bond with maturity in period \( \tau \) under the beliefs of investor \( A \) is \( B_{t,\tau}^s = \mathbb{E}_t^A \xi_{\Lambda,\tau}^s \). We can manipulate the ratio \( \frac{\xi_{\Lambda,\tau}^s}{\xi_{\Lambda,\tau}} \) by inserting Eqs. (1.3.12) and (1.3.16) as follows

\[
\frac{\xi_{\Lambda,\tau}^s}{\xi_{\Lambda,\tau}} = \frac{\int_t^\infty e^{-\rho u} \mathbb{E}_t^A \left[ (M_u)^{-1} \right] + \mathbb{E}_t^A \left[ \lambda_u (M_u)^{-1} \right] \] \, du}{\int_t^\infty e^{-\rho u} \left( \mathbb{E}_t^A \left[ (M_u)^{-1} \right] + \lambda_t \mathbb{E}_t^B \left[ (M_u)^{-1} \right] \right) \, du}

= \frac{\int_t^\infty \mathbb{E}_t^A \left[ e^{-\rho(u-t)} \left( \frac{M_u}{M_t} \right)^{-1} \right] + \mathbb{E}_t^A \left[ e^{-\rho(u-t)} \lambda_u \left( \frac{M_u}{M_t} \right)^{-1} \right] \, du}{\int_t^\infty \left( \mathcal{M} \left( x_t^A, u-t; \hat{\nu}^A, \hat{\nu}_d^A \right) + \lambda_t \mathcal{M} \left( x_t^B, u-t; \hat{\nu}^B, \hat{\nu}_d^B \right) \right) \, du}
\]
Using the previous result and the price of a nominal bond with maturity $\tau \geq t$ at time $t$ under the beliefs of investor A

$$B_{t,\tau}^{S} = \mathbb{E}_{t} \left[ \int_{\tau}^{\infty} \left( \mathbb{E}_{r}^{A} \left[ e^{-\rho(u-r)} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right] + \mathbb{E}_{r}^{A} \left[ e^{-\rho(u-t)} \lambda_{u} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right] \right) du \right]$$

$$= \frac{\int_{\tau}^{\infty} \left( \mathbb{E}_{r}^{A} \left[ e^{-\rho(u-r)} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right] + \mathbb{E}_{r}^{A} \left[ e^{-\rho(u-t)} \lambda_{u} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right] \right) du}{\int_{t}^{\infty} \left( \mathcal{M}(x_{t}^{A}, u-t; \hat{\nu}^{A}, \hat{\nu}_{x}^{A}) + \lambda_{t} \mathcal{M}(x_{t}^{B}, u-t; \hat{\nu}^{B}, \hat{\nu}_{x}^{B}) \right) du}$$

By the tower property of conditional expectations

$$\mathbb{E}_{t}^{A} \left[ e^{-\rho(u-r)} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right] + \mathbb{E}_{t}^{A} \left[ e^{-\rho(u-t)} \lambda_{u} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right] = \mathbb{E}_{t}^{A} \left[ e^{-\rho(u-t)} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right] + \mathbb{E}_{t}^{A} \left[ e^{-\rho(u-t)} \lambda_{u} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right]$$

Furthermore, we can change the measure from investor A's beliefs to investor B's such that

$$\mathbb{E}_{t}^{A} \left[ e^{-\rho(u-t)} \lambda_{u} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right] = \lambda_{t} \mathbb{E}_{t}^{A} \left[ e^{-\rho(u-t)} \lambda_{u} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right]$$

Therefore, the integrand in the numerator of $B_{t,\tau}^{S}$ becomes

$$\mathbb{E}_{t}^{A} \left[ e^{-\rho(u-t)} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right] + \mathbb{E}_{t}^{A} \left[ e^{-\rho(u-t)} \lambda_{u} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right] = \mathbb{E}_{t}^{A} \left[ e^{-\rho(u-t)} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right] + \mathbb{E}_{t}^{A} \left[ e^{-\rho(u-t)} \lambda_{u} \left( \frac{M_{u}}{M_{r}} \right)^{-1} \right]$$

Hence, the price of the nominal bond is given by

$$B_{t,\tau}^{S} = \frac{\int_{\tau}^{\infty} \left( \mathcal{M}(x_{t}^{A}, u-t; \hat{\nu}^{A}, \hat{\nu}_{x}^{A}) + \lambda_{t} \mathcal{M}(x_{t}^{B}, u-t; \hat{\nu}^{B}, \hat{\nu}_{x}^{B}) \right) du}{\int_{t}^{\infty} \left( \mathcal{M}(x_{t}^{A}, u-t; \hat{\nu}^{A}, \hat{\nu}_{x}^{A}) + \lambda_{t} \mathcal{M}(x_{t}^{B}, u-t; \hat{\nu}^{B}, \hat{\nu}_{x}^{B}) \right) du}$$

The nominal yield curve follows from $y_{t,\tau}^{S} = -\frac{\log(B_{t,\tau}^{S})}{\tau-t}$, the expression above and by letting

$$\mathcal{Q}(x_{t}^{A}, x_{t}^{B}, \lambda_{t}, \tau) \equiv \int_{\tau}^{\infty} \left( \mathcal{M}(x_{t}^{A}, u-t; \hat{\nu}^{A}, \hat{\nu}_{x}^{A}) + \lambda_{t} \mathcal{M}(x_{t}^{B}, u-t; \hat{\nu}^{B}, \hat{\nu}_{x}^{B}) \right) du$$

for $\tau \geq t$ with some abuse of notation.

**Dynamics of the nominal yield curve:** The diffusion terms of the nominal yield curve follow from an application of Ito's lemma and by letting, for $l \in \{m, s\}$

$$G(x_{t}^{A}, x_{t}^{B}, \lambda_{t}, \tau; \hat{\nu}_{l}) \equiv \frac{f_{1}G(x_{t}^{A}, \tau; \hat{\nu}^{A}) + (1 - f_{1})G(x_{t}^{B}, \tau; \hat{\nu}^{B})}{\mathcal{Q}(x_{t}^{A}, x_{t}^{B}, \lambda_{t}, \tau)}$$

$$D(x_{t}^{A}, x_{t}^{B}, \tau) \equiv \int_{t}^{\infty} \left[ \mathcal{M}(x_{t}^{B}, u-t; \hat{\nu}^{B}, \hat{\nu}_{x}^{B}) - \mathcal{M}(x_{t}^{A}, u-t; \hat{\nu}^{A}, \hat{\nu}_{x}^{A}) \right] du$$

(1.643)
Chapter 2

Quantifying Private Information about Fundamentals in Treasury Auctions

2.1 Introduction

The role of private information about fundamentals has occupied a central position in the study of financial markets. It is directly related to key questions about market efficiency (Fama, 1970, 1991), market microstructure (Kyle, 1985) and information aggregation (Grossman and Stiglitz (1980); Diamond and Verrecchia (1981); Kyle (1989)). In the context of the primary market for sovereign debt, early work argued that the main incentive for bidders in treasury auctions (i.e. the primary market) is to resell the purchased bonds to investors in the secondary market (Cammack (1991); Spindt and Stolz (1992); Bikhchandani and Huang (1993)). Therefore, these authors concluded, private information about the resale price (the fundamental value) should determine bidding strategies, whereas idiosyncratic differences in valuations could be safely assumed to be negligible. Nyborg et al. (2002) and Bjønnes (2001), among others, provide reduced-form empirical evidence consistent with private information. In contrast, Hortaçsu and Kastl (2012) do not find statistical evidence of private information in Canadian treasury auctions. Hortaçsu et al. (2018) argue that bidders in US Treasury auctions are sophisticated traders with access to overlapping data sources on the value of the bonds being offered, including information on secondary market prices right before the auction. Hence, even if reselling the bonds is a main motivation for bidders, it is unlikely that they are privately informed about resale prices. Instead, heterogeneity in bids is more likely attributable to differences in idiosyncratic valuations. More recently, however, Boyarchenko et al. (2020) study information sharing in US treasury auctions by estimating a model with private information about fundamentals and idiosyncratic valuations. In sum, the importance of private
information in primary debt markets remains an open empirical question.

We contribute to this debate by proposing an empirical method to quantify the amount of private information in treasury auctions structured as divisible good uniform price auctions. The method is based on the estimation of the theoretical model presented in Vives (2010, 2011), where bidders are allowed to be privately informed about fundamentals, but may also receive idiosyncratic valuation shocks. The relative importance of private information is determined by the correlation of bidders’ valuations and the precision of their private signals. These are model parameters that, we show, are identified from data on bids and secondary market prices. We find through an application to the Colombian sovereign debt market that the importance of private information varies with market conditions. In particular, our estimates indicate that the information structure of the market under consideration is less characterized by private information after an exogenous positive shock to the demand for government bonds in the secondary market. Furthermore, we provide model-free empirical evidence that supports this finding.

Bidders in treasury auctions have market power because access to the primary market is generally restricted to a handful of financial institutions. These bidders strategically submit bids below their marginal valuations, a result known in the auctions literature as bid shading or demand reduction. One key result in Vives (2010, 2011) is that private information about fundamentals raises market power above the full-information level. Consequently, the model predicts that holding all else constant an observed reduction in private information would lower market power and bid shading, thereby reducing the borrowing costs for the issuer. However, the model allows for decreasing marginal valuations, and in equilibrium bidders with steeper marginal value curves shade their bids more. The slope of the marginal value function is a parameter of the model that can be interpreted as the cost to the bidder of deviating from a target inventory. Hence, it captures inventory costs. The model then implies that higher inventory costs also result in larger market power. This poses an identification challenge. The same level of bid shading can be rationalized by high private information and low inventory costs or, alternatively, high inventory costs and low private information. Nonetheless, we show that the model parameters that determine private information and inventory costs are separately identified from auctions data (including bids) and a secondary price benchmark.

We apply our method to the Colombian sovereign debt market. In March 2014, J.P. Morgan announced it would boost the weights of Colombian sovereign bonds on two local currency emerging markets debt indices. Williams (2018) documents that this unanticipated rebalancing triggered a surge in the demand for Colombian sovereign debt, which in turn prompted banks in Colombia to reduce their holdings of government bonds. We provide supplementary empirical evidence showing that Colombian sovereign bonds were more actively traded after the shock. Furthermore, the secondary debt market became more liquid and experienced a significant decrease in volatility. We hypothesize that the primary market was in turn likely affected by the rebalancing. Consequently,
we estimate the model separately before and after the shock. We find a substantial reduction in private information about fundamentals as a determinant of observed bids but also a sizeable increase in inventory costs. These two findings have opposite effects on bid shading. In fact, the estimation results indicate that the change in revenue for the treasury is lower than two bp (basis points) of the par value of the bond offering. However, the latter masks the separate effects of private information and inventory costs. We decompose these two effects and find that the reduction in private information alone implies a drop bid shading (increase in revenue) of 34 - 112bp, depending on the number of bidders. In contrast, the increase in inventory costs raises bid shading by 35 - 110 bp.

We provide model-free evidence that supports our results on the diminished importance of private information following the rebalancing. Auction outcomes such as market-clearing (cutoff) prices aggregate information about bidders’ valuations. If private information is prevalent in this market, then the auction cutoff price would reveal bidders’ private signals about fundamentals that were not already captured by observable market prices prior to the auction. Thus, upon observing a realization of the auction price, market participants would revise their own valuation of the offered bond. We show that a high auction cutoff price relative to a pre-auction price benchmark predicts subsequent high secondary market prices. Consistent with an information-revelation channel, we show that this result is significant and persistent at least through a week after the auction. After the rebalancing, however, this effect either shrinks or vanishes altogether, which supports our view regarding the decline in the relative prevalence of private information.

The results of the structural estimation show a large increase in inventory adjustment costs (the slope of the bidders’ marginal value curves). This implies that bidders should be more reluctant to deviate from their inventory target levels after the rebalancing. To test this implication, we estimate a reduced-form model of short term (overnight) changes in inventory based on the literature on inventory risk (Madhavan and Smidt (1993); Friewald and Nagler (2016); Schultz (2017), among others). We find higher one-day mean-reversion in inventories after the rebalancing, which implies lower inventory half-lives. On average, it takes bidders 11.7 days to offload half of their inventories before the rebalancing, but only 7.5 days after the shock.

Our proposed methodology is well-suited to quantify the effects of private information and inventory costs on policies meant to lower treasuries’ borrowing costs. We study two such policies: increasing the number of primary market participants and reducing total bond offering per auction. After the rebalancing, the results of the structural estimation imply that increasing the number of bidders from 10 to 14 (the minimum and maximum observed participation in our sample) reduces bid shading by 4 bp. In contrast, in a hypothetical but empirically feasible scenario with high market power (high private information matching the pre-rebalancing estimates and high inventory costs matching the post-rebalancing result) bid shading drops by 80 bp when the number of bidders increases from 10 to 14. Analogously, we quantify the effect of reducing total bond offering in a
specific auction on market power. From the equilibrium of the model in Vives (2010, 2011), market power increases with bond supply. We find that, following a one standard deviation decrease in total bond supply below its sample mean, market power would fall at most by 4 bp according to our post-rebalancing estimates. In the high market power scenario the reduction in market power lies between 20 and 63 bp depending on the number of bidders.

Debates about whether the aforementioned policies should be pursued require estimates of the prevailing level of market power in the primary market under consideration. Furthermore, more targeted policies intended to address either private information frictions or inventory costs related to market-making services would benefit from methodologies that are able to decompose market power into its determinants to discern which feature is more preponderant in the market. Our paper contributes to both objectives.

We conclude this section with a discussion on related literature. The remainder of this paper is organized as follows: in section 2.2 we summarize the model presented in Vives (2010, 2011). We then show that the model parameters are identified using only bidding data and prices from the secondary debt market in section 2.3, and we lay out our empirical strategy to estimate the parameters of interest. In section 2.4, we describe the main features of the Colombian debt market along with the data we use to estimate the model. In 2.5 we present the results of our structural estimation, a quantitative assessment of the change in bidders’ market power, and a counterfactual analysis to evaluate policies to reduce borrowing costs. In section 2.6, we first provide model-free evidence consistent with our structural estimation results. Then, we discuss the validity of the model assumptions regarding auction participation.

**Related literature** In the auctions literature, questions about the importance of private information have been formulated in terms of the theoretical model that better describes the information structure that determines bidders’ valuations. Private information is consistent with a model with common values. In such models, the *ex post* value of the asset after the auction is the same for all bidders (e.g., its resale price in the secondary market), but it is uncertain to the bidders *ex ante*. Instead, each bidder observes a private signal of the common value that amounts to private information about fundamentals. At the other extreme, the absence of private information corresponds to a model with only private values. In that model, any common component in bidders’ values is common knowledge, hence all information about fundamentals is public, but bidders have different idiosyncratic valuations. The theoretical model in Vives (2010, 2011) encompasses both common and private values. The weight of each of these two components in bidders’ valuations determines the importance of private information.¹

¹Armantier and Sbaï (2006) propose a parametric estimation of a generalized version of the model in Wang and Zender (2002) where bidders have pure common values but are asymmetric. Recent work on non-parametric structural estimation of multiunit auctions under the private values paradigm include Chapman et al. (2007), Kang and Puller

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Our paper is closely related to Boyarchenko et al. (2020). They extend the uniform price common value model of Kyle (1989) and Wang and Zender (2002) by adding a private value component to bidders' valuations. However, they do not rely on an estimation procedure based on bidding data. In contrast to our work, they calibrate the model to auction results and post-auction secondary market prices. In their calibration, the relative importance of the common value component is not recovered from the data and is instead fixed. Hence, in their paper the importance of private information is imposed by assumption. To our knowledge, ours is the first paper to estimate the importance of private information in a divisible good uniform price auction with both private and common values.

Methodologically, our estimation approach is complementary to Hortaçsu and McAdams (2010) and Kastl (2011). Their estimation method is non-parametric and takes into account the fact that bids are discrete. However, it requires the assumption of independent private values. Hence it cannot be used to recover the distribution of bidders' valuations in a setting with private information (common values). In contrast, we follow the model in Vives (2011) that has a unique equilibrium in linear demand schedules. We focus on this equilibrium acknowledging that it imposes strong parametric assumptions and treats bids as continuous linear functions. However, these assumptions allow us to derive a likelihood function, and, consequently, to estimate the model on bidding data and readily available secondary market pricing data alone. Furthermore, unlike non-parametric estimation methods, we can decompose market power into private information and inventory costs.

The results of our application indicate that the information structure is not an immutable characteristic of treasury auctions. Hence, although we find evidence of private information, our finding is not necessarily inconsistent with Hortaçsu and Kastl (2012). Using detailed bidding data from Canadian Treasury auctions, they do not reject the null hypothesis of private values in favor of an alternative common values hypothesis. However, their test applied to a different market or at a different point in time, could in principle reject the null hypothesis, thereby providing statistical evidence of private information.

Our paper also speaks to the empirical literature on the aggregation of private information in sovereign debt markets. Green (2004), Brandt and Kavajecz (2004), Pasquariello and Vega (2007), Valseth (2013), and Czech et al. (2020) show that order-flow predicts future bond yields. They argue that order-flow contributes to price discovery (i.e. aggregation of asymmetric information) in secondary debt markets. While our main focus is on the primary market, we provide empirical evidence that private information is revealed to market participants through the auction’s cutoff.

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2The test proposed by Hortaçsu and Kastl (2012) relies on the econometrician observing how dealers change their bids after they learn their customers’ bids. Therefore, we cannot apply it to our data because there are no customer bids in Colombian treasury auctions.

3Evans and Lyons (2005) and Menkhoff et al. (2016), among others, document a similar result for order-flow and future exchange rates.
price, which in turn predicts future secondary market yields and prices. Furthermore, Green (2004) and Pasquariello and Vega (2007) document that the empirical relationship between order-flow and yields changes with macroeconomic announcements. This result resonates with our findings about the time-varying nature of private information in debt markets.

2.2 Model

We follow Vives (2010, 2011), who presents a model of a uniform price, divisible good auction, with private and common values. $k_t$ units of an asset are sold at each auction $t = 1, \ldots, T$, and bidders are uncertain about the \textit{ex post} value of the asset. For each bidder $i = 1, \ldots, n_t$, the marginal value of buying $x_{i,t}$ units is $v_{i,t}(x_{i,t}) = \theta_{i,t} - \lambda x_{i,t}$, with $\lambda > 0$. $\lambda$ captures decreasing marginal valuation and can be interpreted as an opportunity cost. Bidders do not observe their valuations $\theta_{i,t}$, instead they receive private independent signals $s_{i,t} = \theta_{i,t} + \epsilon_{i,t}$. The distributions of $\epsilon_{i,t}$ and $\theta_{i,t}$ are common knowledge. $\epsilon_{i,t} \sim N(0, \sigma^2_{\epsilon})$ for all $i$ and all $t$, $\text{cov}(\epsilon_{i,t}, \epsilon_{j,t}) = 0$ for $i \neq j$ and $\text{cov}(\theta_{i,t}, \theta_{j,t}) = \rho \sigma^2_{\theta}$ for $i \neq j$, $\text{cov}(\epsilon_{i,t}, \theta_{j,t}) = 0$ for all $i$ and $j$.

Notice that bidders’ valuations can be decomposed into a common value and idiosyncratic independent components, as follows. Let $\theta_{i,t} = \tilde{\phi}_t + \gamma_{i,t}$, where $\tilde{\phi}_t \sim N(\overline{\theta}_t, \rho \sigma^2_{\theta})$, $\gamma_{i,t} \sim N(0, (1 - \rho) \sigma^2_{\gamma})$, $\text{cov}(\tilde{\phi}_t, \gamma_{i,t}) = 0$, and $\text{cov}(\gamma_{i,t}, \gamma_{j,t}) = 0$ for $i \neq j$. $\tilde{\phi}_t$ is a common value, as traditionally defined in the auctions literature. It is common to all bidders and, crucially, for some values of the parameters, bidders are imperfectly and privately informed about its value. This is precisely the feature of the model that captures private information about fundamentals.

The model encompasses extreme cases traditionally considered in the auctions literature. When $\rho = 1$ and $\sigma^2_{\epsilon} > 0$, the model reduces to a pure common value auction since the asset has the same uncertain \textit{ex post} value for all bidders, with probability one. In such a case, bidders are privately informed about the unknown common value of the asset. In contrast, if $\rho < 1$ and $\sigma^2_{\epsilon} = 0$, bidders have pure private values because they are no longer uncertain about their private \textit{ex post} values. Individual valuations still include a common component, but no bidder has private information that is value relevant for other bidders who already know their valuations with certainty, hence private information about fundamentals (about the common component) is irrelevant. Finally, when $\rho = 0$ and $\sigma^2_{\epsilon} > 0$, there is no uncertain common value, the common component $\overline{\theta}_t$ is commonly known, but bidders are not perfectly informed about their \textit{ex post} idiosyncratic values, moreover, their signals are independent conditional on $\overline{\theta}_t$.

In summary, bidders in the model have private information about fundamentals if and only if $\rho > 0$ and $\sigma^2_{\epsilon} > 0$. In such a case, there is a common component $\tilde{\phi}_t$ in bidders valuations that is not perfectly known ($\rho > 0$), and bidders receive different private signals about its value ($\sigma^2_{\epsilon} > 0$). More broadly, the importance of private information in bidders valuations is determined by $\rho$ and
After observing their private signals, bidders submit linear demand schedules of the form
\[ X(s_{i,t}, p) = bt + ats_{i,t} - ct \]. The auctioneer determines the auction cutoff price that clears the market, that is, the price \( \hat{p}_t \) that satisfies \( \sum_{i=1}^n X(s_{i,t}, \hat{p}_t) = k_t \). If there is no such price for a given realization of signals, the auction is canceled and no units of the asset are sold. Following Vives (2011), we only consider symmetric Bayesian Nash equilibria in linear demand schedules. He proves the existence of a unique such equilibrium. We reproduce his Proposition 1 below for convenience.

**Proposition 2.2.1.** Let \( \frac{\sigma_\epsilon^2}{\sigma_\theta^2} \) < \( \infty \). Then there is a unique symmetric linear Bayesian demand function equilibrium if and only if \( n - 2 - M > 0 \), where

\[
M \equiv \frac{\rho \sigma_\epsilon^2 n}{(1 - \rho)(\sigma_\epsilon^2 + \sigma_\theta^2 (1 + (n - 1) \rho))}.
\]

This equilibrium is given by

\[
X(s_{i,t}, p) = \left( E[\theta_i | s_{i,t}, p] - p \right) / (d + \lambda) = b + as_{i,t} - cp,
\]

where \( c = \frac{n - 2 - M}{\lambda(n - 1)(1 + M)} \), \( d = \frac{1}{(n - 1)c} \), \( a = \frac{(1 - \rho)\sigma_\epsilon^2}{(1 - \rho)\sigma_\theta^2 + \sigma_\epsilon^2} (d + \lambda)^{-1} \) and

\[
b = \frac{1}{1 + M} \left( \frac{k}{n} M + \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + (1 + (n - 1) \rho) \sigma_\theta^2} (d + \lambda)^{-1} \right).
\]

In equilibrium, \( \frac{1}{\lambda(1 + M)} > c > 0, a > 0, c \) decreases with \( M \) and with \( \lambda \), and \( d \) is decreasing in \( n \).

**2.2.2 Bid shading, private information and decreasing marginal valuations**

In equilibrium, bidders submit bids below their marginal valuations. This is a feature of several auction models, commonly refer to as bid shading. In fact, the demand function of a price-taking bidder is \( X^{pt}(s_{i}, p) = (E[\theta_i | s_{i}, p] - p) / \lambda \), which is above the equilibrium demand schedule in 2.2.1 because \( d > 0 \). Moreover, it follows from Proposition 2.2.1 and the market clearing condition that the equilibrium cutoff price is \( \hat{p}_t = \frac{1}{n} \sum_{i=1}^n E[\theta_i | s] - (d + \lambda) k/n \). It follows that the difference between the price that bidders pay for the asset and their average marginal valuation is \( dk/n \). Therefore, as discussed in Vives (2010), this quantity measures the amount of bid shading or, alternatively, market power.

In the model, market power is thus explained by three factors: imperfect competition, private information and decreasing marginal valuations. Specifically, \( d \) depends on the number of bidders \( n \), the parameters determining private information (\( \rho \), \( \sigma_\epsilon \), and \( \sigma_\theta \)), and the slope of the marginal valuation function \( \lambda \).
We will first describe the effect of private information on bid shading. The equilibrium in Proposition 2.2.1 is privately revealing in the sense of Allen (1981). In particular, $\mathbb{E}[\theta_i | s_i, p] = \mathbb{E}[\theta_i | s]$, where $s = (s_1, ..., s_n)$ is the vector of all signals. That is, the equilibrium cutoff price conveys the same information to bidder $i$ about its own valuation $\theta_i$ than the signals of all other bidders. This provides incentives for bid shading analogous to the winner’s course in common value auctions. A low cutoff price reveals to bidder $i$ that other bidders received low signals, hence decreasing its ex-post valuation of the asset. Rational bidders protect themselves against this bad news by increasing bid shading.

If $\rho = 0$ or $\sigma_\epsilon = 0$, the equilibrium in Proposition 2.2.1 coincides with a full-information equilibrium where all bidders observe their own signals and the signals of all other bidders. In such a case, $\mathbb{E}[\theta_i | s_i] = \mathbb{E}[\theta_i | s]$, hence the cutoff price no longer contains additional information about bidders valuations. There is still bid shading in the full-information equilibrium (which we denote using a superscript $f$, following Vives (2010)), but it is lower than in the equilibrium with private information ($\rho > 0$ and $\sigma_\epsilon > 0$). Notice that $M > M^f = 0$, therefore $c^f > c > 0$ and $0 < d^f < d$. All things equal, it follows from equation (2.2.1) that larger values of $M$ reduce $c$, therefore bidders submit less elastic demand schedules, and their market power, as measured by $d$, increases.

In the private information equilibrium, $M$ determines the effect of private information on bid shading and market power, through its effect on the slope $c$. However, the latter is also determined by $\lambda$, that is, by how steep the bidders’ decreasing marginal valuations are. Steeper marginal valuations (higher $\lambda$) imply steeper equilibrium demand schedules (higher $c$), which in turn results in higher bid shading (higher $d$).

Finally, as stated in Proposition 2.2.1, $d$ (therefore bid shading as well) is monotonically decreasing in the number of bidders. Moreover, as $n \to \infty$, $d \to 0$, hence the equilibrium with private information converges to the full-information equilibrium.

### 2.3 Identification and estimation

We establish sufficient conditions under which the model in Section 2.2 is identified. For any auction $t$, we assume the econometrician observes the number of bidders $n_t$, the total number of units offered $k_t$, all bids submitted, the auction cutoff price $p^*_t$ and the secondary market price $\bar{p}_t$.

We further assume that the bids correspond to the unique symmetric linear Bayesian equilibrium from Section 2.2, for a given realization of private signals $s_t = (s_{1t}, ..., s_{nt})$. Hence, bids are linear demand schedules of the form $X_{i,t}(p) = \alpha_{i,t} - c_t p$, where $\alpha_{i,t} = b_t + a_t s_{i,t}$. Equivalently, bids can be described as inverse demand schedules: $P_{i,t}(x) = p^0_{i,t} - \frac{1}{c_t} x$, where $p^0_{i,t} = \frac{\alpha_{i,t}}{c_t}$. For a given auction $t$, all the information contained in the bids can be summarized in a vector of intercepts, either $\alpha_t = (\alpha_{1t}, ..., \alpha_{nt})$ or $p^0_t = (p^0_{1t}, ..., p^0_{nt})$, and a slope $c_t$ that is common to all bidders in a symmetric equilibrium.
We assume the econometrician has data from \(T\) independent auctions labeled \(t = 1, \ldots, T\). Pooling bidding data from several different auctions is a common practice in the empirical study of treasury auctions, since the number of potential bidders tends to be low (e.g., Hortaçsu and McAdams (2010), Armantier and Sbaï (2006), Kastl (2011), and Hortaçsu and Kastl (2012)).

In the model, the mean of the bidders’ values \(\theta_t\) is common knowledge for the bidders, but it is unlikely that the econometrician will know it with certainty. This introduces unobserved auction heterogeneity posing a challenge to the identification strategy, since we are combining data from several different auctions. To address this challenge we allow the econometrician to have partial knowledge of \(\theta_t\), captured by a proxy \(\bar{p}_t\). Secondary market prices, which are widely available for sovereign bond markets around the world, are likely good proxies of average valuations. Consequently, besides the assumptions of the model in Section 2.2, we add an assumption stating that secondary market prices are proxies of average valuations.

**Assumption 2.3.1.** Let \(\bar{p}_t\) be the observable secondary market price of the asset. Then \(\theta_t = \bar{p}_t + \eta_t\), where \(\eta_t \sim N(\mu_\eta, \sigma_\eta^2)\) with unknown mean and variance.

Our goal now is to establish the identification of all time-invariant parameters in the model, i.e., \((\rho, \sigma_\epsilon, \sigma_\theta, \lambda)\). We will also discuss the identification of \(\mu_\eta\) and \(\sigma_\eta^2\), although these are not main parameters of interest and are added to the model only to address unobserved heterogeneity. We exploit variation in the number of bidders to prove identification. Specifically, we assume there are arbitrarily many auctions with at least two different numbers of bidders.

To establish identification, we start by noticing that in equilibrium the sensitivity of the intercepts to the signals, \(a_t\), and the slope of the linear bids, \(c_t\), both depend on the number of bidders, \(n_t\), but do not depend on total supply, \(k_t\). Hence, across multiple auctions all with the same number of bidders, \(a_t\) and \(c_t\) remain constant. The same is true of the variables \(M\) and \(d\) in Proposition 2.2.1. In what follows, we will denote the dependency of these variables on \(n_t\) by explicitly expressing them as functions of \(n_t\). That is, we will use the notation \(a(n_t), c(n_t), M(n_t)\) and \(d(n_t)\).

For all \(i\) and \(t\), the intercepts of the inverse demand schedules can be expressed as

\[
p_{i,t}^0 = \bar{p}_t + \frac{M(n_t)}{c(n_t) n_t (1 + M(n_t))} k_t + \frac{a(n_t)}{c(n_t)} (\xi_{i,t} + \epsilon_{i,t})
\]

where \(\xi_{i,t} = \theta_{i,t} - \bar{\theta}_t\) is normally distributed with mean zero and variance \(\sigma_\theta^2\). Therefore, conditional on \(n_t\), \(k_t\) and \(\bar{p}_t\), \(p_{i,t}^0\) follows a normal distribution with

\[
E[p_{i,t}^0 | n_t, k_t, \bar{p}_t] = \bar{p}_t + \frac{M(n_t)}{c(n_t) n_t (1 + M(n_t))} k_t + \mu_\eta
\]

and

\[
\text{Var}[p_{i,t}^0 | n_t, k_t, \bar{p}_t] = \frac{a(n_t)^2}{c(n_t)^2} (\sigma_\theta^2 + \sigma_\epsilon^2) + \sigma_\eta^2
\]
for all bidder $i$, and

$$\text{Cov} \left[p_{i,t}^0, p_{j,t}^0 | n_t, k_t, \bar{p}_t\right] = \frac{a (n_t)^2}{c (n_t)^2} \rho \sigma \sigma_a^2 + \sigma^2$$  \hspace{1cm} (2.3.4)

for $i \neq j$.

The conditional variance and covariances are constant across all auctions with $n_t = \bar{n}$, as well as the slopes of the linear demand schedules $c (\bar{n})$, and the coefficients of the linear conditional expectation: \( \frac{M (\bar{n})}{c (\bar{n}) n_t (1 + M (\bar{n}))} \) and $\mu_\eta$. It follows immediately that the parameter $\mu_\eta$, and the constants $M (\bar{n})$ and $d (\bar{n})$ are separately identified. Moreover, since $c (\bar{n}) = \frac{\bar{n} - 2 - M (\bar{n})}{\lambda (\bar{n} - 1)(1 + M (\bar{n}))}$, then the parameter $\lambda$ is also identified. Variation in the number of bidders allows us to identify all other parameters of the model. We establish this result in Proposition 2.3.2. The proof is left to the Appendix.

Proposition 2.3.2. Given a dataset of $T$ independent auctions $D = \{ n_t, k_t, \bar{p}_t, p_{i,t}^0, c_t \}_{t=1}^T$, if $p_{i,t}^0$ and $c_t$ correspond to the equilibrium linear demand schedules of model 2.2 for each $t$, and assumption 2.3.1 holds, then $\rho$, $\sigma$, $\sigma_\theta$, $\lambda$, $\mu_\eta$ and $\sigma^2_\eta$ are identified when there are arbitrarily many auctions with the same number of bidders $n > 2$, for at least two different values of $n$ denoted $\bar{n}$ and $\hat{n}$.

The identification result above assumes bids are linear (inverse) demand schedules. It shows that the model in Vives (2010, 2011) is identified. Moreover, a symmetric linear Bayesian equilibrium is a common feature of several other theoretical and empirical studies of treasury auctions including Wang and Zender (2002), Armantier and Sbaï (2006), and Boyarchenko et al. (2020). However, in most countries, treasury auction rules require bidders to submit discrete bids consisting of a finite number of price-quantity pairs, as those described in Section 2.4.1. For such cases, we assume the model equilibrium from Section 3 is still a good approximation of the observed bids. More precisely, we assume that the distribution of the highest price submitted by each bidder coincides with the distribution of the intercepts of the inverse linear demand schedules predicted by the model, and that the observed cutoff price is equal to the model’s equilibrium market cutoff price $\hat{p}_t$ plus a normal error due to misspecification. We formally state these two assumptions below and then establish identification of the model parameters under these assumptions.

Assumption 2.3.3. Let $p_{i,t}^0$ denote the highest price submit by bidder $i$. Conditional on the number of bidders $n_t$, total supply $k_t$ and the mean value $\bar{\theta}_t$, $p_{i,t}^0$ follows a normal distribution with mean

$$E \left[p_{i,t}^0 | n_t, k_t, \bar{\theta}_t\right] = \bar{\theta}_t + \frac{M (n_t)}{c (n_t) n_t (1 + M (n_t))} k_t$$  \hspace{1cm} (2.3.5)

and variance

$$\text{Var} \left[p_{i,t}^0 | n_t, k_t, \bar{\theta}_t\right] = \frac{a (n_t)^2}{c (n_t)^2} (\sigma_\theta^2 + \sigma^2)$$  \hspace{1cm} (2.3.6)

Moreover, for any two bidders $i \neq j$

$$\text{Cov} \left[p_{i,t}^0, p_{j,t}^0 | n_t, k_t, \bar{\theta}_t\right] = \frac{a (n_t)^2}{c (n_t)^2} \rho \sigma_\theta^2$$  \hspace{1cm} (2.3.7)
Assumption 2.3.4. $p^c_t = \hat{p}_t + \zeta_t$, and $\zeta_t \sim N(0, \sigma^2)$

Proposition 2.3.5. Under assumptions 2.3.3 and 2.3.4, and given a dataset of $T$ independent auctions $\{n_t, k_t, \bar{p}_t, p^0_t, p^c_t\}_{t=1}^T$, if there are arbitrarily many auctions with the same number of bidders $n > 2$, for at least two different values of $n$ (denoted $\bar{n}$ and $\hat{n}$) the parameters $(\rho, \sigma_\epsilon, \sigma_\theta, \lambda, \mu_\eta, \sigma_\eta)$ are identified.

In the model, the market cutoff price is $\bar{p}_t = \frac{1}{n_t} \sum_{i=1}^{n_t} p^0_{i,t} - \frac{k_t}{n_t c(n_t)}$, hence assumption 2.3.4 implies that conditional on $n_t$, $k_t$, and the vector of price bids $p^c_t$, the observed cutoff price is normally distributed with mean

$$E[p^c_t|n_t, k_t, \bar{p}_t] = \frac{1}{n_t} \sum_{i=1}^{n_t} p^0_{i,t} - \frac{k_t}{n_t c(n_t)} \quad (2.3.8)$$

It follows that $c(\bar{n})$ and $c(\hat{n})$ are identified, even if they are not directly observable. The rest of the proof of Proposition 2.3.5 is identical to that of Proposition 2.3.2.

The parametric assumptions of the model (2.2) allow us to easily estimate its parameters by maximum likelihood. The vector $p^0_t$ is normally distributed conditional on $n_t$, $k_t$ and $\bar{p}_t$. Moreover, the observed cutoff price $p^c_t$ is also normally distributed conditional on $n_t$, $k_t$, $\bar{p}_t$ and $p^0_t$. The respective conditional densities can be used to derived the likelihood function and estimate the model parameters, even in the presence of unobserved auction heterogeneity. The likelihood function is included in the Appendix.

2.4 Data

Our application is the Colombian sovereign debt market. As many other sovereign fixed-income markets, the Colombian debt market is organized around a set of primary dealers (henceforth PDs). Primary dealers are large financial institutions that directly participate in the primary market where the bonds are offered by the issuer, namely, the sovereign government. PDs are expected to participate in the primary market by purchasing securities that they can hold in their balance sheets or resell to other dealers or customers in the secondary market. There are fourteen PDs in our sample. All of them are commercial banks, financial corporations or broker-dealers. For more details on the PDs program in Colombia, see Cardozo (2013) and Williams (2018).

2.4.1 Primary market

Colombian government bonds (TES) are offered at bi-weekly pre-scheduled auctions. TES are coupon-paying government securities issued in local currency with maturity of more than one year. For each bond, the primary market is an auction conducted by the Central Bank of Colombia on behalf of the Colombian Treasury. Only PDs (which we call indistinctly bidders) designated by the
Colombian Treasury can participate in the primary market. Before the auction is held, the Central Bank announces the par value of the bond offering, that is, the bond supply. When an auction opens, every PD may submit a sealed demand schedule that consists of a set of yield-quantity pairs. Equivalently, each demand schedule can be characterized as price-quantity pairs from the submitted yields. A single price-quantity pair within a given bid is a bid-step. When the auction closes, the Central Bank sorts all the individual bid-steps from highest to lowest price. Then, it adds up all the submitted quantities cumulatively. The market clearing or cutoff price is set at the point where aggregate demand (the cumulative sum of submitted quantities) meets bond supply (the total amount offered at the auction). Thus, all pairs above or at the cutoff price are winning pairs. The auctions follow a uniform price mechanism, meaning that PDs pay exactly the cutoff price for all their winning price-quantity pairs regardless of the actual prices (yields) they submitted. The results of the auction are announced on the same day of the offering, which is also when PDs receive their share of won securities.

The Colombian government has restricted the number of different bond issues it offers to a few benchmarks every year. Thus, rather than auctioning off a brand new five-year bond every two weeks, the government usually reopens the offering of an existing bond with maturity of approximately five years on auction days. In fact, unless a TES bond of a given maturity is auctioned off for the very first time, there is an identical TES bond being simultaneously traded in the secondary debt market before and after the auction. Reopening existing bond offerings is a method frequently employed also by Treasuries in Sweden, Norway, and Italy, among others (Nyborg et al. (2002); Valseth (2013); Siaux (2018)). Moreover, some US Treasury bills and notes are reopened frequently (Fleming (2002); Lou et al. (2013)). The PDs may form an accurate estimate of the price at which they can potentially sell the offered bond by observing its pricing in the secondary market. Moreover, given that all PDs observe the trading activity in the secondary market for a to-be-offered bond prior to entering a reopening auction, bidders may use the secondary market price to extract information about the security’s fundamental value to inform their bidding strategies. As we will show in section 2.3, we use a bond’s secondary market price as a commonly known signal of its ex post value when estimating the model.

Bids

Our sample spans all TES auctions between 2013 and 2015. In total, there were 154 bond auctions for nine different issues that were either auctioned off for the first time or reopened. For a given auction, we observe the complete set of yield-quantity pairs every PD submitted. We convert the yields to prices using the bond’s face value (100), coupon payments, time to maturity, and day-count convention (NL/365) consistent with the guidelines of the Colombian treasury. Thus, for bidder $i$ in auction $t$, we have $H$ price-quantity pairs, denoted by $\{p_{i,t,h}, x_{i,t,h}\}_{h=0}^{H-1}$. 

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Table (2.1) presents some summary statistics regarding bidding data. It can be seen that, for instance, in all of our sample, every bidder submits, on average, around four price-quantity pairs in a given auction. Furthermore, an average of 12.4 bidders participate in any auction.

Table 2.1: Summary statistics

Panel A: All sample

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Highest bid over cutoff price (bp)</td>
<td>53.4</td>
<td>34.2</td>
</tr>
<tr>
<td>Number of participants</td>
<td>12.4</td>
<td>1.2</td>
</tr>
<tr>
<td>Number of price-quantity pairs</td>
<td>4.1</td>
<td>1.4</td>
</tr>
<tr>
<td>Total supply (bill. hundr. COP)</td>
<td>1.9</td>
<td>1.0</td>
</tr>
</tbody>
</table>

Panel B: Pre-rebalancing sample

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Highest bid over cutoff price (bp)</td>
<td>49.0</td>
<td>40.7</td>
</tr>
<tr>
<td>Number of participants</td>
<td>12.2</td>
<td>1.3</td>
</tr>
<tr>
<td>Number of price-quantity pairs</td>
<td>3.9</td>
<td>1.6</td>
</tr>
<tr>
<td>Total supply (bill. hundr. COP)</td>
<td>1.8</td>
<td>1.2</td>
</tr>
</tbody>
</table>

Panel C: Post-rebalancing sample

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Highest bid over cutoff price (bp)</td>
<td>57.2</td>
<td>27.1</td>
</tr>
<tr>
<td>Number of participants</td>
<td>12.6</td>
<td>1.0</td>
</tr>
<tr>
<td>Number of price-quantity pairs</td>
<td>4.4</td>
<td>1.0</td>
</tr>
<tr>
<td>Total supply (bill. hundr. COP)</td>
<td>1.9</td>
<td>0.7</td>
</tr>
</tbody>
</table>

Our sample spans all Colombian treasury auctions of all sovereign bonds in locally-denominated currency (COP) between 2013 and 2015. The pre-rebalancing sample corresponds to all auctions held before the J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014. "Highest bid over cutoff price" corresponds to the highest submitted bid by a bidder over the auction’s cutoff or market-clearing price in bp. "Number of participants" corresponds to the effective number of bidders who submitted demand schedules in an auction. "Number of price-quantity pairs" is the total number of price-quantity pairs submitted in an auction. "Total bond supply" is the pre-announced par value of the bond to be issued in the auction in hundreds of billions of COP. The average nominal spot exchange rate in March 2014 was 2,000 COP per 1.00 USD.
Number of bidders per auction

As we describe below in Section 2.3, we take advantage of the variation in the number of bidders that participate in a given auction both before and after the rebalancing announcement on March 19, 2014.\textsuperscript{4} Table (2.2) presents the total number of bond auctions with a given number of bidders both before and after the rebalancing announcement. At most, up to fourteen bidders may participate in any given auction. This value corresponds to the number of PDs designated by the Colombian government in our sample. It is immediately seen from table (2.2) that more than ten bidders participate in the vast majority of auctions before the rebalancing announcement and in all of them afterwards. Thus, at least 71\% of all possible PDs submit bids in the primary market in most bond offerings.

\textbf{Table 2.2: Number of auctions per number of bidders}

\begin{center}
\begin{tabular}{l|cc}
\hline
Number of bidders & Pre-rebalancing & Post-rebalancing \\
                 & (1)     & (2)    \\
\hline
8                 & 2       & 0      \\
9                 & 0       & 0      \\
10                & 7       & 3      \\
11                & 6       & 7      \\
12                & 27      & 22     \\
13                & 19      & 34     \\
14                & 11      & 16     \\
\hline
Total             & 72      & 82     \\
\hline
\end{tabular}
\end{center}

This table presents the total number of auctions with a given number of bidders before and after the J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014. At most, up to fourteen bidders (also known as designated Primary Dealers) may participate in any given auction. Our sample spans all Colombian sovereign bond offerings between 2013 and 2015.

\subsection{2.4.2 Secondary market}

Primary Dealers in the Colombian secondary debt market act as intermediaries between the bond issuer (Treasury) and investors (either end-investors or other dealers), although they usually also keep a fraction of the purchased security in their own books.\textsuperscript{5}

PDs may trade in the secondary market with different sets of customers which can potentially give rise to differences in clients’ order-flow across bidders. In turn, heterogeneous clients’ order-flow....

\textsuperscript{4}See section 2.5 for more details on the rebalancing announcement.

\textsuperscript{5}TES holders are mainly pension funds, commercial banks, brokerage firms, foreign funds, and public institutions.
may result in dealers having private information about fundamental values in Treasury markets, as argued in Boyarchenko et al. (2020).\textsuperscript{6}

We have access to all the transactions in the main inter-dealer broker (IDB) market for TES bonds between 2013 and 2015.\textsuperscript{7} A transaction from the IDB market is recorded as a vector that contains the transaction’s clearing time stamped at the second, its nominal value, its price, and an (anonymized) identifier for the dealers involved in the transaction.

We also gather prices of the offered bonds from Bloomberg. Specifically, we collect Bloomberg’s daily mid-quote for the opening and end-of-day price, respectively, for all nine bonds offered in our sample. These prices are merely suggestive as they are not necessarily prices at which dealers or investors may actually transact. Nevertheless, Bloomberg’s prices for TES bonds are calculated using information from different platforms and trading facilities in the Colombian secondary debt market, so we include them into our information set as well.

We compute the bond’s daily quantity-weighted average price of all transactions in the main IDB market before an auction opens. We use this weighted-price as a measure at which the to-be-offered bond trade in the secondary market around the auction. We interpret this weighted-average price as a proxy for the commonly known information about the fundamental value of the bond. We also perform robustness tests using prices from Bloomberg as an alternative secondary market price benchmark. We exclude bids from three auctions where we do not observe a secondary market price because the bond is auctioned off for the first time, which reduces our effective sample to 151 auctions.

\subsection*{2.5 Application: Rebalancing}

On March 19, 2014 J.P. Morgan announced it would rebalance the weights in its JPM GBI-EM Global and JPM GBI-EM Global Diversified debt indexes. These indexes are composites of locally-denominated government debt instruments issued in emerging markets and at the time where tracked by funds that managed around $200 billion (see Hong and Molinski, 2014). Colombia’s weight in the J.P. Morgan GBI-EM index was boosted with the inclusion of five additional Colombian

\textsuperscript{6}Green (2004), Brandt and Kavajeckz (2004), Pasquariello and Vega (2007), and Czech et al. (2020) investigate the relationship between order-flow imbalances and Treasury returns.

\textsuperscript{7}There are three main different trading and registering systems to transact TES bonds in our sample: the Electronic Negotiation System (known as SEN in Colombia), the Colombian Electronic Market (MEC), and the Colombian Electronic Market Register Module (MEC-R). As discussed in León et al. (2014), SEN, a system managed and operated by the Central Bank, is the principal inter-dealer trading and negotiation platform for TES bonds in Colombia. PDs have exclusive access to this platform, where they can trade and settle anonymously without counterparty limit. More than 50 percent of total trading in TES bonds occurs within this system (Gemloc, 2011). The Electronic Negotiation System is the main IDB we refer to above in the text. See León et al. (2014) for more details on the different trading systems where PDs can transact TES bonds.
southern bonds. The rebalancing increased Colombia’s total weight in the index from 3.24 to 7.69 percent. According to Arslanalp and Tsuda (2015), around 97.7 percent of the weight’s increase was a consequence of the inclusion of the new TES bonds, whereas the remaining portion was due to valuation effects. Romero et al. (2020) estimate that 80% of all offshore investors with significant dealings in Colombia use this index as a benchmark.

Williams (2018) documents that the share of debt held by foreigners in the Colombian debt market rose from 8.5 percent in March 2014 at the time of the announcement to 19 percent in September 2014. Similarly, Arslanalp and Tsuda (2015) estimate that the net foreign purchases of Colombian government bonds were USD $7.36 billion from March to September 2014, a much larger figure than in previous years. Moreover, Williams (2018) provides evidence that the rebalancing had a significant impact on the balance sheet of Colombian PDs. After the rebalancing, PDs reduced their domestic sovereign debt holdings by 7.8 percentage points, as a fraction of their total assets, relative to non-PD commercial banks.

In summary, the J.P. Morgan rebalancing implied a positive demand shock for Colombian sovereign debt, mostly driven by foreign funds tracking the indexes. This shock may have altered the information structure of the primary market for Colombian government bonds by redistributing the share of customer’s order flow that different dealers and PDs have access to. In section 2.5.1 we provide reduced-form empirical evidence of changes in the information structure of the Colombian government debt market based on observed patterns in secondary bond markets around the rebalancing.

2.5.1 Changes in the information structure around the rebalancing: evidence from the secondary market

In this section, we document changes in trading conditions in the Colombian secondary debt market after the rebalancing announcement. We study potential shifts in three separate but related dimensions: total trading activity, liquidity, and price volatility. We discuss below why changes along these dimensions are consistent with variations in the information structure of the market.

We begin by fitting the following panel data model:

\[ \text{trad}_{l,t} = \alpha_l + \beta_r \text{rebal}_t + \gamma' \text{controls}_{l,t} + \varepsilon_{l,t} \quad (2.5.1) \]

where \( \text{trad}_{l,t} \) is a secondary market measure for bond \( l \) on day \( t \) that captures either total trading activity, liquidity, or volatility. \( \text{rebal}_t \) is equal to one after the rebalancing date and zero otherwise. We add bond fixed effects, and control for time-varying bond characteristics (duration and convexity), and market-wide conditions (10-year CDS on the Colombian sovereign government,
and aggregate volatility proxied by the VIX). Only bonds that were auctioned off both before and after the rebalancing announcement are included in the panel.

The estimation results are displayed in table (2.3). We use daily total number of transactions (column 1) and total traded volume at par value (column 2) as measures of total trading activity. For liquidity, we calculate Amihud (2002)’s measure using transaction data from the main inter-dealer market (column 3), and the daily absolute bid-ask spread using Bloomberg’s end-of-day bid and ask prices (column 4).\(^9\) Finally, we construct two measures of daily price volatility. The first one corresponds to realized volatility, namely, the sum of squared intraday returns using adjacent transaction prices from the main inter-dealer market (column 5). The second one is equal to the fitted values of the variance equation of a GARCH (1,1) model on bond daily returns using the end-of-day midquotes from Bloomberg (column 6). Our sample spans the period between 2013 and 2015.

The results in column (1) of table (2.3) indicate that there was a significant increase in the daily number of transactions in the main inter-dealer market after the rebalancing announcement for any of the offered bonds. Column (2) shows similar results for daily traded volume. Relative to the sample mean, there was an increase of approximately 40% in daily total trade for any bond regardless of the chosen indicator.

We further investigate whether this increase in total trade was associated with a change in overall market liquidity.\(^10\) Column (3) in table (2.3) shows a significant reduction in the average Amihud’s illiquidity measure after the rebalancing (approximately 30% relative to the sample mean) for any traded bond. The result is similar when we use instead the absolute bid-ask spread from Bloomberg in column (4), both in terms of direction and magnitude with respect to its own sample mean.

Lastly, we explore whether there was a concurrent change in price volatility of the secondary market after the rebalancing announcement. The estimation results in columns (5) and (6) of table (2.3) show that, regardless of the proxy we pick, secondary market volatility significantly decreased after the rebalancing announcement. The result is particularly sharp for daily realized volatility.

In summary, following the shock, Colombian sovereign bonds are more actively traded, in a more liquid secondary inter-dealer market, with lower price volatility. Previous work in the market microstructure literature has argued that one prominent motive for dealers to trade with each

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\(^9\)These variables have been frequently used in the literature to measure bond market liquidity (see Schestag et al. (2016)). Amihud’s measure is based on changes in trading prices from adjacent transactions, relative to the size of the transaction. We restrict our attention to these two liquidity proxies because our data do not include the sign of any transaction.

\(^10\)Pandolfi and Williams (2019) argue that increases in capital inflows arising from mechanical changes in the composition of sovereign debt indexes bring about liquidity improvements in secondary markets of emerging economies. In a similar vein, (Hegde and McDermott, 2003) find that there is a sustained increase in liquidity of stocks added to the S&P 500.
other is to share their inventory risk (Hansch et al. (1998) and Reiss and Werner (1998)). That is, dealers manage their inventory imbalances, resulting from recent transactions with customers, by trading in the inter-dealer market. Therefore, a substantial rise in inter-dealer trading suggests, at least indirectly, more trading between PDs and their customers.\(^\text{11}\) Consequently, price discovery in the secondary market may be enhanced with the increase in trading volumes that followed the rebalancing announcement (see, for example, Barclay and Hendershott (2015)). Moreover, increased liquidity may lead to more informationally efficient markets by aggregating private information more rapidly (Admati and Pfleiderer, 1988; Chordia et al., 2008). Secondary market volatility may also affect the information structure of the market. For instance, Pasquariello and Vega (2007) argue that the correlation between unanticipated order-flow and future bond yields is higher in periods of high volatility. In their setting, order-flow reveals private information, which is then impounded into market prices.

The trading patterns documented in this section are hence consistent with changes in the information structure. Since private information is not directly observable, we estimate the model in Section 2.2 separately before and after the rebalancing to test for changes in the importance of private information about fundamentals in both environments.\(^\text{12}\)

### 2.5.2 Maximum-likelihood estimation results

We estimate the model separately on bidding data from Colombian treasury auctions before and after the J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014. The pre-rebalancing sample starts on January 1, 2013 and ends on March 19, 2014. The post-rebalancing sample spans March 20, 2014 through December 31, 2015. There are 72 and 82 bond offerings in the pre- and post-rebalancing subsamples, respectively. However, for some auctions there is no secondary market price benchmark available, which reduces our sample sizes to 66 and 81 auctions (799 and 1024 bids) before and after the rebalancing, respectively.\(^\text{13}\) For each parameter in the model, we test the null hypothesis that it does not change after the rebalancing. The results of the estimation are shown in table (2.4). All parameters are statistically different in both sub-samples at the 1% level except for \(\sigma_\zeta\).

\(^{11}\)We cannot provide direct evidence of changes in dealer-customer trading because we do not observe customer orders in our data.

\(^{12}\)We do not claim to provide empirical evidence that the rebalancing announcement was the main cause of the fundamental changes in the Colombian secondary debt market documented here. Instead, we use the announcement date to test the ability of our empirical methodology to detect changes in the information structure of the market, given reduced-form evidence suggesting that such changes at least concurred with the rebalancing announcement.

\(^{13}\)Some auctions correspond to first-time bond offerings. Bonds with maturity in 2018, 2019, 2028, and 2030 were offered for the first time in February 2013, January 2014, January 2013, and January 2015, respectively. We do not use this auctions for the estimation of the model because there are no secondary markets for these bonds prior to the auction.
Table 2.3: Secondary market changes after rebalancing announcement

<table>
<thead>
<tr>
<th>Dependent variable:</th>
<th>Number of trans.</th>
<th>Total volume</th>
<th>Amihud’s meas.</th>
<th>Abs. bid-ask</th>
<th>Volat. (IDB)</th>
<th>Volat. (Bl.)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
<td>(6)</td>
</tr>
<tr>
<td>Post-rebal</td>
<td>18.088***</td>
<td>0.726***</td>
<td>−36.099***</td>
<td>−12.406***</td>
<td>−0.082**</td>
<td>−0.099***</td>
</tr>
<tr>
<td></td>
<td>(3.478)</td>
<td>(0.165)</td>
<td>(10.573)</td>
<td>(1.554)</td>
<td>(0.040)</td>
<td>(0.027)</td>
</tr>
<tr>
<td>Duration</td>
<td>14.176***</td>
<td>0.508**</td>
<td>94.605***</td>
<td>−4.823***</td>
<td>0.003</td>
<td>0.089***</td>
</tr>
<tr>
<td></td>
<td>(4.346)</td>
<td>(0.203)</td>
<td>(12.152)</td>
<td>(1.478)</td>
<td>(0.038)</td>
<td>(0.021)</td>
</tr>
<tr>
<td>Convexity</td>
<td>−2.886***</td>
<td>−0.108***</td>
<td>−8.720***</td>
<td>−0.253</td>
<td>0.009*</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>(0.398)</td>
<td>(0.018)</td>
<td>(1.858)</td>
<td>(0.166)</td>
<td>(0.005)</td>
<td>(0.004)</td>
</tr>
<tr>
<td>CDS (10-year)</td>
<td>−0.345***</td>
<td>−0.015***</td>
<td>0.792***</td>
<td>−0.033**</td>
<td>0.001</td>
<td>0.001***</td>
</tr>
<tr>
<td></td>
<td>(0.037)</td>
<td>(0.002)</td>
<td>(0.110)</td>
<td>(0.016)</td>
<td>(0.0004)</td>
<td>(0.0003)</td>
</tr>
<tr>
<td>VIX</td>
<td>0.080</td>
<td>−0.004</td>
<td>2.687***</td>
<td>0.170</td>
<td>0.009***</td>
<td>0.001</td>
</tr>
<tr>
<td></td>
<td>(0.252)</td>
<td>(0.010)</td>
<td>(0.880)</td>
<td>(0.128)</td>
<td>(0.002)</td>
<td>(0.002)</td>
</tr>
<tr>
<td>Sample mean</td>
<td>59.94</td>
<td>2.53</td>
<td>121.94</td>
<td>36.87</td>
<td>0.12</td>
<td>0.32</td>
</tr>
<tr>
<td>Units</td>
<td>Scalar</td>
<td>HBC</td>
<td>bp/HBC</td>
<td>bp</td>
<td>Perc.</td>
<td>Perc</td>
</tr>
<tr>
<td>Bond FE</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Observations</td>
<td>5,002</td>
<td>5,002</td>
<td>3,384</td>
<td>4,815</td>
<td>4,103</td>
<td>4,456</td>
</tr>
<tr>
<td>R²</td>
<td>0.209</td>
<td>0.193</td>
<td>0.372</td>
<td>0.493</td>
<td>0.122</td>
<td>0.590</td>
</tr>
</tbody>
</table>

Note: (*) Significant at 10% level; (**) significant at 5% level; (***) significant at 1% level. HBC stands for hundreds of billions of COP. This table displays the estimated coefficients of model (2.5.1). Dependent variables are secondary market measures. All dependent variables are at the daily frequency and at the bond level. Dependent variables are: total number of transactions in the main inter-dealer (IDB) market (column (1)); total traded volume in IDB market (column (2)); Amihud (2002)’s liquidity measure (column (3)); absolute bid-ask spread from Bloomberg end-of-day quotes (column (4)); realized volatility calculated as sum of squared of intraday returns using transaction prices from main IDB market (column (5)); fitted values of a GARCH (1,1) model on bond daily returns using the end-of-day midquotes from Bloomberg (column (6)). Post-rebal. is dummy variable equal to one after J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014, and zero otherwise. Duration and convexity are bond-specific at the daily frequency. CDS (10-year) is the 10-year CDS on the Colombian sovereign government. VIX is Chicago Board Options Exchange’s CBOE Volatility Index. CDS and VIX are at the daily frequency and they take the same value for all bonds in the panel on a given trading day.
When $\rho$, the correlation in bidders’ valuations conditional on their commonly known mean $\bar{\theta}_t$, falls substantially after the rebalancing, from 0.67 to 0.22. In the model, all else equal, a lower $\rho$ implies there is more precise public information about fundamentals (or, equivalently, lower variance of the unknown common component in bidders’ valuations). Moreover, the standard deviation of the distribution of private signals, $\sigma_\epsilon$, also falls from 0.86 to 0.49 following the rebalancing. Therefore, bidders are less uncertain about their own valuations. This implies that, from the point of view of any bidder $i$, other bidders are less privately informed about its valuation. Since the variance of bidders’ valuations $\sigma^2_\theta$ remains roughly unaltered (relative to the change in $\sigma_\epsilon$), both the decline in $\rho$ and $\sigma_\epsilon$ imply that private information about fundamentals is less relevant as a determinant of bids after the shock.

Table (2.4) also displays the estimates of the other four parameters of the model that are not related to private information. The estimated value of $\lambda$, which captures a bidder’s valuation of one additional unit of the asset, increases from 0.18 to 2.28. This implies bidders have much steeper decreasing marginal valuations after the rebalancing.

The estimation results also indicate that $\bar{P}_t$ is a biased but otherwise precise proxy of $\bar{\theta}_t$ because, even though $\mu_\eta$ is positive, $\sigma_\eta$ is very close to zero, and not statistically significant, in both subsamples. Finally, $\sigma_\zeta$, the standard deviation of the difference between the observed and the model-implied auction cutoff price due to potential misspecification, drops by 8 bp after the rebalancing.

---

14By assumption, the mean of the bidder’s private valuation, $\bar{\theta}_t$, is common knowledge to all bidders but is unknown to the econometrician who instead only observes a noisy proxy $\bar{P}_t$, as described in section 2.3.
Table 2.4: Estimation results before and after rebalancing

<table>
<thead>
<tr>
<th></th>
<th>Pre-rebalancing</th>
<th>Post-rebalancing</th>
<th>Difference in sub-samples</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.67***</td>
<td>0.22***</td>
<td>0.44***</td>
</tr>
<tr>
<td></td>
<td>(0.04)</td>
<td>(0.07)</td>
<td>(0.08)</td>
</tr>
<tr>
<td>$\sigma_\epsilon$</td>
<td>0.8***</td>
<td>0.38***</td>
<td>0.41***</td>
</tr>
<tr>
<td></td>
<td>(0.06)</td>
<td>(0.13)</td>
<td>(0.14)</td>
</tr>
<tr>
<td>$\sigma_\theta$</td>
<td>0.49***</td>
<td>0.57***</td>
<td>-0.08***</td>
</tr>
<tr>
<td></td>
<td>(0.02)</td>
<td>(0.03)</td>
<td>(0.04)</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.18***</td>
<td>2.28***</td>
<td>-2.1***</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.65)</td>
<td>(0.65)</td>
</tr>
<tr>
<td>$\mu_\eta$</td>
<td>0.37***</td>
<td>0.68***</td>
<td>-0.32***</td>
</tr>
<tr>
<td></td>
<td>(0.05)</td>
<td>(0.1)</td>
<td>(0.11)</td>
</tr>
<tr>
<td>$\sigma_\eta$</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
<td>&lt; 0.01</td>
</tr>
<tr>
<td></td>
<td>(0.15)</td>
<td>(0.18)</td>
<td>(0.24)</td>
</tr>
<tr>
<td>$\sigma_\zeta$</td>
<td>0.34***</td>
<td>0.26***</td>
<td>0.08***</td>
</tr>
<tr>
<td></td>
<td>(0.03)</td>
<td>(0.02)</td>
<td>(0.04)</td>
</tr>
</tbody>
</table>

Number of auctions 66 81 147  
Number of obs. 799 1024 1823

Note: (*) Significant at 10% level; (**) significant at 5% level; (*** ) significant at 1% level. This table presents the results of the estimation of the model before and after the J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014. Columns (1) and (2) report the maximum likelihood estimated parameters of the model described in 2.3. Standard errors reported in parentheses below estimated parameters. Standard errors are calculated using the numerical Hessian from the maximum likelihood estimation in section (2.3). Statistical significance for every parameter was established using the reported estimated value along with its associated standard error through a t-test with degrees of freedom given by the number of auctions in the last row of the table. Column (3) reports the estimated difference between the parameters before and after the rebalancing. To compute this difference, we construct an augmented likelihood function that spans both sub-samples where, we split the set of parameters to be estimated through the augmented likelihood in two groups. The first group corresponds to the vector of parameters in column (1) of table (2.4) for all auctions before the rebalancing. For every auction after the rebalancing, there is a transformed group of parameters defined by the difference in every parameter before and after the rebalancing. We estimate changes in every parameter in both sub-samples using the transformed vector of differences. Standard errors reported in parentheses below estimated parameters. Standard errors constructed using the Hessian of the likelihood function evaluated at the estimated vector of parameters. Number of auctions corresponds to the number of bond offerings with available secondary market pricing information. Number of observations corresponds to the sum of all bidders that participated in every auction with available secondary market pricing information.
2.5.3 Private information, market power and number of bidders

In this section we explore the behavior of the endogenous equilibrium variables in the model before and after the rebalancing. Specifically, based on the maximum likelihood estimates from table (2.4), we calculate $M(n)$, $c(n)$, and $d(n)/n$, for multiple number of bidders $n$, in these two scenarios. We plot the results of this exercise in figure 2.1.

Panel (a) plots $M(n)$ as a function of $n$. As discussed in Section 2.2.2, $M(n)$ captures the amount of bid shading that is attributable to private information. The results indicate that $M(n)$ is higher before the rebalancing announcement than afterwards for any $n$. This result is expected considering that $M(n)$ is a decreasing function of $\rho$ and the ratio $\sigma_{e}/\sigma_{\theta}$, both of which fall after the rebalancing. All else equal, the estimated change in private information thereby implies a reduction in bidders’ market power in the Colombian market after the rebalancing.

Panel (b) in figure 2.1 depicts $c(n)$, the slope (in absolute value) of the equilibrium demand schedule in Eq. (2.2.1), as a function of the number of bidders. Higher values of $c(n)$ imply more elastic demand schedules. For low values of $n$, bidders submit more inelastic demand schedules before the rebalancing. However, for $n \geq 12$ the opposite result obtains. We observe a similar pattern for $d(n)k/n$ in panel (d). As discussed in section 2.2.2, $d(n)k/n$ is a measure of bid shading or, equivalently, market power. Our results indicate that bid shading is slightly larger before the announcement for low $n$. However, this situation reverses as the number of bidders rises. Moreover, the absolute estimated change in bid shading is never larger than two bp.\footnote{Bid shading is expressed in bp (basis points) of the par value of the issue. It can be interpreted as a "discount" relative to the par amount.} Therefore, according to our estimates, bidders’ market power remains mostly unaltered after the rebalancing.

At first glance, the results regarding $c(n)$ and $d(n)k/n$ might seem at odds with our discussion above about the change in $M(n)$. Given that there is evidence of less private information after the rebalancing, we might have expected that bidders would submit more elastic demand curves, thereby reducing their market power. The reason behind this seeming discrepancy is that we also observe a substantial increase in $\lambda$, the slope of the decreasing marginal valuations. When $n$ is large, the increase in $\lambda$ more than offsets the impact that lower values of $M(n)$ have on the elasticity of the demand schedule and thus on bid shading.
Figure 2.1: Model equilibrium as a function of the number of bidders and parameter estimates

This figure plots the equilibrium of the model across multiple $n$, the number of bidders who participate in a given auction. The choice of $n$ in all panels is guided by table (2.2). $M(n)$ and $c(n)$ are calculated according to Eq. (2.2.1) for a given $n$. Bid shading equals $d(n)/nk$ for a given $k$, and $d(n)$ is calculated according to Eq. (2.2.1). $k$ is evaluated at the sample mean of total supply from table (2.1). The parameters $\rho$, $\sigma$, $\sigma_\theta$, and $\lambda$ come from our estimates in table (2.4) before and after the J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014. The blue line corresponds to the $M(n)$, $c(n)$, and bid shading with the estimates from column (1) of table (2.4). The red line depicts 2.3. The red line depicts the same variables with the estimates from column (2) of table (2.4).

To further illustrate the economic impact of these countervailing forces, in figure (2.2) we plot total bid shading, $d(n)k/n$, for different combinations of the estimates in table (2.4). In panel (a), we hold the private information parameters constant at their pre-rebalancing estimated values, and we plot $d(n)k/n$ for two different values of $\lambda$: its pre- and post-rebalancing estimated values, respectively. Clearly, the change in $\lambda$ alone implies a substantial increase in bid shading, ranging
from approximately 37bp - 120 bp, depending on the number of bidders. For comparison, the average absolute bid-ask spread from Bloomberg for all offered bonds in our sample is around 36 bp (see table (2.3)).

In panel (b) of figure (2.2), we keep $\lambda$ constant at its post-rebalancing value, but we let the private information parameters drop from their pre-rebalancing to their post-rebalancing estimates. The reduction in private information has the opposite effect on bid shading than the increase in $\lambda$. The combined effect is depicted in panel (c) of figure 2.1.

![Graphs](a) Change in $\lambda$ (high private info)  
(b) Change in private info (high $\lambda$)

**Figure 2.2:** Decomposition of bid shading into estimates of private information and slope of marginal valuation ($\lambda$)

This figure plots the estimated equilibrium bid shading of the model as a function of the number of bidders. Bid shading is expressed in bp of the par value of the bond offering. Bid shading equals $d(n)/nk$. $n$, the number of bidders who participate in a given auction, comes from table (2.2). $k$ is evaluated at the sample mean of total bond supply from table (2.1). $d(n)$ is calculated according to Eq. (2.2.1) for a given $n$ and combinations of the estimated parameters from table (2.4). High and low private information correspond to scenarios where the amount of private information is set at the estimates of $\rho$, $\sigma_\epsilon$, and $\sigma_\theta$ from the pre- and post-rebalancing samples, respectively. Flatter and steeper marginal valuation correspond to scenarios where $\lambda$ is fixed at pre- and post-rebalancing estimates, respectively.

### 2.5.4 Policy implications

Arguably, a goal of the auctioneer is to increase revenue by sustaining large auction prices. Bidders’ market power is a hindrance to this goal. The model in Section 2.2 predicts an unambiguous reduction in bid shading as the number of bidders increases. Hence, the auctioneer may want to consider a policy to foster primary market participation. Moreover, the structural estimation in Section 2.3 allows us to quantify the change in bid shading resulting from more participation. On the other hand, increasing the maximum number of bidders is likely costly, or not even feasible, as they expect compensation from their role as market makers. Thus, quantifying the benefits
to the bond issuer of increasing competition through lower bidders’ market power is necessary to
determine the optimal number of bidders.\textsuperscript{16}

To illustrate this use of the model estimation for policy analysis, we consider two cases. First,
the estimated post-rebalancing scenario which is characterized by low private information and high
inventory costs. Second, a hypothetical but empirically feasible high market power scenario with
high levels of private information matching the pre-rebalancing estimates, and the same inventory
costs from the post-rebalancing scenario. They are depicted in figure (2.3).

We first focus on bid shading evaluated at the sample mean of bond offering per auction. In
the high market power scenario in panel (a), even though bid shading is never lower than 30 bp
for any number of participants, it would fall by more than 80 bp if the number of bidders raises
from 10 to 14. Therefore, with high market power, fostering market participation is a policy worth
considering. In contrast, in the post-rebalancing scenario in panel (b), bid shading falls by only 4
bp in the same situation. In such situation, designating a small number of PDs would not result
in significantly higher borrowing costs for the government.

Another implication of Vives (2010, 2011) is that the auctioneer should monitor current market
conditions when deciding the size of each bond offering (per auction).\textsuperscript{17} As depicted in Figure 2.3,
reducing the bond offering in the high market power scenario would result in a substantial decrease
in total bid shading for any number of participants. In particular, a one standard deviation decrease
in total bond supply over its sample mean would lower bid shading by 20 bp with fourteen bidders,
and by 63 bp with ten.\textsuperscript{18} In contrast, in the post-rebalancing scenario, bid shading does not fall by
more than 4 bp for any number of bidders between 10 and 14 following a one standard deviation
decrease in bond supply over its sample mean.

\textsuperscript{16} As of 2022, Norway, Sweden, and Luxembourg have, respectively, designated only four, eight, and nine financial
institutions as PDs. In contrast, there are 44 PDs in Germany’s debt market. See https://www.esma.europa.eu/sites/default/files/library/list_of_market_makers_and_primary_dealers.pdf.

\textsuperscript{17} In the long run, total bond issuance is a fiscal policy decision. The policy evaluated here focuses on the size of
each bond offering at a specific auction, given a predetermined fiscal policy.

\textsuperscript{18} A one standard deviation in total bond supply per auction corresponds to roughly 100 hundred billion Colombian
pesos (approx. 50 billion USD as of March 2014). See panel A in table (2.1).
This figure plots the estimated equilibrium bid shading of the model as a function of the number of bidders. Bid shading is expressed in bp of the par value of the bond offering. Bid shading equals \( d(n)/nk \). \( n \), the number of bidders who participate in a given auction, comes from table (2.2). \( k \) is evaluated at the sample mean and one standard deviation (s.d.) below the sample mean of the total bond supply per auction, respectively. The sample means come from table (2.1). \( d(n) \) is calculated according to Eq. (2.2.1) for a given \( n \) and combinations of the estimated parameters from table (2.4). The high market power scenario corresponds to scenarios where the estimates of \( \rho \), \( \sigma_\epsilon \), and \( \sigma_\theta \) are from the pre-rebalancing sample, and \( \lambda \) is set at the post-rebalancing estimate.

### 2.6 Validation exercises

We have a twofold purpose in this section. First, we provide reduced-form evidence that is consistent with our structural estimates in Section (2.6.1). Second, we discuss the assumptions regarding the exogeneity of the decision to participate in any auction and that the effective number of participants is known by all bidders in (2.6.2).

#### 2.6.1 Model-free evidence

In this section we provide model-free empirical evidence that is consistent with the main results of the structural estimation. First, we find sizable evidence of the existence of private information in the primary market before the rebalancing announcement, but significantly less so afterwards. Second, we document significant changes in bidders’ inventory policy that accords well with the increase in the estimates of \( \lambda \), the slope of the marginal valuation function.

### Auction’s cutoff price and secondary market prices

A high realization of the auction cutoff price conveys to all market participants, including dealers in the secondary market that do not participate in the auction, that bidders received high private

![Figure 2.3: Bid shading evaluated at different levels of total bond supply](image-url)
signals, on average. Moreover, in the equilibrium with private information, the auction cutoff price aggregates these private signals containing information about the bond’s fundamental value.

We investigate whether new information aggregated by the auction predicts future secondary market prices. We measure new pricing information as the difference between the auction cutoff price, $p_{ct}$, and the bond secondary market price benchmark right before the auction $p_t$. Recall that $p_t$ is the volume-weighted average price of all transactions of the bond at issue between the market opening time (8 am) and the moment when the auction closes up (10 am). Therefore, $p_{ct} - p_t$ should mostly be driven by new information about bidders’ valuations aggregated by the auction’s cutoff price that is not already impounded into observable secondary market prices prior to the auction.

We test whether $p_{ct} - p_t$ predicts $p_t(\tau)$, for $\tau \in \{1, \ldots, 5\}$, where $p_t(\tau)$ is the bond’s secondary market price $\tau$-days after auction $t^{19}$. If the auction’s cutoff price is indeed revealing private information, then a high value of $p_{ct} - p_t$ would predict a positive $p_t(\tau) - p_t$. Moreover, we would expect the effect on future prices to be persistent over time (Brandt and Kavajecz (2004)).

To test this hypothesis, we pool all auctions in our sample and fit the following model:

$$p_t(\tau) - p_t = \beta_1(\tau)\text{rebal}_t + \beta_2(\tau)(p_{ct} - p_t) + \beta_3(\tau)\text{rebal}_t \times (p_{ct} - p_t) + \beta_4(\tau)n_t + \beta_5(\tau)k_t + \gamma(\tau)’\text{controls}_t + \varepsilon_t \tag{2.6.1}$$

for $\tau \in \{1, \ldots, 5\}$, where $t$ is a bond auction, rebal$_t$ is a post-rebalancing dummy, $n_t$ is the number of participants in auction $t$, $k_t$ is auction’s $t$ total offering, and controls$_t$ are both bond-specific (bond duration and convexity) and also market-wide (10-year CDS on Colombian bonds, VIX, and 1-year rate). Model (2.6.1) also includes bond fixed-effects.

The results, reported in table (2.5), indicate that the impact of $p_{ct} - p_t$ on $p_t(\tau) - p_t$ is positive and statistically significant for all $\tau \in \{1, \ldots, 5\}$ before the rebalancing. That is, a high cutoff auction price relative to the benchmark predicts high secondary market prices several days after the auction.\(^{20}\)

The interaction coefficient $\beta_{p,r}$ in Eq. (2.6.1) measures the change in the predictability of future prices from the auction price, following the rebalancing announcement. $\hat{\beta}_{p,r}(\tau)$ is negative across all $\tau$, and it is statistically significant for $\tau \in \{3, 4, 5\}$. The lower persistence of the post-rebalancing effect is consistent with the auction’s cutoff price aggregating less private information about fundamentals after the rebalancing and thus validates the results from the structural estimation.\(^{21}\)

\(^{19}\)For consistency, we use volume-weighted daily average transaction prices from the main Colombian inter-dealer market $\tau$-days after the auction as measures of secondary market prices.

\(^{20}\)Given that auctions occur every two weeks, by fixing $\tau$ between one and five trading days after the auction, we guarantee that no two adjacent auctions of the same bond overlap.

\(^{21}\)We also estimate a similar model to the one in Eq. (2.6.1) where the dependent variable is the bond yield $\tau \in \{1, \ldots, 5\}$ days after auction $t$. The results remain unaffected. Table (2.8) in the Appendix displays the estimated coefficients.
Table 2.5: Auction price and secondary market prices

<table>
<thead>
<tr>
<th>Dependent variable:</th>
<th>Second. market price ms. benchm. τ-days after auction</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>τ = 1</td>
<td>τ = 2</td>
<td>τ = 3</td>
<td>τ = 4</td>
<td>τ = 5</td>
</tr>
<tr>
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<td>(1)</td>
<td>(2)</td>
<td>(3)</td>
<td>(4)</td>
<td>(5)</td>
</tr>
<tr>
<td>Cutoff price ms. benchm.</td>
<td>1.552***</td>
<td>1.765***</td>
<td>1.637***</td>
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<td>2.857***</td>
</tr>
<tr>
<td></td>
<td>(0.201)</td>
<td>(0.242)</td>
<td>(0.230)</td>
<td>(0.281)</td>
<td>(0.369)</td>
</tr>
<tr>
<td>Post-rebal. dummy</td>
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<td>−0.018</td>
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<td></td>
<td>(0.274)</td>
<td>(0.330)</td>
<td>(0.314)</td>
<td>(0.384)</td>
<td>(0.503)</td>
</tr>
<tr>
<td>Cutoff price ms. benchm. (post-rebal.)</td>
<td>−0.637*</td>
<td>−0.727</td>
<td>−1.018**</td>
<td>−1.866***</td>
<td>−1.737***</td>
</tr>
<tr>
<td></td>
<td>(0.378)</td>
<td>(0.457)</td>
<td>(0.434)</td>
<td>(0.531)</td>
<td>(0.696)</td>
</tr>
<tr>
<td>Number of bidders</td>
<td>−0.113*</td>
<td>−0.189***</td>
<td>−0.137**</td>
<td>−0.186**</td>
<td>−0.135</td>
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<tr>
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<td>(0.057)</td>
<td>(0.069)</td>
<td>(0.066)</td>
<td>(0.080)</td>
<td>(0.105)</td>
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<tr>
<td>Bond offering</td>
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<td>−0.035</td>
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<td>0.126</td>
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<td>(0.088)</td>
<td>(0.107)</td>
<td>(0.101)</td>
<td>(0.124)</td>
<td>(0.163)</td>
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</tbody>
</table>

Bond FE: Yes Yes Yes Yes Yes
Bond controls: Yes Yes Yes Yes Yes
Macro controls: Yes Yes Yes Yes Yes
Observations: 147 147 147 147 147
R²: 0.381 0.373 0.366 0.436 0.373

Note: (*) Significant at 10% level; (**) significant at 5% level; (***) significant at 1% level. Every observation corresponds to the auction of a given bond. Dependent variable is average secondary market price of offered bond τ-days after auction minus price benchmark of to-be-offered bond before auction. Average secondary market price is calculated as volume-weighted average price of all transactions in the main inter-dealer market involving the bond τ-days after it was offered in the primary market. Price benchmark of to-be-offered bond before auction is calculated as the volume-weighted average price of all transactions on an auction-day involving the to-be-offered bond between the market opening time and the auction’s closing time. If there are no transactions in the inter-dealer market in that interval, the opening price from Bloomberg is used instead. "Post-rebal. dummy" is equal to one after J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014, and zero otherwise. "Cutoff price ms. bench." is equal to the auction cutoff price minus price benchmark of to-be-offered bond. "Cutoff price ms. bench. (post-rebal.)" is an interacted term between "Cutoff price ms. bench" and "Post-rebal. dummy". Bond controls include duration and convexity calculated from prices of auction-day. Macro controls include the 10-year CDS on the Colombian sovereign government, Chicago Board Options Exchange’s CBOE Volatility Index (VIX), and the 1-year yield on locally-denominated Colombian government debt calculated by the Central Bank of Colombia. Bond and macro controls are from auction-day.
**Steeper marginal valuation**

The results of the structural estimation in Section 2.5.2 also show a significant increase in $\lambda$ following the rebalancing implying steeper decreasing marginal valuations. In Vives (2011), marginal valuations can also be expressed as a function of the deviation from a target inventory level $x_{i,t}$, as $v(x_{i,t}) = \theta_{i,t} - \lambda(x_{i,t} - \hat{x}_{i,t})$. Hence, $\lambda$ can be interpreted as an inventory adjustment cost. This suggests that bidders became less willing to deviate from their inventory target levels.

To test this implication, we obtained additional data from the Colombian Central Bank on bidders’ daily bond inventory. This data is not used to estimate the auction model. Instead, we use it to examine how dealers control their overnight bond inventory through the following model:

$$\text{inv}_{i,t} - \text{inv}_{i,t-1} = \alpha_i + \alpha_{\text{rebal}}t + \alpha_{\text{inv}}\text{inv}_{i,t-1} + \alpha_{\text{inv, rebal}}t \times \text{inv}_{i,t-1} + \epsilon_{i,t} \quad (2.6.2)$$

where $\text{inv}_{i,t}$ is bidder $i$’s (standardized) aggregate bond inventory on day $t$ and rebal$t$ is a post-rebalancing dummy. We also add bidder fixed-effects $\alpha_i$.

The model above is based on the inventory risk literature (see, for example, Madhavan and Smidt (1993); Friewald and Nagler (2016); Schultz (2017)). $\alpha_{\text{inv}}$ shows how average daily changes in inventory between $t-1$ and $t$ are affected by the average inventory level in $t-1$. Negative values of $\alpha_{\text{inv}}$ imply that bidders’ inventory exhibit mean-reversion. The coefficient $\alpha_{\text{inv, rebal}}t$ captures changes in bidders’ inventory policy following the rebalancing announcement. Armed with the estimates $\hat{\alpha}_{\text{inv}}$ and $\hat{\alpha}_{\text{inv, rebal}}$, we also estimate inventory half-life in the pre- and post-rebalancing samples as $\log(2) / (1 - \hat{\alpha}_{\text{inv}})$ and $\log(2) / (1 - \hat{\alpha}_{\text{inv}} - \hat{\alpha}_{\text{inv, rebal}})$, respectively. According to Madhavan and Smidt (1993); Friewald and Nagler (2016); Schultz (2017), this measures the average number of days that a dealer takes to offload half of their inventory position.

The estimates of Eq. (2.6.2) are reported in panel A of table (2.6). The estimated inventory half-lives before and after the rebalancing, calculated with the coefficients from panel A, are shown in panel B of the same table. Before the rebalancing, the inventory mean-reversion is either equal to -0.061 or -0.058, respectively. This means that, on average, 6.1% or 5.8% of the aggregate bidders’ inventory position is eliminated overnight. This implies that bidders take 11.7 days, on average, to offload half of their inventory holdings. Interestingly, after the rebalancing there is a significant increase in the speed of the mean-reversion parameter in columns (1) and (2) in panel A, thus resulting in a concomitant adjustment of the inventory half-life down to 7.5 days. On average, it takes bidders four fewer days to unwind half of their aggregate inventory position after the rebalancing, which corresponds to a time reduction of roughly 36%.

This change in bidders’ inventory management is consistent with a shift from a principal model of market making towards an agency model.\(^{22}\) In the former, bidders act as market makers who

\(^{22}\)See Duffie (2012); Adrian et al. (2017); Bessembinder et al. (2018), among others, for a related discussion in the aftermath of the adoption of the Volcker rule.
stand ready to accommodate asynchronous trading needs by temporarily warehousing bonds in their books until offsetting orders arrive. They are compensated through the bid-ask spread for the inventory risk they incur as liquidity providers. In contrast, in an agency model, bidders act as brokers who take on less inventory risk by attempting to match buyers and sellers directly. In this scenario, the bid-ask spread would be lower, which indeed is the case in the post-rebalancing sample as documented in section 2.5.1.

2.6.2 Discussion of assumptions

A crucial assumption of the theoretical model of Vives (2010, 2011) is that the number of bidders, $n_t$, is exogenous and known by all bidders. The theoretical model does not address the source of fluctuations in the number of bidders who decide to participate in any given auction. This is of course a restrictive assumption. In this section we discuss each part of this assumption separately.

**Exogenous number of participants in any auction**

Implicit in the theoretical model is the assumption that any bidder’s decision to participate in a given auction is independent from her own signal regarding the *ex post* value of the object. To investigate whether this is actually the case in our setting, we use individual bidder’s inventory prior to entering the auction as a proxy for her *ex ante* valuation of the bond. Intuitively, if a bidder holds a long position on the to-be-offered bond, then the idiosyncratic component in her valuation will be lower, which would encourage the bidder to submit a conservative bid. Conversely, if the bidder holds a short position on the bond, then she might be prompted to submit a more competitive bid to cover her position. With this idea in mind, we first explore empirically whether pre-auction inventory predicts bidding behavior. After establishing the significance of this empirical relationship, we then proceed to investigate whether pre-auction inventory predicts bidder’s participation.

There is an important caveat to address before setting out to test this hypothesis. Bidders’ inventory is idiosyncratic and not necessarily related to private information about bond fundamentals. Unfortunately, we do not have good proxies for bidder’s individual valuation prior to entering the auction that are more directly related to the amount of private information that every bidder possesses. In fact, one of the main contribution of our paper is to propose an econometric methodology to quantify the importance of private information about fundamentals precisely because such variables are difficult to come by. This caveat notwithstanding, our structural estimation results indicate that in both subsamples there is an important idiosyncratic component in bidders’ valuation.

We formally explore whether bidders’ inventory prior to entering the auction affects bidding behavior. To test this, we fit the following reduced-form model based on a modified version of Eq.
Table 2.6: Inventory mean-reversion and half-life after rebalancing announcement

<table>
<thead>
<tr>
<th>Panel A: Inventory mean-reversion</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td><strong>Mean-reversion:</strong> ( \hat{\alpha}_{inv} )</td>
<td>-0.061***</td>
<td>-0.058***</td>
</tr>
<tr>
<td></td>
<td>(0.007)</td>
<td>(0.008)</td>
</tr>
<tr>
<td><strong>Post-rebal. dummy:</strong> ( \hat{\alpha}_r )</td>
<td>0.008</td>
<td>0.009</td>
</tr>
<tr>
<td></td>
<td>(0.006)</td>
<td>(0.009)</td>
</tr>
<tr>
<td><strong>Mean-reversion (post-rebal.):</strong> ( \hat{\alpha}_{inv,r} )</td>
<td>-0.036***</td>
<td>-0.041***</td>
</tr>
<tr>
<td></td>
<td>(0.012)</td>
<td>(0.014)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Inventory half-life (in days)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td><strong>Before rebalancing:</strong> ( \frac{\log(2)}{1 - \hat{\alpha}_{inv}} )</td>
<td>11.7</td>
<td>11.7</td>
</tr>
<tr>
<td><strong>After rebalancing:</strong> ( \frac{\log(2)}{1 - \hat{\alpha}<em>{inv} - \hat{\alpha}</em>{inv,r}} )</td>
<td>7.5</td>
<td>7.5</td>
</tr>
<tr>
<td><strong>Bidder FE</strong></td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td><strong>Observations</strong></td>
<td>7,264</td>
<td>7,264</td>
</tr>
<tr>
<td><strong>R(^2)</strong></td>
<td>0.044</td>
<td>0.044</td>
</tr>
</tbody>
</table>

**Note:** (*) Significant at 10% level; (**) significant at 5% level; (***) significant at 1% level. This table presents the estimated coefficients of model (2.6.2) (panel A), and estimated inventory half-life based on these coefficients (panel B). Aggregate daily inventory corresponds to end-of-day holdings of all TES bonds at the individual level for ten out of fourteen bidders. Aggregate inventories are standardized at the bidder level by subtracting the sample mean and then dividing every resulting observation by the sample standard deviation. "Post-rebal. dummy" is equal to one after J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014, and zero otherwise. The sample ranges from 2013 to 2015.
markdown\(_{i,t}\) = \(\alpha_i + \beta_{\text{inv inventory}}\_i,t + \beta_k k_t + \epsilon_{i,t}\) (2.6.3)

where bidder \(i\)'s markdown in auction \(t\) is given by markdown\(_{i,t}\) = \(p_{0,i,t} - p_t\), where \(p_{0,i,t}\) is the highest bid submitted by bidder \(i\) in auction \(t\), \(\bar{p}_t\) is the secondary market price benchmark in auction \(t\), \(k_t\) is the total bond supply offered, and inventory\(_{i,t}\) is bidder \(i\)'s (de-trended and standardized) inventory on bond \(t\) one day before auction \(t\).\(^{23}\)  Intuitively, a higher markdown captures a more competitive or aggressive bid because \(p_{0,i,t}\) is high relative to the price benchmark. \(\alpha_i\) are bidder fixed-effects.

The results of this model are reported in column (1) of table (2.7). A one standard-deviation increase in aggregate inventory reduces bidders’ markdown by almost 10 percent. Thus, when a bidder is on average long Colombian bonds, she will be prompted to submit less competitive bids, which is consistent with a lower idiosyncratic valuation. Now that we have shown that pre-auction inventory predicts bidding behavior, we test whether it also predicts bidder participation by fitting the following linear probability model:

partic\(_{i,t}\) = \(\alpha_i + \beta_{\text{inv inventory}}\_i,t + \beta_k k_t + \epsilon_{i,t}\) (2.6.4)

where partic\(_{i,t}\) takes the value of one if bidder \(i\) participates in auction \(t\) and zero otherwise. \(\alpha_i\) are bidder fixed-effects. The results of this model are reported in column (2) of table (2.7). It is immediately seen from the first row that we cannot reject the null hypothesis that \(\beta_{\text{inv}}\) is significantly different from zero.\(^{24}\)

We cannot categorically conclude from these two empirical exercises that \(n_t\) is exogenous to bidders’ private signals because, as mentioned above, individual inventory is a predictor of idiosyncratic valuation that is not necessarily related to private information about fundamentals, which is important according to our estimates. However, to the extent that pre-auction inventory is a good proxy for their signals, bidders’ decision to participate in the auction would appear to be unrelated to their valuation.

**Known number of participants in any auction**

We turn our attention here to the discussion of how realistic is to assume that the number of bidders who participate in any auction \(t\), \(n_t\), is known to all auction participants. By regulation,

\(^{23}\)Unfortunately, we only have daily aggregate inventory data at the bidder level for all Colombian government bonds denominated in local currency. Thus, if auctions for the bonds with maturity in 2024 and 2028, for instance, are held simultaneously, the inventory measure will be the same in both auctions. Furthermore, we do not have inventory measures for four (out of fourteen) bidders at any point of time, since these four bidders are not commercial banks and, as such, are not required to report their inventory holdings.

\(^{24}\)We also fitted a logistic model (not reported in this paper) where we could not reject the null hypothesis either.
Table 2.7: Bidder’s markdown and auction participation

<table>
<thead>
<tr>
<th>Dependent variable:</th>
<th>Bidder’s markdown</th>
<th>Bidder participation</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(1)</td>
<td>(2)</td>
</tr>
<tr>
<td>Inventory before auction</td>
<td>−0.088*** (0.018)</td>
<td>0.005 (0.008)</td>
</tr>
<tr>
<td>Total bond supply</td>
<td>0.152*** (0.021)</td>
<td>0.017** (0.008)</td>
</tr>
<tr>
<td>Bidder FE</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>Observations</td>
<td>1,174</td>
<td>1,563</td>
</tr>
<tr>
<td>R²</td>
<td>0.342</td>
<td>0.320</td>
</tr>
</tbody>
</table>

Note: (*) Significant at 10% level; (**) significant at 5% level; (***) significant at 1% level. This table displays the estimated coefficients of the linear models (2.6.3) in column (1), and (2.6.4) in column (2). Dependent variables are bidder’s markdown (column (1)) and bidder participation (column (2)), respectively. Bidder’s markdown is calculated as the difference between a bidder’s highest submitted price in a given auction minus price benchmark. Price benchmark of a bond is calculated as the volume-weighted average price of all transactions on an auction-day involving the to-be-offered bond between the market opening time (8 am) and the moment when the auction closes up (10 am). If there are no transactions in the inter-dealer market in that two-hour interval for a given to-be-offered bond on an auction-day, the opening price from Bloomberg is used instead. Bidder participation takes the value of one when a bidder participates in an auction and zero otherwise. “Inventory before auction” is the (de-trended and standardized) bidder’s aggregate inventory on all Colombian sovereign bonds on the day before the auction denominated in COP. “Total bond supply” is the pre-announced par value of the bond to be issued in the auction in hundreds of billions of COP. Summary statistics for “total bond supply” can be found in table (2.1).
only fourteen PDs are allowed to participate in any Colombian treasury auction. Thus, PDs know that they will be bidding against fourteen bidders at the most. However, as discussed, not all fourteen PDs participate in every auction. This situation begs the question of whether PDs know with certainty the number of auction participants they will be effectively competing against. Rather than evaluating this assumption empirically, which might be an unfeasible task considering our current data, we adopt an indirect route to investigate its implications by testing whether the model’s predictions regarding changes in the number of bidders hold in the bidding data.

Concretely, we know that the endogenous variables that determine the model’s equilibrium in Eqs. (2.2.1) are functions of the deep parameters \( \rho, \sigma, \epsilon, \lambda \), and the number of bidders in auction \( t, n_t \). Thus, the slope of the equilibrium demand curve is in turn a function of \( n_t \). We do not observe directly the equilibrium slope of the demand curve, but we can calculate it using our structural estimates, as we did in figure (2.1). Moreover, we argued in section 2.3 that there is a direct relationship between the market-clearing price \( \hat{p}_t \) and the slope \( c(n_t) \), which holds for any \( n_t \), given by \( \hat{p}_t = \frac{1}{n_t} \sum_{i=1}^{n_t} p_{i,t}^0 - \frac{k_t}{n_{cc}(n_t)}. \) Although this equation is derived from the model’s market-clearing condition and symmetric linear equilibrium, it has the appealing feature that it is solely a function of observed variables such as the average bidders’ intercept for a given auction, total bond supply, number of bidders, and the model-implied market clearing price. Thus, we can estimate \( c(n_t) \) directly from bidding data alone from \( p_{c,t}^c = \hat{p}_t + \zeta_t \), or:

\[
p_{c,t}^c - \frac{1}{n_t} \sum_{i=1}^{n_t} p_{i,t}^0 = \beta_k(n_t) k_t + \zeta_t
\]

for any auction \( t \) with \( n_t \) bidders, where \( p_{c,t}^c \) is observed market cutoff price, and \( \beta_k(n_t) = -\frac{1}{n_{cc}(n_t)}. \) This information was already incorporated into our proposed likelihood function (see section 2.3), but the simple linear regression above does not rely on any of the non-linear dependence between the equilibrium variables and the deep parameters that the model imposes in Eqs. (2.2.1).

Then, using the estimated coefficients of \( \beta_k(n_t) \) from Eq. (2.6.5), we can calculate \( \hat{c}(n_t) \) from \( \hat{\beta}_k(n_t) = -\frac{1}{n_{cc}(n_t)}. \) In figure (2.4) we plot \( c(n_t) \) across \( n_t \) from the results of the structural estimates in table (2.4) (left panel), along with \( \hat{c}(n_t) \) using the estimated coefficients of \( \beta_k(n_t) \) from Eq. (2.6.5) along with \( \hat{\beta}_k(n_t) = -\frac{1}{n_{cc}(n_t)} \) (right panel). We label them ”model” and ”empirical” slopes for the sake of briefness.

We interpret figure (2.4) as suggestive evidence that the data support the model’s predictions regarding the relationship between the number of bidders and the slope of the demand curve. Specifically, we observe that, according to the structural estimation results in the left panel, the ”model” slopes are increasing in \( n_t \) before and after the rebalancing. We observe mostly the same

\[25\] There is survey evidence suggesting that even though bidders do not necessarily disclose to one another whether they will participate in any given auction, some of them are able to infer who will do so based on conversations with their clients ((Cardozo, 2013)).
pattern for the "empirical" slopes, except for \( n_t = 13 \) in the post-rebalancing sample. Furthermore, after the rebalancing, the "model" slopes become less responsive to changes in \( n_t \), or, equivalently, \( c(n_t) \) becomes flatter relative to \( n_t \). Between \( n_t = 10 \) and \( n_t = 14 \), we observe a similar result in the "empirical" slopes although not as smooth as the one in the left panel. Hence, the model’s predictions regarding the relationship between the slope of the demand curve and the number of bidders are supported by empirical counterparties based on observable data and the model’s market-clearing condition and linear equilibrium.

We do not claim to provide decisive evidence in favor of this assumption. However, based on figure (2.4) we can say that the two main predictions of the structural estimation regarding the relationship between number of bidders and the equilibrium slope of the demand curve appear to hold up in the data.

**Figure 2.4:** \( c(n_t) \): model-implied vs empirical demand curve slope

This figure plots \( c(n_t) \) across multiple \( n_t \), the number of bidders who participate in a given auction. Choice of \( n_t \) in both panels is guided by table (2.2). \( c(n_t) \) in panel (a) are calculated according to Eq. (2.2.1), where value of parameters \( \rho, \sigma_\epsilon, \sigma_\theta \), and \( \lambda \) come from estimates in table (2.4) before and after the J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014. Red line in panel (a) depicts \( c(n_t) \) as a function of \( n_t \) using the estimates in column (1) of table (2.4). Blue line in panel (a) depicts \( c(n_t) \) as a function of \( n_t \) using the estimates in column (2) of table (2.4). \( c(n_t) \) in panel (b) are calculated from \( \beta_k(n_t) = -\frac{1}{n_t \cdot c(n_t)} \), using the estimates of \( \beta_k(n_t) \) from the linear model in Eq. (2.6.5) for all \( n_t \). Red line in panel (b) depicts \( c(n_t) \) using estimated \( \beta_k \) from the pre-rebalancing sample for all \( n_t \). Blue line in panel (b) depicts \( c(n_t) \) using estimated \( \beta_k \) from the post-rebalancing sample for all \( n_t \).
2.7 Final remarks

We propose a method to empirically quantify the importance of private information in divisible goods uniform price auctions based on the estimation of a model presented in Vives (2010, 2011). We apply our method to the Colombian sovereign debt market after the rebalancing of two emerging markets debt indices that triggered an increase in the demand for Colombian sovereign bonds. We document that these bonds were more actively traded after the shock, the secondary market became more liquid and experienced a significant decrease in volatility. We estimate the model separately before and after the shock.

We find a substantial reduction in private information about fundamentals as a determinant of observed bids but also a sizeable increase in inventory costs. These two findings have opposite effects on market power. Our estimation results indicate that the change in revenue for the treasury is lower than two bp of the par value of the bond. However, the latter masks the separate effects of private information and inventory costs. We decompose these two effects and find that the reduction in private information implies an increase in revenue of 34-112 bp. In contrast, the increase in inventory costs lowers revenue by 35-110 bp.

We provide model-free evidence that supports the reduction in the importance of private information and the increase in inventory adjustment costs. If private information about fundamentals is prevalent in this market, then an auction’s cutoff price would reflect bidders’ private signals about fundamentals that were not already captured by observable market prices prior to the auction. We show that auctions with high cutoff prices relative to a pre-auction benchmark predict high subsequent secondary market prices. After the rebalancing, this effect either shrinks or vanishes altogether, which is consistent with a decline in the importance of private information. Furthermore, we find faster overnight mean-reversion in inventories after the rebalancing. On average, it takes bidders 11.7 days to offload half of their inventories before the rebalancing, but only 7.5 days after the shock.

We quantify the effects of private information and inventory costs on policies meant to lower treasuries’ borrowing costs. After the rebalancing, our structural estimates imply that increasing the number of bidders from 10 to 14 reduces bid shading by 4 bp. In contrast, in a hypothetical high private information and inventory scenario, market power falls significantly when the number of bidders increases from 10 to 14.
2.8 Appendix

Proof of Proposition 2.3.2

Let’s assume that there are infinitely large sets of auctions, each with a fixed number of bidders per auction \( \bar{n} \) and \( \hat{n} \), respectively, where \( \bar{n} > 2, \hat{n} > 2 \) and \( \bar{n} \neq \hat{n} \). It follows directly from equation (2.3.2) that \( \mu_\eta \) is identified and \( M(n) \) is identified for \( n \in \bar{n}, \hat{n} \). Hence, and \( d(n) \) and \( \lambda \) are also identified. Now let,

\[
K(n) = \frac{M(n)}{n(1 + M(n))} = \frac{\rho\sigma^2_\epsilon}{(1 + (n - 1)\rho)\left(\sigma^2_\epsilon + \sigma^2_\theta(1 - \rho)\right)}
\]

(2.8.1)

It follows that both \( K(\bar{n}) \) and \( K(\hat{n}) \) are identified. Moreover,

\[
\frac{K(\bar{n})}{K(\hat{n})} = \frac{1 + (\hat{n} - 1)\rho}{1 + (\bar{n} - 1)\rho},
\]

(2.8.2)

hence \( \rho \) is identified.

The difference between the conditional variance and covariance of the intercepts can be expressed as

\[
\text{Var}[p_{i,t}|n, k_t, \bar{p}_t] - \text{Cov}[p_{i,t}, p_{j,t}|n, k_t, \bar{p}_t] = \frac{a(n)^2}{c(n)^2} \left(\sigma^2_\theta + \sigma^2_\epsilon\right) - \frac{a(n)^2}{c(n)^2} \rho\sigma^2_\theta
\]

\[
= \left(1 + (1 - \rho)\frac{\sigma^2_\theta}{\sigma^2_\epsilon}\right) \left(1 + (n - 1)\rho\right)^2 \frac{\sigma^2_\theta}{\sigma^2_\epsilon} \frac{\sigma^2_\epsilon}{\sigma^2_\theta}
\]

(2.8.3)

Moreover, from Proposition 2.2.1,

\[
M(n) = \frac{\rho\sigma^2_\epsilon n}{(1 - \rho)\left(\sigma^2_\epsilon + (1 + (n - 1)\rho)\sigma^2_\theta\right)}. \tag{2.8.4}
\]

Lets now consider two cases. If \( M(\bar{n}) = M(\hat{n}) = 0 \), it follows from the definition of \( M \) that \( \sigma^2_\epsilon = 0 \). In such case, \( \sigma^2_\theta \) is also identified from 2.8.3.

If instead, \( M(\bar{n}) > 0 \) and \( M(\hat{n}) > 0 \), then \( \sigma^2_\epsilon > 0 \) and

\[
\left(\frac{\rho\bar{n}}{(1 - \rho) M(\bar{n}) - 1}\right)(1 + (\bar{n} - 1)\rho)^{-1} = \frac{\sigma^2_\theta}{\sigma^2_\epsilon}, \tag{2.8.5}
\]

hence the ratio \( \frac{\sigma^2_\theta}{\sigma^2_\epsilon} \) is identified. It then follows from 2.8.3 that \( \sigma^2_\theta \) and \( \sigma^2_\epsilon \) are separately identified.

From Proposition 2.2.1, \( a(\bar{n}) = \frac{(1 - \rho)\sigma^2_\theta}{(1 - \rho)\sigma^2_\theta + \sigma^2_\epsilon}(d(\bar{n}) + \lambda)^{-1} \), hence \( a(\bar{n}) \) is also identified, and the parameter \( \sigma^2_\theta \) is then also identified from the conditional variance of the intercepts. This completes the proof that the model parameters \((\rho, \sigma_\epsilon, \sigma_\theta, \lambda)\) and the incidental parameters \((\mu_\eta, \sigma^2_\eta)\) are all identified.
Likelihood function

For each auction $t$ the probability density function of $p^0_t$ is given by

$$f^0_t (p^0_t | n_t, k_t, \bar{p}_t; \Gamma) = \frac{1}{\sqrt{(2\pi)^{n_t} | \Sigma_t|}} \exp \left( -\frac{1}{2} \left( p^0_t - \mu_t \right)^T \Sigma_t^{-1} \left( p^0_t - \mu_t \right) \right)$$

where $\Gamma = (\rho, \sigma, \sigma_, \lambda, \mu_\eta, \sigma_\eta^2)$, $\mu_t$ is the vector of conditional means $E[p^0_{i,t} | n_t, k_t, \bar{p}_t]$ and $\Sigma_t$ is the variance-covariance matrix with diagonal elements $\text{Var}[p^0_{i,t} | n_t, k_t, \bar{p}_t] = \frac{a(n_t)^2}{c(n_t)^2} (\sigma_\theta^2 + \sigma_\epsilon^2) + \sigma_\eta^2$, for all $i$, and off-diagonal elements equal to $\text{Cov}[p^0_{i,t}, p^0_{j,t} | n_t, k_t, \bar{p}_t] = \frac{a(n_t)^2}{c(n_t)^2} \rho \sigma_\theta^2 + \sigma_\eta^2$ for all $i \neq j$.

Similarly, the conditional density of $p^c_t$ is

$$f^c_t (p^c_t | n_t, k_t, \bar{p}_t, p^0_t; \Gamma) = \frac{1}{\sqrt{2\pi \sigma_\zeta^2}} \exp \left( -\frac{1}{2} \left( p^c_t - \mu^c_t \right)^2 / \sigma_\zeta^2 \right)$$

where $\mu^c_t$ is the conditional mean $E[p^c_{i,t} | n_t, k_t, p^0_t]$. For a sample of $T$ independent auctions $\{n_t, k_t, \bar{p}_t, p^0_t, p^c_t\}_{t=1}^T$, the log-likelihood function is

$$L(\Gamma) = \sum_{t=1}^T \log f^0_t (p^0_t | n_t, k_t, \bar{p}_t; \Gamma) + \log f_t (p^c_t | n_t, k_t, \bar{p}_t, p^0_t; \Gamma)$$
Additional reduced form evidence

Auction’s cutoff price and secondary market prices

Table 2.8: Auction price and secondary market yields

<table>
<thead>
<tr>
<th>Dependent variable:</th>
<th>Second. market yields ms. benchm. τ-days after auction</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
<th>(4)</th>
<th>(5)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>τ = 1</td>
<td>τ = 2</td>
<td>τ = 3</td>
<td>τ = 4</td>
<td>τ = 5</td>
<td></td>
</tr>
<tr>
<td>Cutoff price ms. benchm.</td>
<td>-0.486***</td>
<td>-0.504***</td>
<td>-0.491***</td>
<td>-0.608***</td>
<td>-0.650***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.094)</td>
<td>(0.094)</td>
<td>(0.095)</td>
<td>(0.093)</td>
<td>(0.093)</td>
<td></td>
</tr>
<tr>
<td>Post-rebal. dummy</td>
<td>-1.383***</td>
<td>-1.345***</td>
<td>-1.341***</td>
<td>-1.402***</td>
<td>-1.363***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.128)</td>
<td>(0.128)</td>
<td>(0.130)</td>
<td>(0.127)</td>
<td>(0.127)</td>
<td></td>
</tr>
<tr>
<td>Cutoff price ms. benchm. (post-rebal.)</td>
<td>0.813***</td>
<td>0.809***</td>
<td>0.845***</td>
<td>0.960***</td>
<td>0.945***</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.177)</td>
<td>(0.177)</td>
<td>(0.179)</td>
<td>(0.176)</td>
<td>(0.175)</td>
<td></td>
</tr>
<tr>
<td>Number of bidders</td>
<td>-0.068**</td>
<td>-0.059**</td>
<td>-0.066**</td>
<td>-0.061**</td>
<td>-0.067**</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.027)</td>
<td>(0.027)</td>
<td>(0.027)</td>
<td>(0.027)</td>
<td>(0.027)</td>
<td></td>
</tr>
<tr>
<td>Bond offering</td>
<td>0.074*</td>
<td>0.076*</td>
<td>0.088**</td>
<td>0.078*</td>
<td>0.064</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.041)</td>
<td>(0.041)</td>
<td>(0.042)</td>
<td>(0.041)</td>
<td>(0.041)</td>
<td></td>
</tr>
</tbody>
</table>

| Bond FE | Yes | Yes | Yes | Yes | Yes |
| Bond controls | Yes | Yes | Yes | Yes | Yes |
| Macro controls | Yes | Yes | Yes | Yes | Yes |
| Observations | 147 | 147 | 147 | 147 | 147 |
| R²       | 0.916 | 0.917 | 0.915 | 0.918 | 0.918 |

Note: (*) Significant at 10% level; (**) significant at 5% level; (***) significant at 1% level. This table displays the estimated coefficients of a model analogous to that of Eq. (2.6.1) with the bond yield as dependent variable. Every observation corresponds to the auction of a given bond. Dependent variable is average secondary market yield of offered bond τ-days. Average secondary market yield is converted from volume-weighted average price of all transactions in the main inter-dealer market involving the offered bond τ-days after the bond was offered in the primary market. "Post-rebal. dummy" is equal to one after J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014, and zero otherwise. "Cutoff price ms. benchm." is equal to the auction cutoff price minus price benchmark of to-be-offered bond. "Cutoff price ms. benchm. (post-rebal.)" is an interacted term between "Cutoff price ms. benchm" and "Post-rebal. dummy". Bond controls include duration and convexity calculated from prices of auction-day. Macro controls include the 10-year CDS on the Colombian sovereign government, Chicago Board Options Exchange’s CBOE Volatility Index (VIX), and the 1-year yield on locally-denominated Colombian government debt calculated by the Central Bank of Colombia. Bond and macro controls are from auction-day.
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