On the Asset Pricing Implications of Incomplete Information in Sovereign Debt Markets

by

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Para Willy y Saul, que no pudieron presenciar
la culminacion de estas correr as, y
Claire y Renate, que s pudieron

Abstract

In the first chapter, we investigate the asset pricing implications of disagreement about the unobservable process that drives monetary policy. We construct a monetary economy with two investors who have heterogeneous priors about the unobservable expected growth of money, namely, the underlying monetary stance. The key friction is that information about this unobservable variable is symmetric but incomplete. Investors estimate it from observed realizations of money supply in a Bayesian framework. Investors value real money holdings intrinsically, which gives rise to a demand for money. In equilibrium, the demand for money and investors' estimates endogenously determine the nominal short rates and expected inflation through a portfolio rebalancing channel. Disagreement about money growth is one driver of the conditional variation in all nominal interest rates, expected inflation, and inflation risk-premium. Through a parameterized example, we show that the volatility of nominal interest rates tend to rise with increases in disagreement. We show that the relation between disagreement about money growth and asset prices crucially depends on the persistence of the unobservable growth rate of money. We further use the model to investigate the consequences of allowing for heterogeneous interpretations of public information (*e.g.* central bank statements) on interest rates. We introduce a public signal on which investors may have different views. One steadfast investor believes the signal conveys relevant information about the unobservable process, while a doubtful one believes it contains only noise. Depending on the persistence of the unobservable process, interest rates may be more or less volatile in an economy populated by one steadfast investor and one doubtful investor that in an economy with two doubtful investors.

In the second chapter, we propose a method to measure the importance of private information about fundamentals as a determinant of market power in the primary market for sovereign debt. The method is based on the estimation of a divisible good uniform price auction model presented in Vives (2010, 2011). We establish conditions under which the model is identified from bidding data. Through an application to the Colombian debt market, we show that private information decreases and inventory costs rise after the rebalancing of two emerging markets debt indices. These two findings have opposite effects on bid shading. The estimation results indicate that the change in revenue for the treasury is lower than two basis points of the par value of the bond offering. However, this result masks the economically meaningful impact that private information

and inventory costs separately have on market power. We use the model estimates to quantify the effect of these changes on bidders' market power, and to evaluate the effectiveness of policies meant to reduce borrowing costs.

Finally, in the third chapter, we provide model-free evidence that supports the estimated reduction in the importance of private information and the increase in inventory costs from our structural estimates. If bidders are privately informed in this market, then an auction's cutoff price would reflect bidders' private information about fundamentals that were not already captured by observable market prices. We show that auctions with high cutoff prices relative to a pre-auction benchmark predict high future secondary market prices. After the rebalancing, this effect either shrinks or vanishes altogether, consistent with a decline in the importance of private information. Furthermore, we find higher mean-reversion in inventories after the rebalancing. On average, it takes bidders 11.7 days to offload half of their inventories before the rebalancing, but only 7.5 days afterwards. Finally, we provide reduced-form evidence that is consistent with the assumptions regarding primary market participation from the theoretical model and our empirical identification and estimation strategies.

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Chapter 1

Disagreeing about money growth in a monetary economy

1.1 Introduction

Over the past 30 years, and especially since the financial crisis in 2008, central banks have increasingly relied on tools such as forward guidance to guide the market participants' views on the future path of interest rates and the unobservable monetary policy stance (Campbell et al., 2017). Blinder et al. (2008) argue that \[N]owadays, it is widely accepted that the ability of a central bank to affect the economy depends critically on its ability to influence market expectations about the future path of overnight interest rates, and not merely on their current level." In a similar vein, King et al. (2008) assert that monetary policy is now mostly concerned about managing the expectations of the private sector.

Nevertheless, these efforts to manage private sector's views have not resulted in a concomitant decrease in disagreement among investors about the future path of interest rates. Empirical evidence indicates that interest rate expectations remain dispersed across market participants. Andrade et al. (2019) report the presence of substantial heterogeneity in the inflation and Federal Funds rate forecasts across participants in the Blue Chip Financial Forecasts survey. Similarly, Caballero and Simsek (2020) document significant disagreement between The Fed's predictions and forward rates that are difficult to reconcile with traditional macroeconomic models. These levels of disagreement are likely to bring important asset pricing implications. For instance, Ehling et al. (2018) theoretically and empirically show that disagreement about inflation has a strong impact on the yield curve.

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Monetary authorities track the private sector's view on monetary policy stance and interest rate expectations. They are particularly interested in registering shifts in market views. For instance, the New York Fed performs the Survey of Primary Dealers (SPD) and the Survey of Market Participants that reads, \The objective of the Survey of Primary Dealers [. . .] is to gain insight into the expectations of primary dealer rms. [...] [R]espondents have been queried about their expectations for the future level of the federal funds rate. [...] Survey results [...] are used by Federal Reserve sta in their evaluations of market expectations for the economic outlook, monetary policy, and the nancial markets. [...] [O]ccasionally, the Desk asks respondents to update their responses immediately following an FOMC meeting to gauge how expectations have changed in response to new information."[1]

Given these considerations, we argue that the development of a benchmark for evaluating the asset pricing implications of disagreement about the monetary policy is desirable. In this paper, we construct a monetary model to investigate the relation between interest rates, expected in ation, and the underlying monetary policy stance. Our monetary model is embedded into an exchange economy with exogenous paths for consumption and nominal money supply, both of which are observable. By contrast, we assume that the expected growth rate of money supply is an unobservable and stationary stochastic process. Investors use their heterogeneous priors and observations of changes in the money supply to estimate the expected growth rate of money. In the model equilibrium, the investors' estimates, on which they base their investment decisions, become the driver of the conditional variation in interest rates and expected in ation. Through their dependence on the investors' estimates, our model replicates the empirical relation between short nominal interest rate and in ation.[2]

We relate the latent monetary policy stance to the unobservable expected growth rate of money in our model.[3] The latter may capture in reduced-form smooth changes over time of the underlying monetary policy stance, as Campbell et al. (2014) and Song (2017) argue regarding the case in the US. Intuitively, realizations of the nominal money supply contain relevant information from which investors can learn about the conduct of monetary policy. The monetarist literature has highlighted that changes in monetary aggregates indeed convey information about key unobservable variables, which is not already contained in, for instance, changes in the nominal short rate (McCallum and Nelson, 2010; Ireland, 2017).[4]

We assume that investors derive utility from consumption and real money holdings. The intra-temporal rate of substitution between consumption and real balances creates an endogenous demand

[1] See https://www.newyorkfed.org/markets/survey_market_participants.html for details.

[2] The empirical relation between short nominal interest rates and in ation is what Carvalho and Nechio (2014) call one of the basic principles underlying the Taylor rule

[3] King et al. (2008) assert that private agents do not know the underlying nature of the central banks' policies.

[4] See section (1.2.5) for a thorough discussion on the validity of relating the underlying stance of the monetary policy authority to the expected growth rate of money in our model.

for real money balances. In turn, this optimality condition determines the nominal short rate, namely, the nominally risk-free interest rate that investors earn from investing in a bank account. Thus, the nominal short rate corresponds to the opportunity cost of holding money associated with standard models of the demand for money, which completes our monetary economy.[5] The price level and interest rates in this economy endogenously adjust, so that the markets for consumption and real balances clear for any consumption and nominal money supply path. Therefore, a portfolio rebalancing channel arises endogenously in our model.

In a simple single-investor economy, the mechanism behind this channel is as follows: if the investor forms a high estimate of the expected growth rate of money, she would prefer to borrow to hold more real money balances today instead of tomorrow when her marginal utility will be lower. For markets to clear, the nominal short rate needs to contemporaneously increase. The higher interest she will earn will afford her to increase future consumption, thus raising expected inflation. The level and conditional volatility of nominal interest rates across different maturities are determined in turn by her estimates through her inter-temporal decisions. By contrast, the real interest rates and the instantaneous inflation risk premium are constant.

To study the asset pricing implications of disagreement about money growth, we introduce another investor into the economy. We assume that both investors observe the same public information, but they have heterogeneous priors about the volatility of the unobservable expected growth rate of money. Therefore, the key friction in the model is that information about the monetary policy stance is symmetric but incomplete. Investors update their estimates of the expected growth rate of money based on realizations of the money supply. However, their heterogeneous priors result in the formation of heterogeneous estimates. Disagreement about money growth is priced in the real and nominal state-price densities. In turn, it drives expected inflation, the inflation risk premium, and nominal interest rates. In parameterized examples, we show that the volatility of nominal interest rates tends to rise with increases in the *absolute value* of disagreement, while the inflation risk premium is an increasing function of its *level*. The effect of disagreement on the level of nominal interest rates is ambiguous. Crucially, we assert that the overall impact of disagreement on asset prices hinges on the persistence of the expected growth rate of money. In particular, the relation between disagreement and equilibrium volatility and risk premium is weakened when the unobservable process is less persistent. In such case, investors' estimates are more heavily influenced by the long-run mean of the underlying process, on which both investors agree, than by their respective heterogeneous estimates.

Our model can be adapted to investigate the effects of heterogeneous interpretations of public information about the underlying monetary policy stance. For instance, central banks that attempt to guide market views on the future path of interest rates rely on unconventional sources (*e.g.*,

[5] See McCallum (1989) for a comprehensive treatment on the subject. See Ireland (2009); Lucas Jr and Nicolini (2015); Belongia and Ireland (2017) for recent works on the demand for real money holdings.

statements, speeches, and the Fed dots, among others) to communicate policy-relevant information to market participants. However, market participants may interpret the same information differently because they have heterogeneous estimation models or priors (Andrade et al., 2019; Caballero and Simsek, 2020; Sastry, 2021). Motivated by this argument, we further examine the effect on our monetary model of introducing a public signal that conveys potentially useful and non-redundant information about the unobservable expected growth rate of money. We assume that a steadfast investor believes that the signal is informative. On the opposite side, a doubtful investor believes that the signal conveys only noise. The steadfast investor's estimate is then subject to two countervailing forces. Due to her beliefs, she believes she learns about the expected growth rate of money from observing the signal. Consequently, she strongly reacts to the signal when updating her estimate of the underlying process and less strongly to the information conveyed by changes in money growth. By contrast, the doubtful investor bases revisions to her estimate solely on realizations of the latter. Hence, both investors have different estimation models and heterogeneously interpret the same public information. The signal itself becomes a risk factor that both investors are exposed to, despite that only the steadfast investor deems it informative.

We show in two parameterized economies that the nominal interest rates are higher in a mixed economy with a steadfast and a doubtful investor than in an economy where both investors are doubtful and there is no disagreement. The higher the credibility the steadfast investor attaches to the signal, the higher the equilibrium nominal rates. However, this effect is economically minor. By contrast, the informational tradeoff between money growth and signal stands out when comparing the volatility of interest rates in both economies. When the unobservable process is persistent, revisions to the steadfast investor's estimate after observing the signal are less pronounced than when it is less persistent. Thus, when the state of the economy is more persistent, the volatility of nominal rates in the mixed economy is lower than the volatility in the economy without disagreement and vice versa.

We conclude this section with a discussion of the related literature. The remainder of Chapter I is organized into several sections. In Section (1.2), we introduce a single-investor economy to investigate the portfolio rebalancing channel through which interest rates and expected inflation are determined in the model. We proceed to describe the investor's filtering problem and characterize the model's equilibrium. We conclude the section with an in-depth discussion on whether our monetary economy is an appropriate model to study monetary policy. In Section (1.3), we introduce another investor with different priors into the economy, thus producing an endogenous disagreement variable. We discuss some general properties of the equilibrium of the disagreement economy and some differences with respect to the single-investor economy. In Section (1.4), we calibrate the model to study the single-investor and disagreement economies in more detail. We conclude with a summary of our main results and a brief discussion on directions for future research.

Related literature Our paper is closely related to the literature on asset prices and heterogeneous beliefs. Ehling et al. (2018) and Xiong and Yan (2010) study the asset pricing impact of disagreement about inflation. Ehling et al. (2018) focus on disagreement about the expected growth rate of inflation, whereas Xiong and Yan (2010) examine disagreement about the long-run inflation target. Buraschi and Whelan (2012) investigate the implications of disagreement about the long-run expected growth rate of consumption and its implications on the term structure of interest rates. Contrary to these works, our focus is on the nominal sector of the economy, especially on the underlying monetary policy stance. In particular, by endogenizing the demand for real money holdings, nominal interest rates in our model are jointly determined through a portfolio rebalancing channel. Thus, nominal interest rates are not subordinated by real interest rates in our model.

Croitoru and Lu (2015) also explore the asset pricing implications of disagreement in a monetary economy similar to ours. However, we allow for the expected growth rate of money to be governed by a stationary and stochastic process. This feature makes the level and volatility of nominal interest rates and expected inflation time-varying, regardless of whether investors have heterogeneous beliefs. Importantly, we relate the expected growth rate of money to the underlying monetary policy stance, which changes over time, as argued by Campbell et al. (2014) and Song (2017). Moreover, our model can naturally accommodate the signaling information channel of monetary policy documented by Melosi (2017). The heterogeneous confidence about the informativeness of the public signal in our model is also related to the analysis of Sastry (2021). Other important works in this literature include Scheinkman and Xiong (2003), Gallmeyer and Hollifield (2008), Dumas et al. (2009), and Baker et al. (2016), although their focus is on the stock market and the real sector.

In the macro-finance literature, Caballero and Simsek (2020) examine the disagreement about the path of interest rates when markets are opinionated. Andrade et al. (2019) analyze monetary models in which agents have heterogeneous priors. Their focus is on forward guidance announcements about the future macroeconomic outlook and how they can potentially be interpreted differently by the private sector. Agents' disagreement in their models stems from aggregate demand and not from the monetary policy stance, as in our model.

Our monetary model draws from a resurgent literature that emphasizes the interactions between the demand for real money holdings and monetary policy. Lucas Jr and Nicolini (2015); Belongia and Ireland (2016, 2017, 2019); Benati et al. (2021); Belongia and Ireland (2022) and Benati et al. (2021) restore the demand for money to the academic debate about the conduct of monetary policy. They construct new monetary aggregates that have a stable empirical relation with inflation and interest rates. Belongia and Ireland (2022) argue that an explicit money growth rule delivers similar results to those of standard interest-rate rules in terms of its capacity to stabilize inflation and output. Furthermore, money growth rules are particularly relevant when the zero-lower bound becomes binding. The focus of our paper is on asset prices as opposed to macroeconomic considerations.

Finally, the equilibrium of the single-investor economy shares some properties with some popular single-factor interest rate models such as that of Cox et al. (1985). In the model equilibrium, the nominal interest rates are non-negative, time-varying, and heteroskedastic in the state variable. Our model has more parameters than Cox et al. (1985); thus, it can potentially allow for a rich array of dynamics. An important difference with that literature is that the underlying state variable is unobservable; hence, the driver in the conditional variation of the interest rates is the estimate of the unobservable state as opposed to an observed variable.

1.2 A single-investor monetary economy

This section introduces a monetary economy where the price level, inflation and interest rates are all determined endogenously. The framework throughout the paper is a continuous-time pure exchange economy model. Uncertainty in the model is summarized by the filtered probability space $(\Omega; \mathcal{F}; \{\mathcal{F}_t\}, P)$. We first study a monetary economy with a single-representative investor (investor i) who derives utility from a perishable consumption good and real cash balances. Assuming that money enters directly the utility function of the investor can be rationalized by the the fact that cash holdings facilitate consumption transactions using money. Models with money-in-the-utility-function have been adopted by Bakshi and Chen (1996); Lucas (2001); Basak and Gallmeyer (2001); Buraschi and Jiltsov (2005, 2007), and Balduzzi (2007), among others.

Before introducing disagreement, we introduce a the single-investor economy to gain a better understanding on the properties of the equilibrium of our monetary economy. We investigate the relation between the consumption, the demand for money, interest rates and prices with the unobservable expected rate of money growth (*i.e.* monetary policy stance). Thus, we can isolate the effects that disagreement itself introduces into the model equilibrium.

1.2.1 Model setup

The model has two sectors: real and nominal. The real sector is represented by a consumption process denoted by $C = (C_t)$, and the nominal sector consists of a nominal money process denoted by $M = (M_t)$, where M_t is the dollar amount of total monetary aggregates held at time t. We assume that aggregate consumption follows an exogenous processes of the form

$$dC_t = C_t [\mu_c dt + \sigma_c dz_{c;t}]. \tag{1.2.1}$$

All the uncertainty in the real sector of this economy is driven by the Brownian motion process $z_{c;t}$ which proxies for real shocks that affect consumption.

As in Woodford (1995), M_t corresponds to investor i's direct or indirect holdings of the monetary base, namely, the sum of her currency holdings and deposits. We also refer to M_t as the nominal money supply. Furthermore, we assume that M_t is governed by the following stochastic process

$$dM_t = M_t\left[x_t dt + \sigma_m dz_{m;t}\right]; \tag{1.2.2}$$

where x_t denotes the expected growth rate of money. We assume that x_t follows an Ornstein-Uhlenbeck process given by

$$dx_t = \kappa(\bar{x} - x_t)\,dt + \sigma_x dz_{x;t}. \tag{1.2.3}$$

Thus, x_t drives the variation in the conditional expectation of the money supply growth. The Brownian motions $z_{c;t}$, $z_{m;t}$ and $z_{x;t}$ are uncorrelated by assumption.[6] Furthermore, we assume that the investor does not observe x_t, although she knows it follows an Ornstein-Uhlenbeck process. Moreover, she knows the persistence parameter κ and long-run mean $\bar{x}$, but does not observe the true volatility parameter σ_x. As argued later, σ_x plays a crucial role in the relation between disagreement about money growth and the asset prices.

Throughout this paper we refer to the system of Eqs. (1.2.2)-(1.2.3) as monetary policy. Furthermore, we indistinctly refer to Eq.(1.2.2) as the unobservable, underlying, or latent state of the economy. The expected growth rate of nominal money supply is stochastic and stationary. The exogeneity assumption is related to the monetarist literature, whereby traditionally the government exogenously specifies the path of money supply (see Woodford (1995) and references therein). We remit the reader to section (1.2.5) for a more thorough discussion on the appropriateness of our notion of a monetary policy stance based on the model above.

1.2.2 Public signal and money growth

We introduce a public source of information similar to the one in Dumas et al. (2009); Xiong and Yan (2010), but adapted to a monetary economy where the unobservable variable is the expected growth rate of money. Concretely, the investor observes a public signal, s_t, that may convey information about x_t. The investor believes that the signal obeys the stochastic differential equation

$$ds_t = \Phi_i dz_{x;t} + \sqrt{1 - \Phi_i^2}\, dz_{s;t}; \tag{1.2.4}$$

[6] We could assume, for instance, that consumption and nominal money supply are correlated by letting $dz_{c;t}$ affect the dynamics of dM_t. The model can easily be adapted to allow for this possibility, although the main results of the paper would remain unaffected. We focus on the case where all three processes are uncorrelated for the sake of clarity and tractability.

where $z_{s,t}$ is a Brownian motion uncorrelated with $z_{c,t}$, $z_{m,t}$ and $z_{x,t}$. Investor i believes that the signal s_t has a correlation parameter $\rho_i \in [0,1]$ with the unobservable process x_t through their common dependence on the Brownian motion $z_{x,t}$. Thus, ρ_i is a subjective belief about the informativeness of the signal. When $\rho_i \to 1$, investor i believes that the signal is perfectly correlated with the innovation $z_{x,t}$, and, thereby, she can infer the unobservable state of the economy by observing a realization of the signal, s_t. Conversely, when $\rho_i \to 0$, investor i believes that s_t is pure noise. Therefore, she believes it conveys no relevant information.

Intuitively, the signal captures that, besides observed changes in monetary aggregates, the investor believes there are other sources from which she can learn about the unobservable process that drives money growth. The alternative sources could be related to statements released by central banks, speeches from board members, or explicit forward guidance, among others. Different investors may have heterogeneous subjective beliefs about the degree of informativeness of the public signal. Accordingly, they may react differently to the flow of information conveyed by the signal (see section (1.3) for more details). Importantly, given that ρ_i is a subjective belief by assumption, the monetary authority does not have control over the informativeness of the signal. Rather, it is taken for granted as an input.

Optimal filtering

Following standard filtering theory (see theorem 12.7, Liptser and Shiryaev, 2001), the money supply dynamics under investor's i beliefs are

$$dM_t = M_t \left(x_t^i dt + \sigma_m dz_{m,t}^i \right). \tag{1.2.5}$$

Investor i's optimal estimator of the expected growth rate x_t^i has dynamics

$$dx_t^i = \kappa \left(\bar{x} - x_t^i \right) dt + \hat{\sigma}_m^i dz_{m,t}^i + \hat{\sigma}_s^i ds_t, \tag{1.2.6}$$

where $\hat{\sigma}_m^i$ and $\hat{\sigma}_s^i$ are defined below. From Eqs. (1.2.5) and (1.2.6)

$$dz_{m,t}^i = \frac{1}{\sigma_m} \left(\frac{dM_t}{M_t} - x_t^i dt \right).$$

Therefore, $z_{m,t}^i$ is a surprise component of the change in money supply growth relative to the estimate x_t^i adjusted by σ_m. As new information about M_t becomes available, investor i updates her estimate x_t^i according to Eq. (1.2.6). We assume that the investor observes the money supply process for a sufficiently long time such that the variance of her estimate, $\left(\hat{\sigma}_m^i \right)^2$ has reached its steady state level.[7] Therefore

$$\hat{\sigma}_m^i = \sigma_m \left(\sqrt{\kappa^2 + \left(\frac{\sigma_{x,i}}{\sigma_m} \right)^2 \left(1 - \rho_i^2 \right)} - \kappa \right). \tag{1.2.7}$$

[7] See Scheinkman and Xiong (2003); Dumas et al. (2009); Xiong and Yan (2010); Ehling et al. (2018).

Investor i's posterior variance is $\hat{}^{i}_{m}\ ^{2}$. It is a measure of the uncertainty she attaches to her estimate x^i_t. The parameter $_{x;i}$ corresponds to the investor's prior estimate of the unobservable parameter $_x$. In turn, $\hat{}^i_s$ measures investor i's uncertainty about the informativeness of s_t about x_t with

$$\hat{}^i_s = \quad _{x;i}\ _i. \tag{1.2.8}$$

Investor i's total uncertainty about x^i_t is denoted by $\hat{}^i$ with $\hat{}^i \quad \sqrt{(\hat{}^i_m)^2 + (\hat{}^i_s)^2}$. From Eqs. (1.2.7) and (1.2.8), an increase in the correlation coe cient between the signal s_t and $z_{x;t}$ has two opposite e ects on $\hat{}^i$. On the one hand, a higher $_i$ increases $\hat{}^i_s$ and the total perceived uncertainty $\hat{}^i$. On the other hand, $\hat{}^i_m$ and $\hat{}^i$ fall as $_i$ increases. Intuitively, higher values of $_i$ imply that the investor assigns a larger weight to the signal as a source of information about x_t relative to her own updating model. Thus, optimally she reacts less strongly to her own surprise $z^i_{m;t}$ and more strongly to the signal. In the extreme case when the investor believes that $_i = 1$, then $\hat{}^i_m = 0$ and she would only update her estimate x^i_t based on the public signal. On the other extreme, when $_i = 0$, the investor believes that the signal is pure noise and she optimally ignores it, so that $\hat{}^i_s = 0$. In contrast to $_i$, a higher estimate of $_{x;i}$ raise $\hat{}^i$ through both, $\hat{}^i_m$ and $\hat{}^i_s$.[8]

1.2.3 Single investor problem

We assume that markets are complete. Investor i solves an in nite time horizon optimization problem and derives utility from consumption and the real value of the monetary holdings, *i.e.*, $m_t \quad M_t/P_t$, where P_t is the unit dollar price of the consumption good. At time t the investor has the opportunity to invest in a money account with nominal rate of return of R_t. Her optimization problem is

$$\begin{aligned} \sup_{(C_t;m_t)} \quad & E \int_0^{1} e^{\ \ t} U(C_t; M_t/P_t) \\ \text{s.t.} \quad & E \int_0^{1} \ _t \left(C_t + \frac{M_t}{P_t} R_t \right) dt \quad W_0; \end{aligned} \tag{1.2.9}$$

where $_t$ denotes the unique real state-price density.

The rst-order condition of the problem with respect to consumption and money holdings are, respectively

$$\begin{aligned} e^{\ \ t} U_c(C_t; M_t/P_t) &= y\ _t; \\ e^{\ \ t} U_m(C_t; M_t/P_t) &= y\ _t R_t; \end{aligned}$$

[8] Andrade et al. (2019) and Sastry (2021) also introduce monetary models where agents have heterogeneous priors.

where y is the Lagrange multiplier for the optimization problem. Combining the first-order conditions, we obtain an expression for the nominal short interest rate

$$R_t \quad = \quad \frac{U_m\,(C_t; M_t{=}P_t)}{U_c\,(C_t; M_t{=}P_t)}: \tag{1.2.10}$$

The intuition for this result is straightforward: if an investor has an extra dollar, she can invest it in the money account for which she will earn a nominal dollar interest of R_t. She may use these earnings to purchase $\frac{R_t}{P_t}$ units of the consumption good. This action would raises her marginal utility by $U_c \;\; C_t; \frac{M_t}{P_t} \;\; \frac{R_t}{P_t}$. Otherwise, she can keep the same dollar and have her utility increased by $U_m \;\; C_t; \frac{M_t}{P_t} \;\; \frac{1}{P_t}$. At the optimum, these increments must equate one another.

1.2.4 Equilibrium

An investment of one dollar at time t in the money account generates a continuous payment stream at the rate of R_t dollars over time. The corresponding real investment at time t is $1{=}P_t$ and the real dividend at time u is $R_u{=}P_u$ for $u \quad t$. Using the state-price density $\;_t$ to price this investment, the following relation obtains

$$\frac{1}{P_t} \quad = \quad \mathrm{E}_t \quad \mathrm{Z} \,_t^{\,1} \;\; \frac{\;_u}{\;_t}\frac{R_u}{P_u}\mathrm{d}u \quad : \tag{1.2.11}$$

Equation (1.2.11) may be interpreted as the familiar asset pricing equation, where the real price of a unit of money ($1{=}P_t$) equals the expected present value of its implicit future dividends $\frac{R_u}{P_u}$. In this context, $\frac{R_u}{P_u}$ represents the real value of the transaction services provided by holding money.

By the definition of the state-price density $\;_t$ and the optimality condition for R_t (1.2.10), the previous expression becomes

$$\frac{1}{P_t} \quad = \quad \mathrm{E}_t \quad \mathrm{Z} \,_t^{\,1} \;\; e^{\quad (u \quad t)}\frac{U_m\,(C_u; m_u)}{U_c\,(C_t; m_t)}\frac{1}{P_u}\mathrm{d}u \quad : \tag{1.2.12}$$

As pointed out by Basak and Gallmeyer (2001), Eq. (1.2.12) is a backward stochastic differential equation. For tractability, we assume that the investor has logarithmic utility given by

$$U\,(C_t; m_t) \quad = \quad ' \ln{(C_t)} + (1 \quad ')\ln{(m_t)}\,; \;\; 0 \quad ' \quad 1;$$

where $'$ is the share of expenditure on consumption.[9] With this functional form, Eq. (1.2.12) becomes

$$\frac{1}{P_t} \quad = \quad \frac{1 \quad '}{'}\mathrm{E}_t \quad \mathrm{Z} \,_t^{\,1} \;\; e^{\quad (u \quad t)}\frac{C_t}{M_u}\mathrm{d}u \quad :$$

[9] Ireland (2004) shows that the estimated cross-derivative of consumption and real money holdings in a linearized IS equation is close to zero. McCallum and Nelson (2010) argue that, in many cases, utility can be treated as approximately separable in money and other arguments based on the previous study, among others.

Re-arranging terms

$$\frac{1}{P_t} = \frac{1-\phi}{\phi}\frac{C_t}{M_t}\int_t^{\infty} \mathrm{E}_t\left[e^{-\rho(u-t)} \left(\frac{M_u}{M_t}\right)^{-1}\right] \mathrm{d}u. \tag{1.2.13}$$

Using 1.2.13 and the dynamics for M_t and x_t in Eqs. (1.2.2) and (1.2.3), we calculate the equilibrium price level and nominal short rate of this economy in the Proposition (1.2.1).

Proposition 1.2.1. *The equilibrium price level and short nominal rate in the monetary economy without disagreement are given, respectively, by*

$$\frac{1}{P_t} = \frac{1-\phi}{\phi}\frac{C_t}{M_t}; \tag{1.2.14}$$

$$R_t = Q\left(x_t^i\right)^{-1} \geq 0; \tag{1.2.15}$$

where $Q\left(x_t^i\right) \equiv \int_t^{\infty} M\left(x_t^i; u-t\right) du$,

$$M\left(x_t^i; u-t\right) = \mathrm{E}_t\left[e^{-\rho(u-t)} \left(\frac{M_u}{M_t}\right)^{-1}\right] = \exp\left(-A(u-t) - B(u-t)\,x_t^i\right); \tag{1.2.16}$$

$$B(u-t) = \frac{1}{\kappa}\left(1 - e^{-\kappa(u-t)}\right); \tag{1.2.17}$$

$$\begin{aligned} A(u-t) = {} & \left(\bar{x} - \frac{\left(\hat{\sigma}^i\right)^2}{2\kappa^2} - \frac{\hat{\sigma}^i_x}{\kappa}\right)\left[(u-t) - B(u-t)\right] + \\ & \left(\rho - \sigma_m^2\right)(u-t) + \frac{\left(\hat{\sigma}^i\right)^2}{4\kappa}B(u-t)^2. \end{aligned} \tag{1.2.18}$$

From Eq. (1.2.16), it can be seen that $M\left(x_t^i; u-t\right) > 0$ for any x_t^i, which implies that $Q\left(x_t^i\right)$ is non-negative.[10] Re-arranging terms in Eq. (1.2.14) and combining with Eq. (1.2.15)

$$\ln\left(\frac{M_t}{P_t}\right) = \ln(m_t) = \text{constant} + \ln(C_t) - \ln(R_t). \tag{1.2.19}$$

Eq. (1.2.19) has the natural interpretation of a demand for real money holdings: higher levels of income (consumption in equilibrium) increase the demand for money, while high nominal interest rates decrease it. The relation between R_t and m_t is pinned down by $Q\left(x_t^i\right)$, which in turn is a function of x_t^i. From Eq. (1.2.19), $Q\left(x_t^i\right)$ is proportional to the demand of real money holdings per unit of income, which we refer to as the normalized demand for money hereafter. Therefore, the normalized demand for money is in itself determined by the estimate x_t^i. Suppose that investor i infers that x_t will be high, either due to a large realization of the nominal money supply, $\mathrm{d}M_t$, or signal $\mathrm{d}s_t$. According to Eq. (1.2.16), $M\left(x_t^i; u-t\right)$ must be lower. Therefore, the investor optimally decides to reduce her real money holdings in period t, so that $Q\left(x_t^i\right)$ falls. For markets

[10] Observe that $M\left(x_t^i; u-t\right)$ is similar to the equilibrium bond price in the seminal interest rate model of Vasicek (1977).

to clear, the nominal short rate R_t must rise to induce the investor to substitute cash balances today for nominal interest-bearing assets in the future. Thus, the nominal short rate in this economy is the opportunity cost of holding cash at hand.

Eq. (1.2.1) shows that, in equilibrium, the nominal short rate is endogenously determined from the optimality condition (1.2.10) and the processes (1.2.1), (1.2.2) and (1.2.3). From Eq. (1.2.15), the estimate of the expected growth rate of money, x_t^i, drives the time-variation in R_t.[11] Moreover, given that $\frac{@Q(x_t^i)}{@x_t^i} < 0$, R_t is an increasing function of x_t^i. Indeed, x_t^i fully describes the dynamics of all nominal interest rates and expected in ation in this economy.

Proposition 1.2.2. *The dynamics of the price level, nominal short rate, and the real and nominal state-price densities in the monetary economy without disagreement are*

$$\frac{dP_t}{P_t} = {}_t dt + {}^p_c dz_{c;t} + {}^p_{m;t} dz^i_{m;t} + {}^p_{s;t} ds_t; \quad (1.2.20)$$

$$dR_t = {}^R_t dt + {}^R_{m;t} dz^i_{m;t} + {}^R_{s;t} ds_t; \quad (1.2.21)$$

$$\frac{d \ _t}{_t} = rdt \quad {}_c dz_{c;t}; \quad (1.2.22)$$

$$\frac{d \ ^\$_t}{^\$_t} = R_t dt \quad {}^\$_{m;t} dz^i_{m;t} \quad {}^\$_{s;t} ds_t; \quad (1.2.23)$$

where ${}^\$_t = \frac{_t}{P_t}$ *is the unique nominal state-price density. The expected in ation rate,* ${}_t$*, and the di usion terms* ${}^p_c$*,* ${}^p_{m;t}$*, and* ${}^p_{z;t}$ *are given by Eqs. (1.6.5), (1.6.6), (1.6.7), and (1.6.8), respectively. The drift and di usion terms* ${}^R_t$ *and* ${}^R_{z;t}$ *are given by Eqs. (1.6.9), and (1.6.10), respectively. The real short rate, r, is given by* $r = \ + \ _c \ \ ^2_c$ *and* ${}_c = \ _c$*. Finally, the nominal short rate,* R_t*, and the nominal prices of risk* ${}^\$_{m;t}$ *and* ${}^\$_{s;t}$ *are given, respectively, by Eqs. (1.6.16), (1.6.14), and (1.6.15) in the Appendix.*

From Proposition (1.2.2), there are three risk factors in this economy: a real risk factor associated with consumption, $z_{c;t}$, and two nominal risk factors, $z^i_{m;t}$ and s_t, which are associated with investor i's adapted shock and the public signal, respectively. Therefore, whenever ${}_i > 0$, the signal becomes another risk factor for which investor i requires compensation. In turn, the real short rate, r, is constant and independent of the nominal parameter ${}_m$ and of any realization of x_t^i. Moreover, the economy's single real price of risk, ${}_c$, is constant, while the nominal risk factors do not a ect the real state-price density, ${}_t$. Therefore, the monetary sector does not a ect consumption and other real variables the economy. This result is a consequence of the log-utility assumption, which equalizes the substitution and income e ects. In turn, the real risk factor is not priced in by the nominal state-price density, ${}^\$_t$. Therefore, in the model's equilibrium, the nominal and real sectors of the economy are determined separately.

[11] For the sake of conciseness, we choose not to state the explicit dependence of R_t on x_t^i unless necessary.

Furthermore, the drift of the nominal state-price density must be equal to the short rate defined in Eq. (1.2.15). Accordingly, as we show in the Appendix

$$R_t \;=\; {}^{2}_{m} + x^i_t + \frac{G \;\; x^i_t}{Q \;\; x^i_t} \quad \bar{x} \quad x^i_t \quad \hat{}^{i}_{x} \quad \frac{\hat{}^{i\;\;2}}{2}\frac{H \;\; x^i_t}{Q \;\; x^i_t};$$

where $G \;\; x^i_t$ and $H \;\; x^i_t$ are defined when proving proposition (1.2.2) in the Appendix.[12] A higher time discount increases the nominal short rate, whereas a higher value of $_m$ lowers the nominal rate due to risk-aversion, as in standard models. Furthermore, x^i_t first has a linear impact on R_t. When nominal money supply is expected to grow at a faster rate, it takes a larger R_t to induce the investor to invest in the money account. This result is similar to the linear impact of an increase in the expected growth rate of consumption, $_c$, on the short real rate, r, arising from the Euler equation.

The remaining terms are adjustments accounting for the stochastic and unobservable nature of x_t. The first one of them is due to the mean-reversion of the estimate x^i_t. For instance, a high value of x^i_t brings the process farther away from its long-run mean, $\bar{x}$. As a result, x^i_t is pulled back towards $\bar{x}$. The speed of mean-reversion is modulated by $\frac{G(x^i_t)}{Q(x^i_t)}$, which corresponds to the negative of the elasticity of the demand for real cash balances to changes in x^i_t. $\frac{G(x^i_t)}{Q(x^i_t)}$ is bounded below by zero and above by ^{1}. The mean-reversion of R_t is large enough for R_t to never become negative, even for very low values of x^i_t.[13] Second, the steady-state volatility $\hat{}^{i}_{x}$ reduces R_t due to the filtering problem that investor i faces when estimating x_t. The last term comes from Jensen's inequality. It is determined by the steady-state posterior variance $\hat{}^{i\;\;2}$ and $\frac{H(x^i_t)}{Q(x^i_t)}$. Therefore, R_t is bounded above by x^i_t plus a constant. Finally, due to the independent determination of the real and nominal sectors, $_c$ and $_c$ do not influence the nominal short rate.

We observe from Eqs. (1.2.21) and (1.2.23) that the drift and diffusion terms in the dynamics of the nominal short rate and state-price densities are a function of x^i_t, and therefore, stochastic. Thus, x_t is a driver of the conditional expectation and variance of the nominal sector.[14] We introduce a discussion about the expected inflation rate, $_t$, in the comparative analysis of the model in section (1.4).

Proposition 1.2.3. *The instantaneous inflation risk premium in period t, IRP_t, and the real and*

[12]It can be shown that $\frac{G(x^i_t)}{Q(x^i_t)}$ and $\frac{H(x^i_t)}{Q(x^i_t)}$ are bounded below by zero and above by $\frac{1}{}$ and $\frac{1}{\;2}$, respectively. Moreover, they are inverted sigmoid functions of x^i_t.

[13]This mechanism is similar to the pull in the widely used interest rate model from Cox et al. (1985), where the interest rate can never become negative either. As can be seen from $^{R}_{m;t}$ and $^{R}_{s;t}$ in proposition (1.2.2), the stochastic nature of the volatility of the nominal interest rate is another trait that our model shares with Cox et al. (1985).

[14]This result is not present in Croitoru and Lu (2015), where time-variation of the system arises only after the introduction of disagreement.

nominal yield curves between periods t and , $y_{t;}$ and $y^{\$}_{t;}$, are given, respectively, by

$$IRP_t = {}^{2}_{c};\qquad (1.2.24)$$

$$y_{t;} = + {}_{c} \quad {}^{2}_{c};\qquad (1.2.25)$$

$$y^{\$}_{t;} = \frac{\log Q \ \ x^i_{t;}}{t} + \frac{\log Q \ \ x^i_t}{t};\qquad (1.2.26)$$

where $Q \ \ x^i_{t;} \quad \int^{1} M \ \ x^i_{t};u \quad t \ du$. *Furthermore, the dynamics of the nominal yield curve are*

$$dy^{\$}_{t;} = {}^{y}_{t;}\ dt + {}^{y}_{m;t;}\ dz^i_{m;t} + {}^{y}_{s;t;}\ ds_t:\qquad (1.2.27)$$

The di usion terms ${}^{y}_{m;t;}$ *and* ${}^{y}_{s;t;}$ *are equal to*

$$ {}^{y}_{m;t;} = \frac{\hat{}^i_m}{t} \left(\frac{G \ \ x^i_{t;}}{Q \ \ x^i_{t;}} \quad \frac{G \ \ x^i_t}{Q \ \ x^i_t} \right) ;\qquad (1.2.28)$$

$$ {}^{y}_{s;t;} = \frac{\hat{}^i_s}{t} \left(\frac{G \ \ x^i_{t;}}{Q \ \ x^i_{t;}} \quad \frac{G \ \ x^i_t}{Q \ \ x^i_t} \right) ;\qquad (1.2.29)$$

where $G \ \ x^i_{t;} \quad \int^{1} B \ \ (u \quad) \ M \ \ x^i_{t};u \quad t \ du$.

From Eq. (1.2.24), the instantaneous in ation risk premium is positive. Hence, investing in the nominally risk-free money account is risky in real terms because the in ation rate is itself stochastic, as can be seen from Eq. (1.2.20).[15] Nevertheless, the instantaneous in ation risk premium is constant and independent of x^i_t, as is the real yield curve, $y_{t;}$, for any t. These results are a further consequence of the log-utility assumption, which implies that the cross-derivative of the utility function with respect to consumption and real money holdings is zero (Bakshi and Chen, 1996).

In contrast, the nominal yield curve is time-varying and fully characterized by realizations of x^i_t, just like the nominal short rate. For $> t$, it follows from Eq. (1.2.26) and the de nition of $Q \ \ x^i_{t;}$ in proposition (1.2.3) that $\int^{1} M \ \ x^i_{t};u \quad t \ du < \int^{1}_{t} M \ \ x^i_{t};u \quad t \ du$. Therefore, $y^{\$}_{t;} > 0$ for any realization of x^i_t. Additionally, the di usion terms of the nominal yield curve ${}^{y}_{m;t;}$ and ${}^{y}_{s;t;}$ are in turn functions of x^i_t, such that the instantaneous volatility of the yield curve is heteroskedastic. It is di cult to state more general properties about the shape of the nominal yield curve and its dynamics. Instead, we rely on a comparative numerical analysis of the level and volatility of the nominal in the single-investor economy in section (1.4).

[15] The stochastic nature of the price level also implies that the Fisher relation between the short term nominal and real interest rates does not hold in this economy.

1.2.5 Discussion of monetary policy in the model

Thus far, we have refrained from discussing whether our model conforms to an appropriate framework for the study of monetary policy. Our implicit assumption is that the monetary system, given by Eqs. (1.2.2)-(1.2.3), constitutes a good proxy for the unobservable monetary policy stance. In a seminal paper, Taylor (1993) argues that the Federal Reserve conducts monetary policy by gradually adjusting the short-term nominal interest rate to changes in inflation and some measure of the output gap. Since then, most work in the macroeconomics and macro-finance tradition develop models with a production sector that study monetary policy under a Taylor rule. In contrast to these works, our model is framed as an exchange economy where prices are determined endogenously, whereas consumption (output in equilibrium) and nominal money supply are exogenous. This assumption rules out any meaningful measure of the output gap that is directly derived from the model.[16]

Instead, our model pertains to the monetarist and quantitative-theoretic tradition according to which the aggregate measures of money play an important role in the conduct of monetary policy. This literature emphasizes the importance of money as a monetary policy instrument over the nominal short rate.[17] Studies within this tradition argue that, to the extent that the relation between money and prices or interest rates (*i.e.,* the demand for real money holdings) is stable, monetary policy should be tied to money growth targets.[18]

As shown in Eq. (1.2.19), a demand for real money holdings naturally arises in our model: for a given level of consumption, the demand for money adjusts to changes in the nominal short rate. In turn, the nominal short rate responds to variations in the expected growth rate of money supply, x_t. Even though investor i cannot directly observe x_t, she knows it follows an Ornstein-Uhlenbeck process. Investor i's estimate, x_t^i, inherits the stationarity of x_t. Moreover, as argued in the previous section, we know that R_t is bounded below by zero and above by x_t^i plus a constant. Therefore, the demand for real money holdings is stable. Given that the conditional variation in expected inflation π_t is in turn fully characterized by x_t^i, a stable and positive relation between R_t and π_t also follows. Thus, our model is capable of replicating at least the first part of what Carvalho and Nechio (2014) refer to as the basic principles underlying the Taylor rule: short nominal interest rates tend to increase with inflation.

[16] In addition, it is unfeasible to introduce a Taylor rule in our model due to the estimation problem faced by the investors. Standard filtering theory from Liptser and Shiryaev (2001) requires a state-space specification where measure and state variables are already solved for. On the contrary, a Taylor rule establishes an explicit relation between inflation and the nominal short rate, which are endogenous variables.

[17] Lucas Jr and Nicolini (2015) argue that one driver of the shift towards the nominal short rate as a policy instrument was the deterioration of the stability of the empirical relations between monetary aggregates, prices, and interest rates. They construct an updated monetary aggregate that accounts for changes in regulation that affected the substitutability between cash and bank deposits.

[18] See Friedman (1988), McCallum (1989), Ireland (2017), among many others.

We also assume that the expected growth rate of money, x_t, is a stochastic and unobservable process; hence, the nominal interest rates and expected inflation are, in turn, stochastic and time-varying.[19] This result is important because, as we argue in upcoming sections, the persistence of x_t exerts a substantial influence on the impact of disagreement on interest rates and inflation. As for the unobservable nature of x_t, McCallum and Nelson (2010) assert that under some circumstances, money growth might produce more accurate signals about the unobservable monetary policy stance than interest-rate rules. For instance, money growth may contain valuable information about the unobservable natural rate of interest rates. They argue that monetary aggregates reveal relevant information for future aggregate demand that is not conveyed by current output and interest rates.[20] While we only speculate that x_t might be related to the natural rate of interest rates or other latent variables containing pertinent information for future macroeconomic outcomes, the filtering problem in Section (1.2.2) is consistent with the mechanism described by McCallum and Nelson (2010).[21] In particular, investors in our model learn about the unobservable process x_t, the sole driver of conditional variation in the economy, from the information revealed by money growth over time. When producing her estimate, the investor accounts for the mean-reversion of x_t, which influences her expectations about the future path of money growth and interest rates.

Overall, we believe that our model constitutes a sensible starting point to investigate the asset pricing implications of disagreement about monetary policy, to the extent that money growth is indeed capable of capturing the latter. We certainly acknowledge the limitations of the answers that our model can provide, given its simple setup as an exchange economy.

1.3 A monetary economy with disagreement

In this section, we introduce disagreement about monetary policy among investors. To this end, we assume the existence of two investors who have heterogeneous beliefs about the dynamics of the expected growth rate of the money x_t. As before, we assume that both investors observe C_t, with dynamics given by Eq. (1.2.1) and M_t, having the incomplete information filtration

[19] In reduced-form, the time-varying nature of x_t may accommodate smooth changes in the policy stance of the monetary authority. Campbell et al. (2014); Song (2017) document significant shifts in the aggressiveness of monetary policy and inflation in the U.S. Malmendier et al. (2021) argue that decisions made by new appointees to the board of the directors of a Central Bank may be substantially influenced by their previous experiences with inflation.

[20] Belongia and Ireland (2019) argue that interest-rate rules are of limited value to learn about the monetary policy stance when the zero lower bound becomes a binding constrain on nominal short rates, as has been the case since 2008.

[21] Moreover, the asset pricing results in our model follow from the adjustment of the relative prices of real money holdings, consumption, and bonds to the estimate x_t^i. This result is consistent with the view of Allan Meltzer on the monetary transmission mechanism, which relies on changes in relative prices and imperfect information. See Ireland (2017) for a more thorough discussion of Allan Meltzer's vast work on monetary policy and its implications for the current academic and policy debate.

$\mathcal{F}_t^M \in \mathcal{F}_t$, $t \in [0, \infty)$. Furthermore, we assume that investors have equivalent probability measures $\mathbb{P}^i$, $i \in \{A, B\}$. We assume that both investors deduce σ_m from the quadratic variation of M_t, but can only draw inferences about x_t from realizations of M_t. Specifically, we assume that investors know that x_t follows an Ornstein-Uhlenbeck process as the one in Eq. (1.2.3), but they may have different priors about the volatility of the expected growth rate of the money supply, which yields different estimates about the process x_t. Put differently, we assume that investors agree on the long-run mean $\bar{x}$ and the speed of mean reversion κ in Eq. (1.2.3), but may have different beliefs about σ_x, the volatility of the unobservable expected growth rate of money. Investor A knows investor B's estimate $\sigma_{x,B}$ and vice versa. However, they believe that their own estimate is correct. That is, they "agree to disagree." We refer to this model as the disagreement economy for the remainder of this paper.

1.3.1 Investor's problem

Here, we recast the optimization problem (1.2.9) for investor i where $i \in \{A, B\}$. Now, the information set on which every investor conditions their decisions are based on their own priors and their estimates. The investor's problem is very similar to the one from the single-investor

$$
\begin{aligned}
\sup_{(C_{i,t}, m_{i,t})} \quad & \mathbb{E}^i \left[\int_0^\infty e^{-\rho t} U\left(C_{i,t}, M_{i,t}/P_t\right) \right] \\
\text{s.t.} \quad & \mathbb{E}^i \left[\int_0^\infty \xi_{i,t} \left(C_{i,t} + \frac{M_{i,t}}{P_t} R_t \right) \mathrm{d}t \right] \leq W_{i,0}.
\end{aligned} \tag{1.3.1}
$$

Assuming log-utility, the optimality conditions of the problem are

$$
y_i \xi_{i,t} = \varphi e^{-\rho t} C_{i,t}^{-1}, \tag{1.3.2}
$$

$$
y_i \xi_{i,t} = (R_t)^{-1} (1 - \varphi) e^{-\rho t} m_{i,t}^{-1}, \tag{1.3.3}
$$

where y_i is the Lagrange multiplier from the static budget constraint in investor i's optimization problem (1.3.1).

1.3.2 Sharing rules derivation

As in Basak (2005) and Ehling et al. (2013), the equilibrium sharing rules are derived using a state-dependent representative investor with utility given by

$$
\begin{aligned}
U(C_t, m_t; \lambda_t) = \max_{\{C_{A,t} + C_{B,t} = C_t,\, m_{A,t} + m_{B,t} = m_t\}} & \left[\varphi \ln(C_{A,t}) + (1 - \varphi) \ln(m_{A,t}) \right] + \\
& \lambda_t \left[\varphi \ln(C_{B,t}) + (1 - \varphi) \ln(m_{B,t}) \right],
\end{aligned}
$$

where $\lambda_t > 0$ is a stochastic welfare weight that captures the impact of disagreement among investors on risk sharing. The equilibrium allocation must solve the representative investor's problem above implying that

$$
{}_{t} \;=\; \frac{C_{A;t}^{\;1}}{C_{B;t}^{\;1}} = \frac{y_1 \;\; {}_{A;t}}{y_2 \;\; {}_{B;t}}; \tag{1.3.4}
$$

where the second equality comes from the optimality individual condition in Eq. (1.3.2). For $u > t$, the random variable $\;_u$, weighted by $\frac{y_2}{y_1}$, is the Radon-Nikodym derivative of investor B's beliefs with respect to investor A's beliefs; *i:e:*, $\frac{\mathrm{dP}^B}{\mathrm{dP}^A}$.

From Girsanov's theorem, innovations of investors A and B are linked by

$$
\mathrm{d}z_{m;t}^{B} \;=\; \mathrm{d}z_{m;t}^{A} \qquad {}_{t}\mathrm{d}t; \tag{1.3.5}
$$

where $\;_t = \frac{x_t^B \quad x_t^A}{\;_m}$. $\;_t$ captures the level of disagreement between x_t^B and x_t^A per unit of uncertainty of the nominal money supply. We also refer to $\;_t$ as the beliefs spread.

As pointed out by Basak (2005), since $\;_t$ follows directly from the exogenous endowment process and investors' priors, we may treat it as an exogenous parameter if we do not impose any equilibrium restrictions on it.

Proposition 1.3.1. *The dynamics of the stochastic weight $\;_t$ are*

$$
\frac{d \;\; {}_{t}}{{}_{t}} \;=\; {}_{t} dz_{m;t}^{A}: \tag{1.3.6}
$$

Applying Ito's lemma to $\;_t$ and re-arranging terms, we compute the dynamics of the disagreement process

$$
d \;\; {}_{t} \;=\; {}_{t}dt + \;\; {}_{;m}dz_{m;t}^{A} + \;\; {}_{;s}ds_t; \tag{1.3.7}
$$

where $\;\frac{{}_m + \hat{}_{m;}^{B}}{{}_m}$, $\;_{;m}\;\frac{\hat{}_m^B \;\; \hat{}_m^A}{{}_m}$ and $\;_{;s}\;\frac{\hat{}_s^B \;\; \hat{}_s^A}{{}_m}$, where $\hat{}_m^i$ and $\hat{}_s^i$ are de ned in Eqs. (1.2.7) and (1.2.8), respectively, for i 2 $fA;Bg$. The equilibrium consumption allocations for investors A and B are given by

$$
C_{A;t} \;=\; f\,(\;\;_t)\,C_t; \tag{1.3.8}
$$

$$
C_{B;t} \;=\; (1 \quad f\,(\;\;_t))\,C_t; \tag{1.3.9}
$$

where $f_t \quad f\,(\;\;_t) = \frac{1}{1+\;\;_t}$. The equilibrium consumption allocations for investors A and B are given by

$$
m_{A;t} \;=\; f\,(\;\;_t)\,m_t; \tag{1.3.10}
$$

$$
m_{B;t} \;=\; (1 \quad f\,(\;\;_t))\,m_t: \tag{1.3.11}
$$

The real state-price densities for investors A and B are, respectively

$$
\frac{{}_{A;t}}{{}_{A;0}} \;=\; e^{\quad t} \left(\frac{f\,(\;\;_t)}{f\,(\;\;_0)}\right)^{1} \left(\frac{C_t}{C_0}\right)^{1}; \tag{1.3.12}
$$

$$
\frac{{}_{B;t}}{{}_{B;0}} \;=\; e^{\quad t} \left(\frac{1 \quad f\,(\;\;_t)}{1 \quad f\,(\;\;_0)}\right)^{1} \left(\frac{C_t}{C_0}\right)^{1}: \tag{1.3.13}
$$

The stochastic weighting process ${}_t$ captures the impact of disagreement about the money growth process on the equilibrium allocations. The uncertainty of ${}_t$ is driven by the adapted nominal shock $z^A_{m;t}$, while Eq. (1.3.5) links $z^A_{m;t}$ and $z^B_{m;t}$. The disagreement process, ${}_t$, obeys an Ornstein-Uhlenbeck process with long-run mean equal to zero. The diffusion terms ${}_{;m}$ and ${}_{;s}$ depend on the steady-state posterior variances $(\hat{}^i_m)^2$ and $(\hat{}^i_s)^2$ for $i \in \{A; B\}$.[22] For disagreement to persist over time, either $\hat{}^A_m \neq \hat{}^B_m$ or $\hat{}^A_s \neq \hat{}^B_s$. Therefore, investors must have different estimates about the volatility of x_t (*i.e.*, ${}_{x;A} \neq {}_{x;B}$) or about the informativeness of the public signal s_t (*i.e.*, ${}_A \neq {}_B$). Otherwise, the disagreement process ${}_t$ becomes deterministic, as can be seen from Eq. (1.3.7). For the remainder of this paper, unless stated otherwise, we assume that ${}_{x;A} \neq {}_{x;B}$ or ${}_A \neq {}_B$, so that ${}_t$ is stochastic.

Suppose that ${}_{x;A} > {}_{x;B}$, so that investor A has a higher prior estimate of the volatility of x_t than B. Therefore, $\hat{}^A_m > \hat{}^B_m$ and ${}_{;m} < 0$. If investor A observes a large and positive dM_t that is difficult to reconcile with her previous estimate x^B_t, she would interpret it as a large money surprise $dz^A_{m;t}$. Then, she would revise her estimate x^A_t upwards, thereby decreasing $d{}_t$. Thus, investor A becomes relatively more optimistic about x_t.[23] Alternatively, suppose that ${}_A = 0; {}_B > 0$. Thus, investor B believes that the signal conveys relevant information about x_t, whereas A believes it is pure noise, which implies that ${}_{;s} = \hat{}^B_s > \hat{}^A_s = 0$. A positive ds_t then means that investor B infers good news about the unobservable expected growth rate of money from the signal. Consequently, she revises her estimate x^B_t upwards such that $d{}_t$ rises in turn. Accordingly, investor B becomes relatively more optimistic about x_t.

1.3.3 Equilibrium

The asset pricing equation that values an investment of one dollar in real terms under investor i's beliefs is

$$\frac{1}{P_t} = E^i_t \int_t^{\infty} \frac{{}_{i;u}}{{}_{i;t}} \frac{R_u}{P_u} du \; ; \; i \in \{A; B\}. \tag{1.3.14}$$

Combining the first-order conditions (1.3.2) and (1.3.3), we obtain the optimality condition for the nominal short rate R_t

$$R_t = \frac{1 - {}'}{{}'} C_{i;t} \left(\frac{M_{i;t}}{P_t} \right) .$$

Inserting the previous equation into Eq. (1.3.14), we obtain an analogous equation to (1.2.13) under the beliefs of investor i, where the price level is determined by

$$\frac{1}{P_t} = \frac{1 - {}'}{{}'} E^i_t \int_t^{\infty} e^{(u - t)} \frac{C_{i;t}}{M_{i;u}} du \; ; \; i \in \{A; B\}. \tag{1.3.15}$$

[22] See Section (1.2.2) for details on the derivation of the optimal filtering problem faced by investor i.

[23] We adopt the term optimistic for convenience to indicate that the expected growth rate of money is expected to rise. We employ the term pessimistic conversely.

Specifically, under the beliefs of investor A the equation above becomes

$$\frac{1}{P_t} = \frac{1-\phi}{\phi} \mathrm{E}_t^A \left[\int_t^{\infty} e^{-\rho(u-t)} \frac{C_{A,t}}{M_{1,u}} du \right].$$

Proposition 1.3.2. *The equilibrium price level and nominal short rate are, respectively*

$$P_t = \frac{\phi}{1-\phi} Q\left(x_t^A, x_t^B, \xi_t\right)^{-1} \left(\frac{C_t}{M_t}\right)^{-1}; \tag{1.3.16}$$

$$R_t = Q\left(x_t^A, x_t^B, \xi_t\right)^{-1} \geq 0; \tag{1.3.17}$$

where $Q\left(x_t^A, x_t^B, \xi_t\right) \equiv \int_t^{\infty} \left[f_t M\left(x_t^A, u-t; \hat{\gamma}^A, \hat{\gamma}_x^A\right) + (1-f_t) M\left(x_t^B, u-t; \hat{\gamma}^B, \hat{\gamma}_x^B\right)\right] du$ *and* $M\left(x_t^A, u-t; \hat{\gamma}^A, \hat{\gamma}_x^A\right) \equiv \mathrm{E}_t^i \left[e^{-\rho(u-t)} \left(\frac{M_u}{M_t}\right)^{-1} \right]$ *is defined in Eq. (1.6.17) for* $i \in \{A, B\}$.

From Proposition (1.3.2), $Q\left(x_t^A, x_t^B, \xi_t\right)$ can be decomposed into two components

$$Q\left(x_t^A, x_t^B, \xi_t\right) = f_t Q\left(x_t^A; \hat{\gamma}^A, \hat{\gamma}_x^A\right) + (1-f_t) Q\left(x_t^B; \hat{\gamma}^B, \hat{\gamma}_x^B\right); \tag{1.3.18}$$

where $Q\left(x_t^i\right) \equiv \int_t^{\infty} M\left(x_t^i, u-t; \hat{\gamma}^i, \hat{\gamma}_x^i\right) du$ for $i \in \{A, B\}$.[24] Suppose that investor B estimates a high x_t in period t. For the reasons discussed in the single-investor economy in section (1.2), this means that $Q\left(x_t^B\right)$ would be low. Then, from Eqs. (1.3.17) and (1.3.18), the nominal short rate in the disagreement economy rises in turn. $Q\left(x_t^i\right)$ is proportional to the normalized demand for real cash balances in an economy entirely populated by investor i. Therefore, $Q\left(x_t^A, x_t^B, \xi_t\right)$ is proportional to the weighted-average of the demand for money that would prevail in the single-investor economies of A and B, where the weights of both investors are given by the sharing rules f_t and $1-f_t$, respectively. To facilitate the comparison between the single-investor economy in section (1.2) and the disagreement economy in this section, table (1.1) draws a mapping between some of the auxiliary variables in the single-investor economy and their corresponding counterparties in the disagreement economy. The latter are, in turn, averages of hypothetical single-investor economies weighted by the sharing rules.

We proceed to compute the dynamics of the price level, nominal short rate, and the real and nominal state-price densities of the disagreement economy in proposition (1.3.3). Then, we compare the results to those of the single-investor economy laid out in section (1.2).

Proposition 1.3.3. *The dynamics of the price level, nominal short rate, and nominal and real state-price densities are*

$$\frac{dP_t}{P_t} = \mu_t dt + \sigma_c^P dz_{c,t} + \sigma_{m,t}^P dz_{m,t}^A + \sigma_{s,t}^P ds_t; \tag{1.3.19}$$

$$dR_t = \mu_t^R dt + \sigma_{m,t}^R dz_{m,t}^A + \sigma_{s,t}^R ds_t; \tag{1.3.20}$$

$$\frac{d\xi_{A,t}}{\xi_{A,t}} = -r dt - \theta_c dz_{c,t} - \theta_{m,t} dz_{m,t}^A; \tag{1.3.21}$$

$$\frac{d\xi_{A,t}^{\$}}{\xi_{A,t}^{\$}} = -R_t dt - \theta_{m,t}^{\$} dz_{m,t}^A - \theta_{s,t}^{\$} ds_t; \tag{1.3.22}$$

[24] We omit the explicit dependence of $Q\left(x_t^i\right)$ on the posterior variances $\hat{\gamma}^i, \hat{\gamma}_x^i$ for the sake of conciseness.

where $^{\$}_{A;t} = \frac{_{A;t}}{P_t}$ *is investor A's nominal state-price density. The expected inflation rate,* $_t$*, and the diffusion terms* $^{p}_{c;t}$*,* $^{p}_{m;t}$*, and* $^{p}_{s;t}$ *are given by Eqs. (1.6.29), (1.6.30), (1.6.31), and (1.6.32), respectively. The drift and diffusion terms* $^{R}_{t}$*,* $^{R}_{m;t}$ *and* $^{R}_{s;t}$ *are given by Eqs. (1.6.34), (1.6.35), and (1.6.36), respectively. The real short rate, r, and the real prices of risk,* $_c$ *and* $_{m;t}$*, are given respectively by* $r = \ + \ _c \quad ^2_c$*,* $_c = \ _c$*, and* $_{m;t} = \ (1 \quad f_t) \ _t$*. The nominal short rate,* R_t*, and the nominal prices of risk* $^{\$}_{m;t}$ *and* $^{\$}_{s;t}$*, a vector of nominal prices of risk are given, respectively, by Eqs. (1.6.40), (1.6.41), and (1.6.42) in the Appendix.*

There are similarities and differences between the single-investor economy and the disagreement economy above. First, the same three risk factors are present in the disagreement economy. Furthermore, the real short interest rate and the single real risk factor are also constant. However, the nominal risk factor $z^{A}_{m;t}$ now affects the real-state price density, with time-varying price of risk measured by $_{m;t}$. Intuitively, disagreement about the money growth process leads to beliefs about the state of the economy and sharing rules that fluctuate over time. Given that the sharing rules determine real cash balances and consumption allocations, the prices of the risky real assets of this economy (*e.g.* stocks) are indeed impacted by the level of disagreement in the economy. This result is also present in Ehling et al. (2018) even with log-utility.

Before continuing with the analysis the impact of disagreement on the dynamics of the variables in proposition (1.3.3), we discuss the disagreement adjustment term $\quad x^{A}_{t}; x^{B}_{t}; \ _t$ defined in Eq. (1.6.21). We call it a disagreement adjustment because it affects the level of the nominal short rate, expected inflation, and the instantaneous inflation risk premium. In turn, it impacts the diffusion terms of the dynamics of the price level, the nominal short rate, state-price density and yield curve. Suppose that $x^{B}_{t} > x^{A}_{t}$, such that $_t > 0$. Thus, $M \ x^{A}_{t}; u \quad t; \hat{}^{A}; \hat{}^{A}_{x} \ < M \ x^{B}_{t}; u \quad t; \hat{}^{B}; \hat{}^{B}_{x}$ when $\hat{}^{A}$ and $\hat{}^{B}$ are of similar magnitude.[25] By construction, it then follows that $D \ x^{A}_{t}; x^{B}_{t} \ < 0$. When $x^{B}_{t} < x^{A}_{t}$ and $_t < 0$, following an analogous reasoning it can be shown that $D \ x^{A}_{t}; x^{B}_{t} \ > 0$. Therefore, regardless of the sign of $_t$, when $x^{A}_{t} \neq x^{B}_{t}$ it is the case that $_t D \ x^{A}_{t}; x^{B}_{t} \ < 0$. Furthermore, the disagreement adjustment term depends on the product of the sharing rules $(1 \quad f_t) f_t$, which is maximized at $f_t = 0{:}5$, that is, when both investors have an equal share of the endowment processes.

The drift of the dynamics of $^{\$}_{A;t}$, given by Eq. (1.6.13), must coincide with the nominal short rate in Eq. (1.3.17). From table (1.1) and Eq. (1.6.40), we observe that all but the last two terms of the nominal short rate in the disagreement economy correspond to weighted-averages of the nominal short rates in hypothetical single-investor economies. The last two terms are $_{m;t}$, whose sign depends on $_t$, and $\quad x^{A}_{t}; x^{B}_{t}; \ _t$, where the latter tends to lower the nominal short rate. Both terms stem from the disagreement about x_t. They imply that, in general, the equilibrium nominal

[25] We show this in section (1.4) for some parameterized example.

short rate in the disagreement economy does not equal the weighted-average of the nominal short rates of the single-investor economies. Unless, $_t = 0$, this result is at odds with that of Xiong and Yan (2010). In fact, the presence of these two terms induce additional volatility of the nominal interest rate not present in Xiong and Yan (2010). In sum, the overall e ect of disagreement on the nominal short rate is ambiguous.

Finally, we compare the di usion terms in proposition (1.3.3) to those from weighted-averages of single-investor economies in table (1.1). Given that investors agree on the consumption process and the public signal (although they may have heterogeneous subjective beliefs about its informativeness), the disagreement adjustment term $x_t^A; x_t^B; \ _t$ does not appear in the di usion terms associated with them. Therefore, the di usion terms associated with consumption and public signal correspond to weighted-averages of hypothetical single-investor economies, as can be seen in table (1.1). On the contrary, it appears in the di usion terms associated with investor A's adapted shock $z_{m;t}^A$, even in the case of the real state-price density. Therefore, any e ect of the beliefs spread $_t$ on the volatility of the variables of interest arises from $x_t^A; x_t^B; \ _t$, $_{m;t}$, and $z_{m;t}^A$. We revisit the analysis of the volatility of the short rate in section (1.4).

Proposition 1.3.4. *In the economy with disagreement, the instantaneous in ation risk premium in period t, IRP_t, and the real and nominal yield curves between periods t and , $y_{t;}$ and $y_{t;}^{\$}$, are*

$$IRP_t = {}_c^2 + {}_{m;t} \quad {}_m \left[\frac{G \ x_t^A; x_t^B; \ _t; \hat{}_m}{Q \ x_t^A; x_t^B; \ _t} + \ x_t^A; x_t^B; \ _t \right]; \tag{1.3.23}$$

$$y_{t;} = \ + \ _c \ {}_c^{2;} \tag{1.3.24}$$

$$y_{t;}^{\$} = \frac{\log Q \ x_t^A; x_t^B; \ _t;}{t} + \frac{\log Q \ x_t^A; x_t^B; \ _t}{t}; \tag{1.3.25}$$

where $Q \ x_t^A; x_t^B; \ _t; \quad \int_1 \frac{M\left(x_t^A; u \ t; \hat{}^A; \hat{}_x^A\right) + \ _t M\left(x_t^B; u \ t; \hat{}^B; \hat{}_x^B\right)}{1 + \ _t} du$. Furthermore, the nominal yield curve has dynamics

$$dy_{t;}^{\$} = {}_{t;}^{y} \, dt + {}_{m;t;}^{y} \, dz_{m;t}^1 + {}_{s;t;}^{y} \, dz_{s;t};\tag{1.3.26}$$

with

$$\begin{aligned} {}_{m;t;}^{y} &= \frac{1}{t} \left[\frac{G \ x_t^A; x_t^B; \ _t; \ ; \hat{}_m}{Q \ x_t^A; x_t^B; \ _t;} \quad \frac{G \ x_t^A; x_t^B; \ _t; \hat{}_m}{Q \ x_t^A; x_t^B; \ _t} \right. \\ &\quad \left. \frac{f_t (1 \quad f_t) \ _t}{t} \quad D \ x_t^A; x_t^B; \quad + D \ x_t^A; x_t^B \quad \right]; \end{aligned} \tag{1.3.27}$$

$${}_{s;t;}^{y} = \frac{1}{t} \left[\frac{G \ x_t^A; x_t^B; \ _t; \ ; \hat{}_s}{Q \ x_t^A; x_t^B; \ _t;} \quad \frac{G \ x_t^A; x_t^B; \ _t; \hat{}_s}{Q \ x_t^A; x_t^B; \ _t} \right]; \tag{1.3.28}$$

with $G \ x_t^A; x_t^B; \ _t; \ ; \hat{}_l$ for $l \in \{m; s\}$, and $D \ x_t^A; x_t^B;$ are given by Eqs. (1.6.43) and (1.6.44), respectively.

Contrary to the single-investor economy, the instantaneous inflation risk premium is time-varying in the disagreement economy. Despite the log-utility assumption, investors require a compensation to invest in nominal bonds over real bonds. The compensation varies with their estimates of the unobservable money growth process x_t. As expected, $\left(x_t^A; x_t^B; {}_t\right)$ affects the inflation risk premium, with its total effect being multiplied by ${}_{m;t}$, whose sign depends on sign of ${}_t$ so there is an ambiguous prediction about the impact of $\left(x_t^A; x_t^B; {}_t\right)$ on IRP_t. Instead, suppose that ${}_t > 0$ such that investor B is more optimistic about x_t than A. Thus, ${}_{m;t} \left(x_t^A; x_t^B; {}_t\right)$ would tend to be positive, thus increasing the inflation risk premium. Intuitively, from the standpoint of investor A, who is relatively more pessimistic about x_t than B, it takes a higher inflation risk premium to induce her to invest in nominal bonds. If ${}_t < 0$, then the opposite result obtains. Moreover, if the economy experiences a positive shock to money supply growth, then investor B would confirm her more optimistic view on x_t, thereby increasing her weight $1 \; f_t$, which in turn affects ${}_{m;t}$. This feedback exacerbates the effect of the initial bet about x_t between investors A and B.

As in the single-investor economy, the real yield curve is constant and flat. In contrast, it is immediately seen from Eq. (1.3.25) that the nominal yield curve is a function of disagreement. We draw a similar conclusion regarding its conditional volatility from the diffusion terms in Eqs. (1.3.27) and (1.3.28).

1.4 Calibration and comparative analysis

We calibrate the model to examine the equilibrium of the economies in Sections (1.2) and (1.3). The parameters are selected to be roughly consistent with those in Ehling et al. (2013, 2018). The model's calibration is summarized in Table (1.2). Throughout this section, we focus our analysis on three key parameters, namely , ${}_{x;i}$, and ${}_i$. The persistence parameter is known by investors, whereas ${}_{x;i}$ is an estimate of the volatility of the unobservable process x_t. In turn, ${}_i$ is the subjective belief about the correlation between the public and x_t according to investor i. Given the importance of these three parameters in our model, we allow them to vary across exercises. Consequently, they are not necessarily set at the values in Table (1.2).

In Section (1.4.1), we first evaluate the impact of the parameters of interest on the level of the nominal interest rates and expected inflation, and the volatility of nominal interest rates in the single-investor economy. In Section (1.4.2), we assume that investors A and B disagree about their estimates ${}_{x;i}$ but express the same beliefs about ${}_i$. In Section (1.4.2) we assume that investors disagree about ${}_i$ but have the same estimate ${}_{x;i}$.

1.4.1 Single-investor economy

In this section, we analyze the equilibrium of the single-investor economy characterized in Propositions (1.2.2) and (1.2.3). We focus on the estimate of the expected growth rate of money, x_t^i, because it is the sole driver of conditional variation in this economy. We calculate the model equilibrium for the variables of interest as a function of x_t^i. The domain of x_t^i is set within two standard deviations below and above its unconditional mean, $\overline{x}$ in Eq. (1.6.2), for u t.

Figure (1.1) plots the nominal short rate, R_t, and expected in ation rate, $_t$ as a function of x_t^i across multiple values of three key parameters of interest: , $_{x;i}$, and $_i$. Regardless of the parameter of choice, R_t and $_t$ are both increasing in x_t^i. When investor i infers that money supply will grow at a faster pace in the future, she will lower her demand for real money holdings today relative to tomorrow. For markets to clear, R_t must increase. From the optimality condition in Eq. (1.2.10), the interest to be earned in the future from the reduction in her cash balances in t, will result in additional future consumption, such that $_t$ increases in turn. Moreover, R_t is a non-negative convex function of x_t^i. For high realizations, $\frac{G(x_t^i)}{Q(x_t^i)}$, the elasticity of the demand for real money holdings to x_t^i, approaches zero and R_t becomes approximately linear in x_t^i.

In turn, reducing the persistence of x_t, attens the responsiveness of R_t and $_t$ to the estimate x_t^i. For high levels of , the future values of the expected growth rate will likely be close to $\overline{x}$ and, thus, will be less in uenced by the current latent state of the economy, x_t. In the limit, as increases, R_t and $_t$ become at because x_t^i would be set at $\overline{x}$ at all times regardless of the shocks that hit the economy. Furthermore, increasing the estimate $_{x;i}$ of the volatility parameter $_x$ shifts the nominal short rate down across all x_t^i. Thus, in an economy where the investor has more uncertainty about the volatility of x_t, she will prefer to invest more money in the bank account. This e ect results in lower values of R_t. She subsequently substitutes real cash balances for consumption, the expected growth rate of which is constant and observable, thus raising expected in ation. Finally, the higher the level of $_i$, the higher the nominal interest rate R_t across all x_t^i. When investor i believes that the public signal conveys valuable information about x_t, she is less willing to invest in the money account. Thus, for markets to clear, the nominal short rate increases and the expected in ation falls. Intuitively, according to Eq. (1.10), higher values of $_i$ reduce $\hat{}_x$, the risk adjustment due to the unobservable nature of x_t. Although R_t increases in equilibrium, this e ect is economically negligible.

From Figure (1.2), we observe that the instantaneous volatility of the nominal short rate is an increasing function of x_t^i. Furthermore, the nominal short rate becomes more volatile as rises. As for the uncertainty parameter $_{x;i}$, we know that more uncertainty unambiguously results in higher posterior variance $\hat{}_m^{i\;2}$, which in turn shifts upwards the volatility curve of R_t for any x_t^i. In turn, increases in $_i$ reduce the investor's posterior variance $\hat{}_m^{i\;2}$, but instead increase $\hat{}_s^{i\;2}$. Thus, the investor responds more strongly to any realization of the public signal. In the equilibrium, the increase in $\hat{}^{i}_{s}\ ^{2}$ more than offsets the reduction in $\hat{}^{i}_{m}\ ^{2}$; and volatility rises with $_{i}$ for any x^i_t.

We plot in Figure (1.3) the nominal yield curve between t and for two levels of x^i_t: low and high, which correspond to values below and above $\overline{x}$, respectively. The first important result is that the slope of the yield curve is a function of the estimate x^i_t. In particular, when x^i_t is low relative to $\overline{x}$, the yield curve is upward-sloping, whereas it is downward-sloping for high values of x^i_t relative to $\overline{x}$. Intuitively, when x^i_t is low, the nominal short rate is concomitantly low for the reasons discussed above. In addition, given that the expected growth rate of money is mean-reverting, the investor expects that x^i_t will be pulled up towards the long-run mean in $> t$. Thus, the nominal interest rate between t and is expected to rise in the future, such that the yield curve slopes upwards. Larger deviations of x^i_t from $\overline{x}$ imply stronger pulls towards the long-run mean, thereby further raising its slope. Conversely, when x^i_t is high, the investor expects the growth rate of money to be pulled down to $\overline{x}$, and the opposite result for the slope of the yield curve follows. We also observe that larger values of make the yield curve less responsive to x^i_t. Additionally, more uncertainty $_{x;i}$ shifts downward the yield curve for all t, and a higher degree of confidence $_{i}$ has the opposite effect. As these three effects are analogous to those above alluding to the nominal short rate, so we do not reiterate the same discussion here for the sake of briefness.

We conclude this section by analyzing the instantaneous volatility of the yield curve between t and for low and high levels of x^i_t in Figure (1.4). The main takeaway is that the volatility is decreasing in the horizon of the yield curve, regardless of the parameter configuration. Moreover, higher estimates x^i_t increase the volatility of the interest rates between and t. The impact of varying the three parameters of interest on the instantaneous volatility of the yield curve is analogous to that of the volatility of the nominal short rate.

1.4.2 Disagreement economy

In this section, we first explore the asset pricing implications of disagreement when one investor is more confident than the other one regarding the uncertainty parameter ($_{x;B} < _{x;A}$). We then examine a scenario where investor B believes that the public signal is informative, but investor A does not (*i.e.*, $_{B} > 0;\ _{A} = 0$). This comparative analysis is based on the disagreement economy described in Section (1.3).

Investors A and B must have different priors about the volatility of the expected growth rate of money supply or the informativeness of the signal. If they had the same priors, then they would converge to the same beliefs about x_t for any initial level of disagreement, as can be seen in Eq. (1.3.7). For the remainder of this paper, without loss of generality, we adopt the convention that investor B is either more confident, in which case $_{x;B} < _{x;A}$, or assigns a larger credibility to the signal, such that $_{B} > _{A}$. Furthermore, as previously argued, $_{t}$ directly follows from the exogenous endowment process and investors' priors; hence, it can be treated as an exogenous variable. Additionally, conditional on information up to period t, the stochastic weighting $_t$ is itself a state variable that determines the investors' consumption and real money holdings allocations through the sharing-rule $f(\ _t)$, which endogenously fluctuates between zero and one. Thus, in the disagreement economy, the set of state variables expands to include $_t$ and $f(\ _t)$. In the parameterized economy, we construct a grid for $_t$ within two standard deviations below and above zero, the unconditional mean of $_t$ (see Eq. (1.3.7)). In turn, we select key values for the sharing rule, such that $f(\ _t) \in \{0.1; 0.5; 0.9\}$.

No public signal

We initially assume there is no public signal and $_{x;A} > \ _{x;B}$, that is, investors have different estimates about the volatility of the process x_t.[26] Figure (1.5) plots the impact of disagreement $_t$ on the disagreement adjustment term $\left(x_t^A; x_t^B;\ _t\right)$ and the nominal short rate R_t for three different sharing rules. The first result is that the adjustment term is an inverted parabola in $_t$. When $_{x;B}$ is close to $_{x;A}$, its vertex is approximately zero, indicating that $\left(x_t^A; x_t^B;\ _t\right)$ is decreasing in the absolute value of disagreement, namely $|\ _t|$.[27] Otherwise, $\left(x_t^A; x_t^B;\ _t\right)$ is decreasing in $|\ _t|$, except in a small neighborhood around its vertex. Stated differently, $\left(x_t^A; x_t^B;\ _t\right)$ tends to fall as disagreement increases because either investor B is becoming more optimistic relative to investor A, or vice versa. Additionally, for a given level of disagreement, $\left(x_t^A; x_t^B;\ _t\right)$ is largest (in absolute value) when the sharing rules are equal, *i.e.,* $f(\ _t) = 0.5$.

By contrast, the impact of the disagreement on the nominal short rate is ambiguous. The reason for this is that, as evidenced in Eq. (1.6.40), two components determine the nominal short rate: the weighted-average of the nominal interest rates that would prevail in hypothetical single-investor economies, and the disagreement terms $\left(x_t^A; x_t^B;\ _t\right)$ and $_{m;t}$. The impact of disagreement in the former is ambiguous. For instance, fixing x_t^A and then increasing (decreasing) x_t^B such that $_t$ rises (falls) in turn, leads to a high (low) R_t. The decrease in $\left(x_t^A; x_t^B;\ _t\right)$ that tends to follow an increase in $_t$ is then dominated.

In Figure (1.6), we study the effect of disagreement on the instantaneous volatility of the nominal short rate and inflation risk premium. First, we observe that the volatility of R_t plots a parabola in $_t$, with its vertex slightly shifted to the left of $_t$. Its shape resembles that of the disagreement adjustment $\left(x_t^A; x_t^B;\ _t\right)$, except that now it is (mostly) increasing in the absolute value of $_t$. Therefore, a larger level of disagreement in either direction tends to make the nominal short rate more volatile. As before, the impact of disagreement on volatility is maximum when $f(\ _t) = 0.5$, for any given $_t$. The mechanism behind this result is the usual one in the literature

[26] Alternatively, we could have assumed that $_A = \ _B = 0$ and the same results would follow because investors would optimally choose to ignore the signal. This assumption of different estimates about the volatility of the unobservable process is similar to that of Ehling et al. (2018)

[27] The vertex is tilted towards the left of $_t = 0$ because, by assumption, $_{x;A} > \ _{x;B}$.

of heterogeneous beliefs. Upon the observation of a positive shock that raises the nominal money supply, the more optimistic investor (*i.e.,* the one with the largest estimate x_t^i) confirms her bullish view on x_t and, hence, bets on an increase in interest rates. The weight of the more optimistic investor in the economy then endogenously grows. This feedback effect raises the nominal interest rate in the economy by a larger magnitude. Furthermore, the more dissimilar the investors' weights in the economy, the more influential the views of the dominant investor in the determination of asset prices.

As for the instantaneous inflation risk premium, it is an increasing function of $_t$. By contrast, in the single-investor economy it is constant. To understand this result, observe that when $_t > 0$, investor B is more optimistic about x_t than A. In turn, consistent with her optimistic view on x_t, investor B expects higher interest rates in the future than investor A. Hence, she expects to earn more interest from investing in the money market. For the money market to clear, investor A requires an inflation risk premium to invest in nominal short bonds. Otherwise, she would prefer to invest in risk-free real short bonds that are unaffected by the beliefs' spread. A higher value of $_t$ would prompt investor A to command a larger compensation in case investor B turns out to be right. We also observe that the magnitude of the instantaneous inflation risk premium decreases as $f(\ _t)$ rises. By assumption, investor A has less certainty about the volatility parameter $_x$ than B, such that $_{x;A} > \ _{x;B}$. Thus, when her weight in the economy is low (*e.g.,* $f(\ _t) = 0.1$), the instantaneous inflation risk premium must be higher to induce her to invest in the bank account for markets to clear.

Impact of disagreement under different levels of persistence of expected growth rate

We investigate in this section the impact of the persistence of the expected growth rate of money, , on the relation between disagreement and asset prices in the disagreement economy. In Figure (1.7), we depict the same four variables as before (disagreement adjustment $x_t^A; x_t^B;\ _t$, nominal short rate R_t, and instantaneous inflation risk premium and nominal short rate volatility) as a function of $_t$ across multiple values of . For the sake of space, we only report the plots when $f(\ _t)$ is fixed at 0.5, but the results remain unaffected across different sharing rules. We first observe that both the disagreement adjustment term and the instantaneous volatility of the nominal short rate become less responsive to changes in $_t$ as increases. Furthermore, the response of the instantaneous inflation risk premium to $_t$ is also lower for high values of , whereas it is ambiguous in the case of the nominal short rate.

This argument can be stated formally by looking at Eq. (1.6.1). For $u > t$, low values of imply that the term $e^{\ (u\ t)}$ in the conditional expectation of x_u^i is large, thereby making x_u^i proportionally more dependent on the current estimate x_t^i. By contrast, when is high $e^{\ (u\ t)}$, rapidly declines. Thus, the conditional expectation of x_u^i becomes more heavily influenced by the long-run mean $\bar{x}$ than by x_t^i. Investors would undoubtedly continue to "agree to disagree" in this scenario and still interpret realizations of the money supply growth through the lens of their own models and priors. However, their updated estimates would uctuate over $\bar{x}$, a parameter they both agree on, as opposed to their own respective previous estimates, on which they have di erent views. Therefore, the feedback e ect between investment decisions and endogenous uctuations of wealth is lessened.

In Figure (1.8), we analyze the relation between money growth disagreement and the level and instantaneous volatility of the nominal yield curve across multiple values of . We conduct this analysis for three di erent levels of $_t$: no disagreement, low and high belief spreads, where the last two correspond to two standard deviations below and above the unconditional mean of $_t$ (*i.e.,* zero disagreement), as can be seen from Eq. (1.3.7). Furthermore, we x x_t^A at 90% of its long-run mean. We observe that the slope of the yield curve depends on the level of disagreement and the persistence parameter. First, the low levels of $_t$ indicate that x_t^B is low. Thus, the investor B expects for x_t to be pulled up towards $\bar{x}$ in the future, which implies that she anticipates high future interest rates. The result is an upward-sloping nominal yield curve between periods and t, regardless of the value of . Conversely, a downward-sloping nominal yield follows from a high value of $_t$.

The intermediate case in Figure (1.8) illustrates an interesting situation. In this scenario $x_t^B = x_t^A = 0{:}9\bar{x}$ and disagreement is equal to zero. In turn, both investors have estimates below their long-run mean $\bar{x}$. Given the mean-reversion of x_t, we might expect that the nominal interest rates between and t would rise. This case indeed holds for most values of in the plot. However, for the lowest , the opposite result emerges. Thus, even though both investors believe that money growth will be higher in the future, the yield curve slopes downwards. We interpret this situation as follows: despite the bullish view on future monetary policy stance that both investors share, the high persistence of the unobservable monetary policy stance implies that the individual estimates x_t^i will fail to be pushed up rapidly enough for long-term interest rates to rise. This situation is compounded by the fact that the estimates are close to their long-run mean.[28]

Heterogeneous interpretation of public signal

In this section, we consider a situation where investor B believes that the public signal, s_t, conveys relevant information about money growth, *i.e.,* $_B > 0$, whereas investor A believes it is not informative. Thus, investor B is steadfast about her ability to partly infer x_t from the public signal. From Eq. (1.2.6), investor B's model then relies on the signal, realizations of the money supply, and her own priors to adjust her estimate of x_t. By contrast, investor A is doubtful about the signal as she believes it conveys only noise and, consequently deciding to optimally ignore it. In this scenario, investors e ectively employ di erent estimation models based on the same public

[28] This scenario bears some resemblance to the \Greenspan's Conundrum", whereby rising short-term interest rates are associated with falling long-term interest rates.

information. We investigate the extent to which the adoption of di erent estimation models a ects the equilibrium asset prices and volatility. To isolate the e ect of having heterogeneous subjective beliefs about the informativeness of the signal, we assume for the remainder of this section that $_{x;A} = {}_{x;B}$.[29]

Using table (1.2), we calculate the equilibrium in two economies: a mixed economy where $_{B} \in (0;1)$; $_{B} = 0$, and a doubtful economy without disagreement with $_{A} = {}_{B} = 0$. For every case, we subsequently calculate asset prices and volatility when the beliefs spread is zero, low, and high. We set the beliefs spread within two standard deviations below and above zero for the low and high cases, respectively.[30] Afterwards, we compute the ratio of a variable of interest (*e.g.,* the nominal short rate) in the mixed economy to its corresponding value in the doubtful economy. If the resulting ratio for R_t is equal to, say, 1.06, then R_t is 1.06 larger in the mixed-economy than in the benchmark homogeneous economy. We calculate these ratios for a grid of $_{B}$ between zero and one across multiple values of . We plot the resulting ratios for the nominal short rate, R_t, and the 10-year nominal rate,$y^{\$}_{t;t+10}$, in Figure (1.9), and for the instantaneous volatility of the nominal short rate and 10-year nominal rate in Figure (1.10).

We rst observe from Figure (1.9) that both the nominal short and 10-year rates are higher in the mixed economy than in the doubtful economy without disagreement. Moreover, the ratio is increasing in $_{B}$, which means that the higher credibility investor B attaches to the signal, the larger the level of nominal interest rates relative to the benchmark economy. Given that investor B believes she can learn about x_t from s_t, she is less willing to invest in nominally risk-free assets. The nominal interest rates must rise in equilibrium for markets to clear. We also observe that the wedge between the economies is larger for low values of . However, the overall e ect is economically small, regardless of the value of and the beliefs spread.

In Figure (1.4.2), we study the volatility of the nominal interest rates. For low values of , the instantaneous volatilities of the nominal short and 10-year rates are smaller in the mixed-economy with $_{B} > 0$ than one where $_{B} = 0$. In fact, the wedge between both economies is larger than 10% for all $0 < {}_{B} < 1$ and all three levels of the belief spread. By contrast, for a high level of ,

[29] This exercise is similar to that of Scheinkman and Xiong (2003); Dumas et al. (2009); Xiong and Yan (2010), among others. These papers assume that one investor is overcon dent about the informativeness of a public signal relative to a prudent investor who correctly infers that the public signal is uninformative about the unobservable state of the economy. In the context of monetary policy, this exercise is related to the signaling e ects documented by Melosi (2017). We interpret it as a situation where investors have heterogeneous interpretations about the same public signal that provides them with potentially useful information about the underlying monetary policy stance. For instance, when two investors disagree about whether a central bank statement is e ectively conveying information about the future path of interest rates.

[30] To control for the beliefs spread, we assume that $_{t}$ is the same in the mixed economies and doubtful economies. Of course, a non-zero beliefs spread in the agreement economy cannot be sustained inde nitely because $_{A} = {}_{B}$ and $_{x;A} = {}_{x;B}$. Thus, both investors will converge to the same estimate eventually. Given that we perform comparative statics in this section, we temporarily ignore this property.

the opposite result emerges. As argued in Section (1.2.2), two countervailing forces affect investor B's filtering problem when she believes that the public signal conveys relevant information. On the one hand, from Eq. (1.2.6), higher $_B$ will decrease her reliance on her own surprise $z^B_{m;t}$, thereby reducing the updated volatility estimates $\hat{}^B_x$ and $\hat{}^B_m$. On the other hand, she will respond more strongly to any realization of s_t, thus increasing $\hat{}^B_s$. Each of these estimates increases volatility; hence, the overall impact of varying $_B$ depends on which effect eventually prevails. Our results indicate that the answer once again hinges on the persistence of the underlying money growth process. Investor B updates $\mathrm{d}x^B_t$ when she receives the signal flow $\mathrm{d}s_t$ regardless of the value of . However, when is low, the current state x^i_t carries significant influence on the magnitude of the revision. Therefore, $\mathrm{d}x^B_t$ does not respond as strongly to $\mathrm{d}s_t$. On the contrary, when the underlying money growth is less persistent, $\mathrm{d}x^B_t$ strongly reacts to any realization of the signal, thereby making the dynamics of x^B_t and, hence, the nominal interest rates more volatile. Finally, the effect of $_B$ on the volatility wedge in either scenario is monotonic in $_B$.

1.5 Final remarks and further discussion

We investigate the asset pricing implications of disagreement about the unobservable process driving monetary policy. We present a monetary economy embedded into an exchange economy with exogenous paths for consumption and nominal money supply. The monetary economy is populated by two investors with heterogeneous priors about the unobservable expected growth of money, which we identify as the underlying monetary policy stance. The key model friction is that information about the expected growth rate of money is symmetric but incomplete. Investors estimate the expected growth rate of money by combining their priors with the observed realizations of the money supply. Investors value holding real cash balances intrinsically, which creates a demand for money. In equilibrium, the demand for money and investors' estimates endogenously determine the nominal short rate and expected inflation for any given path of money supply through a portfolio rebalancing channel.

Disagreement about money growth drives the conditional variation in all nominal interest rates, expected inflation, and their volatility. Through a parameterized example, we show that the volatility of nominal interest rates tends to rise with increases in the absolute value of disagreement. In turn, the inflation risk premium increases with the view of the more optimistic investor. The effect on the nominal interest rates is ambiguous. The relation between disagreement about money growth and asset prices crucially depends on the persistence of the unobservable growth rate of money. For instance, when the unobservable process is less persistent, disagreement about money growth has a weaker impact on asset prices.

Our model is also well suited to address questions about the consequences of having heterogeneous interpretations about unconventional monetary policy on interest rates. To this end, we introduce into the model a public signal on which investors may have di erent views. One steadfast investor believes that the signal conveys relevant information about the unobservable process, whereas a doubtful one believes that the signal is pure noise. The volatility of the nominal rates of an economy populated by one steadfast investor and one doubtful investor may be larger or smaller than the volatility of an economy with two doubtful investors and no disagreement. Whichever economy is more volatile depends, in turn, on the persistence of the unexpected growth rate of money.

Despite all the reasonable caveats of our paper, we believe that our simple monetary model provides a sensible starting point to investigate the asset pricing implications of disagreement about monetary policy in more general settings. In particular, the assumption that investors' utility function is logarithmic makes the model tractable. We show that disagreement about money growth determines nominal interest rates, expected in ation, and in ation risk premium even with log utility. However, the e ects of disagreement on the real bond market are largely muted due to this assumption. As documented by Ehling et al. (2018), this is an economic relation with strong empirical support. By considering more general risk-aversion con gurations, the model may become a building block to accommodate a broader research question. For instance, the model could include a properly de ned real sector with investment on physical capital, which then could be used for investigating the impact of disagreement about money growth on the stock market. This topic has received little attention in the literature.

Another avenue for future research is related to the study of disagreement about other types of policies such as quantitative easing (QE). As underscored by Ireland (2017), the three rounds of QE are directed at raising the monetary base to prevent in ation from decreasing. Moreover, QE purportedly works through a change in the relative price of the private sector's portfolio, namely, a portfolio rebalancing channel (Bernanke (2013)). Both components, monetary growth and a portfolio rebalancing channel, play a preponderant role in our model. Moreover, Beckworth (2017) argues that one crucial component of the e ectiveness of QE is the capacity of the Federal Reserve to commit to a permanent expansion of the monetary base. As we have argued throughout the paper, the persistence of the underlying monetary policy stance is a key determinant of our results. For all these reasons, we believe that our model can be extended to address this pertinent question.

1.6 Appendix

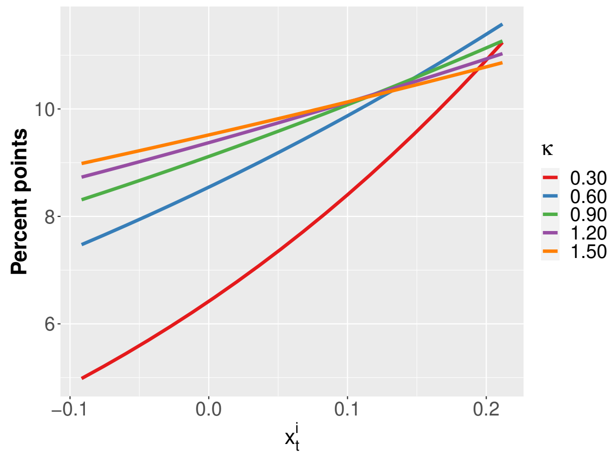
1.6.1 Tables and gures

Table 1.1: Variable correspondence between single-investor and disagreement economies

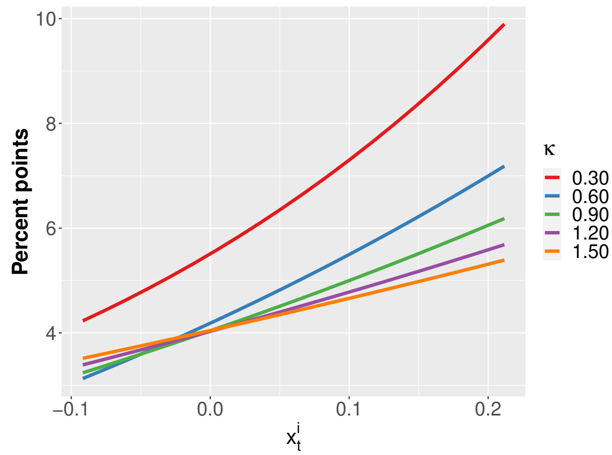
Single-investor economy		Disagreement economy	
Variable	De nition	Variable	De nition
$M\ \ x_t^i;u\ \ \ t$	$\mathrm{E}_t^i\ \ e^{\ \ (u\ \ t)}\ \ \frac{M_u}{M_t}\ \ ^{1}$	$M\ \ x_t^i;u\ \ \ t;\hat{}^i;\hat{}_x^i$	$\mathrm{E}_t^i\ \ e^{\ \ (u\ \ t)}\ \ \frac{M_u}{M_t}\ \ ^{1}$
$Q\ \ x_t^i$	$\int_t^{1}\ M\ \ x_t^i;u\ \ \ t\ \ du$	$Q\ \ x_t^A;x_t^B;\ \ _t$	$\frac{\int_{\sqcup}^{\infty} M\left(x_t^A;u\ \ t;\hat{}^A;\hat{}_x^A\right)+\ \ _t M\left(x_t^B;u\ \ t;\hat{}^B;\hat{}_x^B\right)du}{1+\ \ _t}$
$G\ \ x_t^i$	$\int_t^{1} B\ \ (u\ \ \ t)\ M\ \ x_t^i;u\ \ \ t\ \ du$	$G\ \ x_t^i$	$\int_t^{1} B\ \ (u\ \ \ t)\ M\ \ x_t^i;u\ \ \ t;\hat{}^i;\hat{}_x^i\ \ du$
$H\ \ x_t^i$	$\int_t^{1} B\ \ (u\ \ \ t)^2\ M\ \ x_t^i;u\ \ \ t\ \ du$	$H\ \ x_t^i$	$\int_t^{1} B\ \ (u\ \ \ t)^2\ M\ \ x_t^i;u\ \ \ t;\hat{}^i;\hat{}_x^i\ \ du$
		$D\ \ x_t^A;x_t^B$	$\int_t^{1}\ \ M\ \ x_t^B;u\ \ \ t;\hat{}^B;\hat{}_x^B\ \ \ \ M\ \ x_t^A;u\ \ \ t;\hat{}^A;\hat{}_x^A\ \ \ du$
		$x_t^A;x_t^B;\ \ _t$	$(1\ \ \ f_t)\ f_t \frac{D\left(x_t^A;x_t^B\right)}{Q\left(x_t^A;x_t^B;\ \ _t\right)}\ \ \ _t$
	$G\ \ x_t^i\ \ \ \ \overline{x}\ \ \ x_t^i\ \ \ \ \hat{}_x^i$	$G\ \ x_t^A;x_t^B;\ \ _t$	$f_t G\ \ x_t^A\ \ \ \ \overline{x}\ \ \ x_t^A\ \ \ \ \hat{}_x^A\ \ +(1\ \ \ f_t)\ \ \ \overline{x}\ \ \ x_t^B\ \ \ \ \hat{}_x^B$
	$\hat{}_l^i G\ \ x_t^i\ \ ;$ for $l\ 2\ fm;sg$	$G\ \ x_t^A;x_t^B;\ \ _t;\hat{}_l$	$f_t G\ \ x_t^A\ \ \ \hat{}_l^A+(1\ \ \ f_t)\ G\ \ x_t^B\ \ \ \hat{}_l^B;$ for $l\ 2\ fm;sg$
	$\hat{}^i\ \ ^2 H\ \ x_t^i$	$H\ \ x_t^A;x_t^B;\ \ _t;\hat{}$	$f_t H\ \ x_t^A\ \ \ \ \hat{}^A\ \ ^2+(1\ \ \ f_t)\ H\ \ x_t^B\ \ \ \ \hat{}^B\ \ ^2$

Table 1.2: Parameter calibration

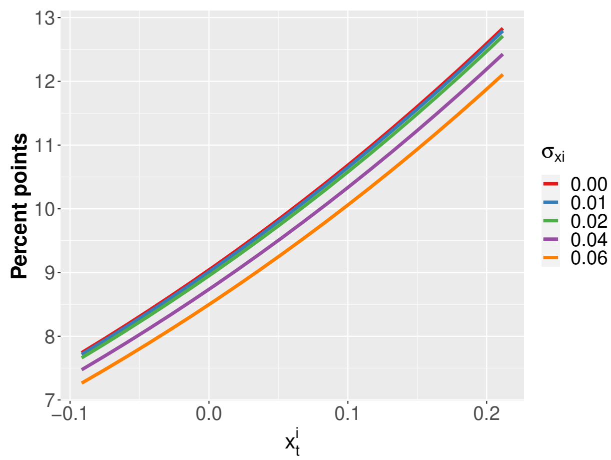
Parameter	Description	Value
	Investors	
	Time preference parameter	0.017
$f(\ _0)$	Initial cons/money hold. allocation	0.5
	Consumption process	
$_c$	Expected cons. growth	0.0172
$_c$	Volatility of cons.	0.0332
	Money supply process	
$_m$	Money supply total vol.	0.013
$\overline{x}$	Long-run mean	0.050
	Mean reversion	0.5
$_x$	Vol. of expected money growth (true)	0.05
	Unobservable process	
$_{x;A}$	Vol. of expected money growth for investor *A*	0.06
$_{x;B}$	Vol. of expected money growth for investor *B*	0.02



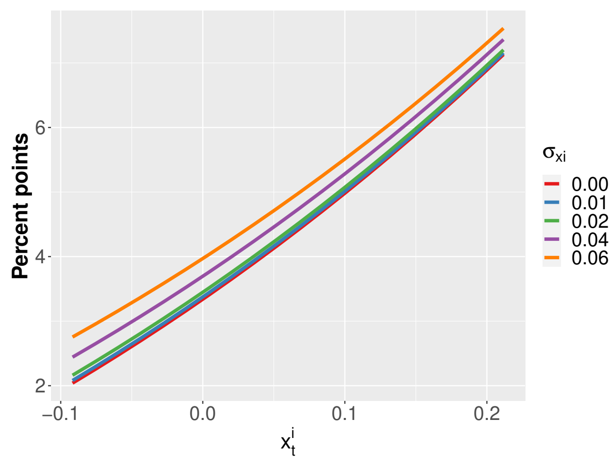
(a) Nominal short rate: R_t



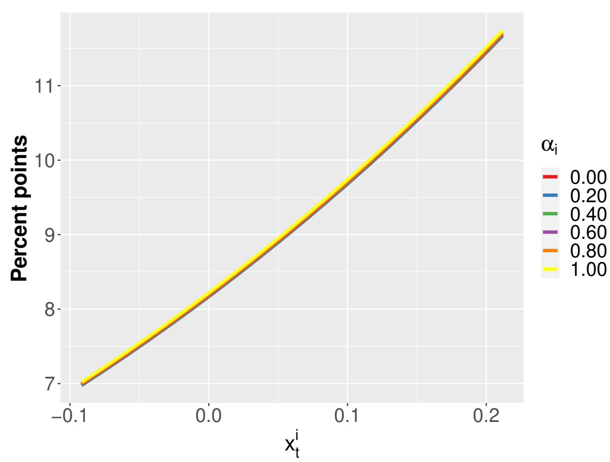
(b) Expected in ation rate: $_t$



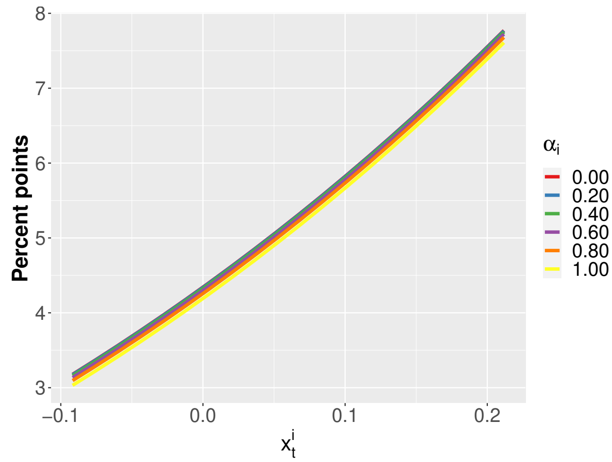
(c) Nominal short rate: R_t



(d) Expected in ation rate: $_t$



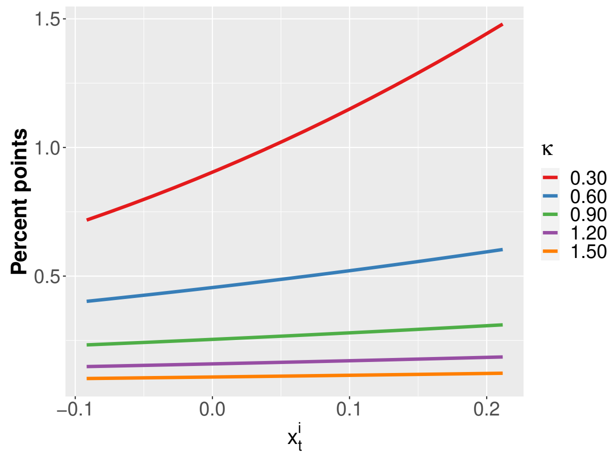
(e) Nominal short rate: R_t



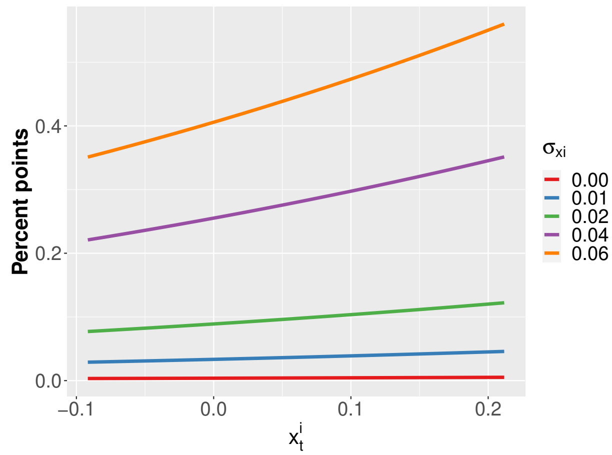
(f) Expected in ation rate: $_t$

Figure 1.1: Nominal short rate and expected in ation as a function of x_t^i

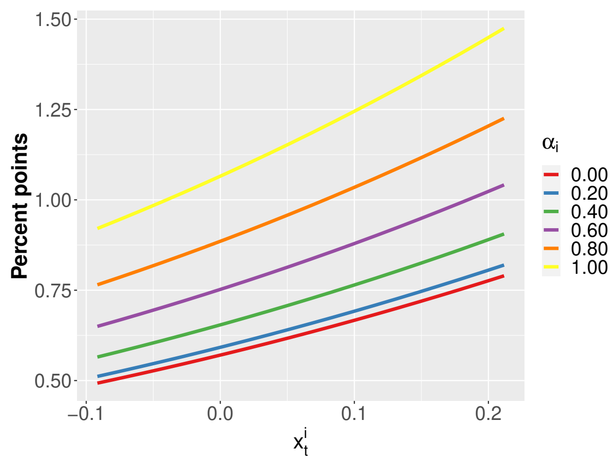
Note: This gure plots the nominal short rate and expected in ation as a function of estimates the expected growth rate of money, x_t^i, in the single-investor economy described in section (1.2). The parameters come from table (1.2). The grid of x_t^i is within two standard deviations below and above its unconditional mean, $\bar{x}$. See Eq. (1.6.1).



(a) Varying



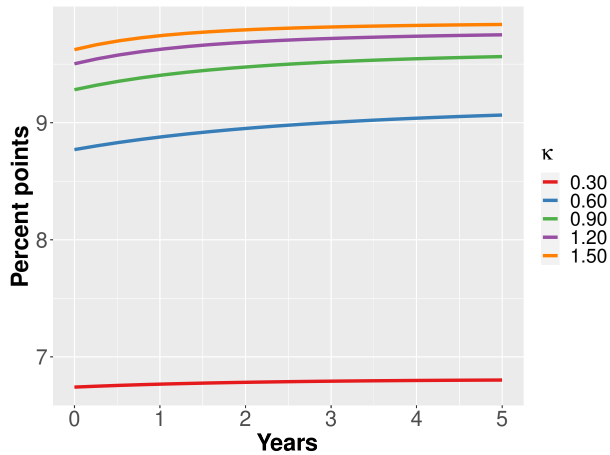
(b) Varying $_{x;i}$



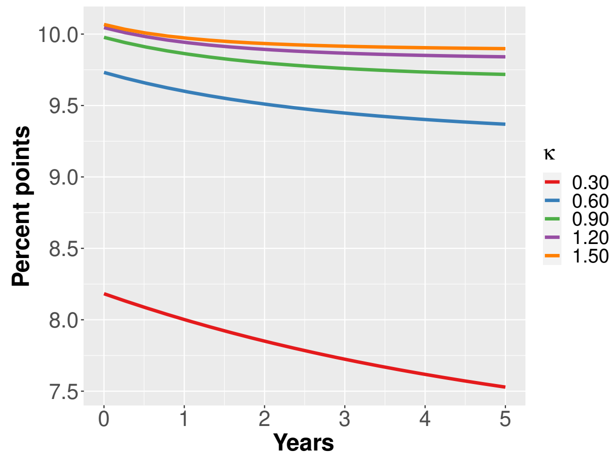
(c) Varying $_i$

Figure 1.2: Nominal short rate instantaneous volatility as a function of estimates of expected growth rate of money, x_t^i

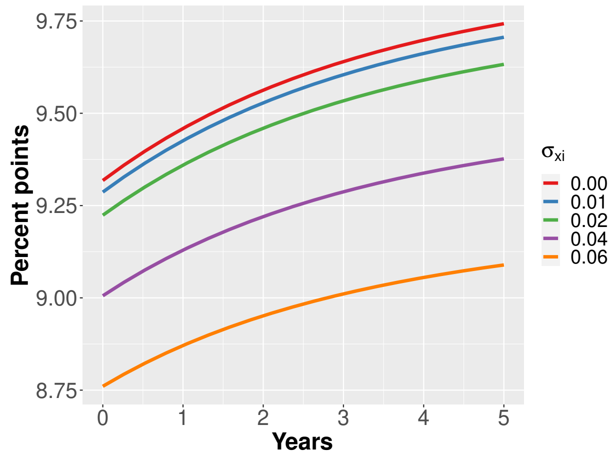
Note: This gure plots the instantaneous volatility of the nominal short rate, R_t, as a function of estimates of the expected growth rate of money, x_t^i, in the single-investor economy described in section (1.2). The parameters come from table (1.2). The grid of x_t^i is within two standard deviations below and above its unconditional mean, $\bar{x}$. See Eq. (1.6.1).



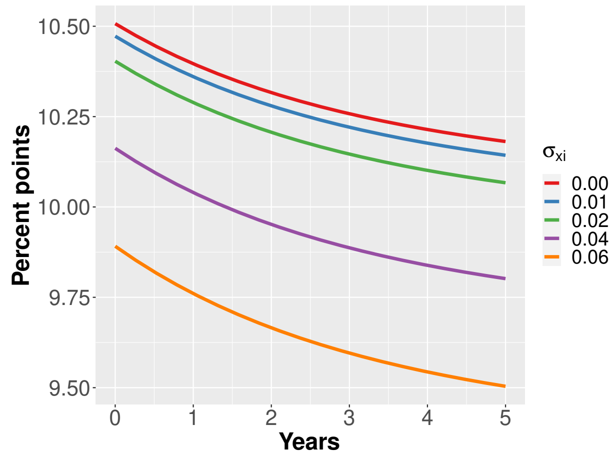
(a) Low x^i_t



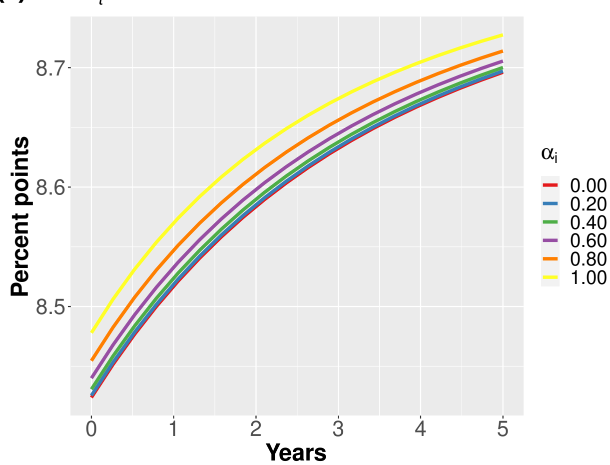
(b) High x^i_t



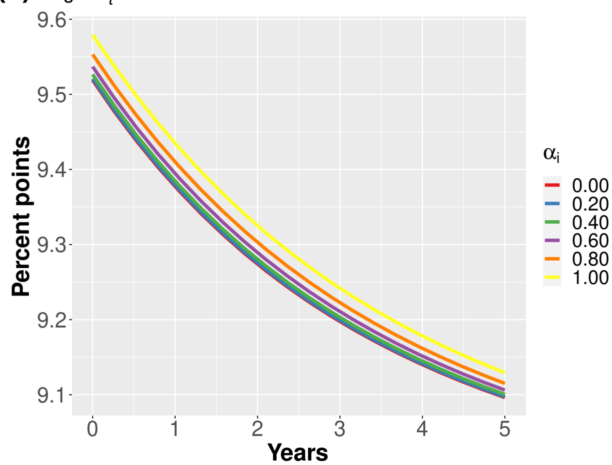
(c) Low x^i_t



(d) High x^i_t



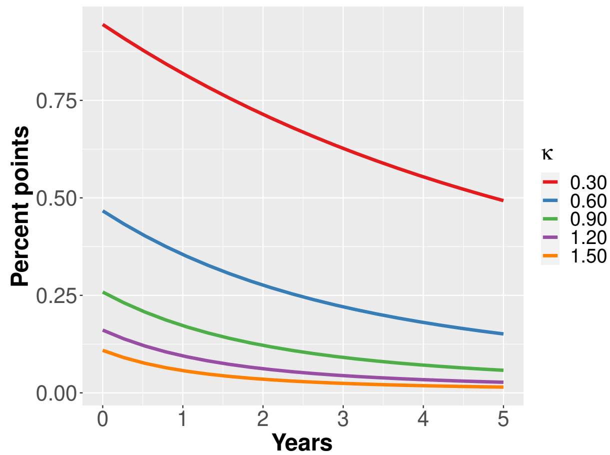
(e) Low x^i_t



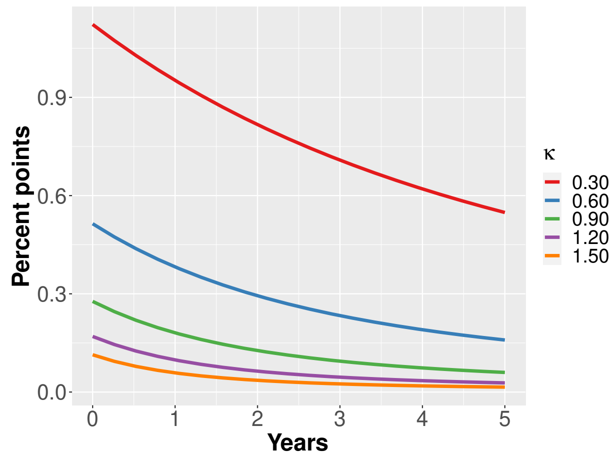
(f) High x^i_t

Figure 1.3: Nominal yield curve as a function of estimates of expected growth rate of money, x^i_t

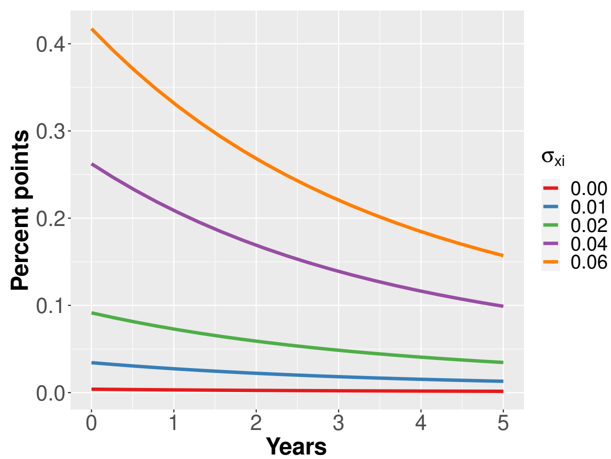
Note: This gure plots the level of the nominal yield, $y^{\$}_{t;}$, across two values of the expected growth rate of money, x^i_t, in the single-investor economy described in section (1.2). The parameters come from table (1.2). Low and high values of x^i_t are given by $0{:}5\overline{x}$ and $1{:}5\overline{x}$, respectively.



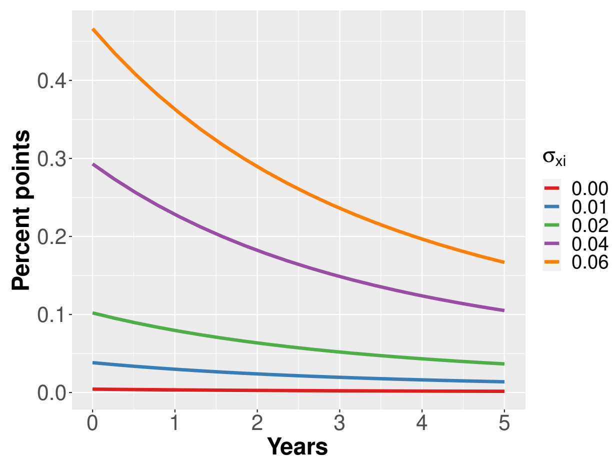
(a) Low x_t^i



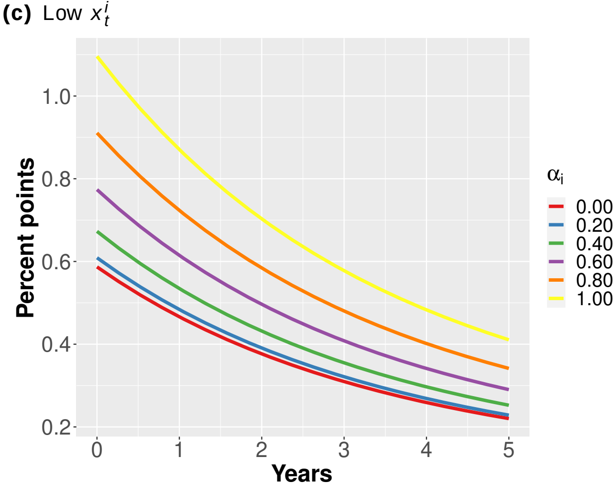
(b) High x_t^i



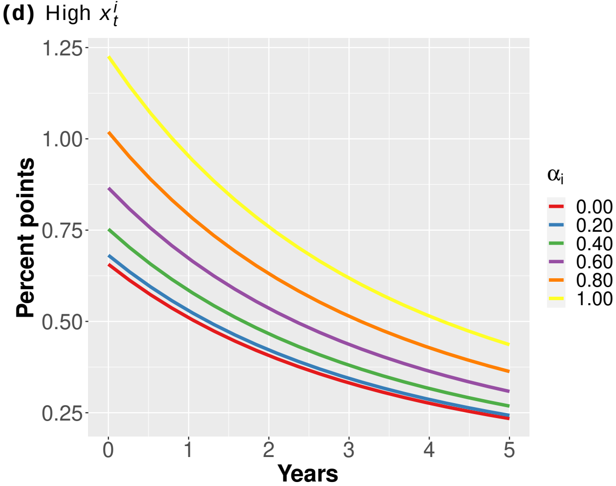
(c) Low x_t^i



(d) High x_t^i



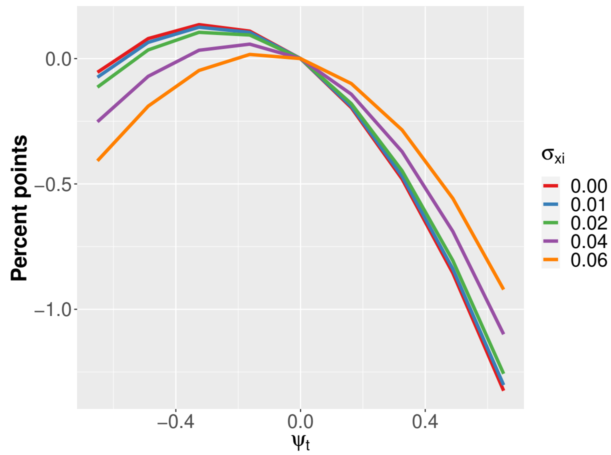
(e) Low x_t^i



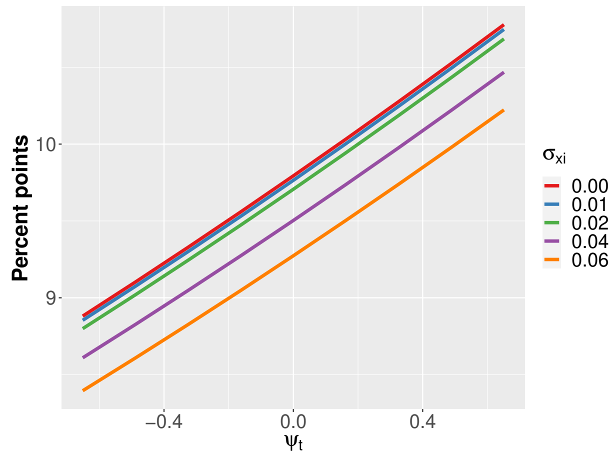
(f) High x_t^i

Figure 1.4: Instant. volatility of nominal yield curve as a function of x_t^i

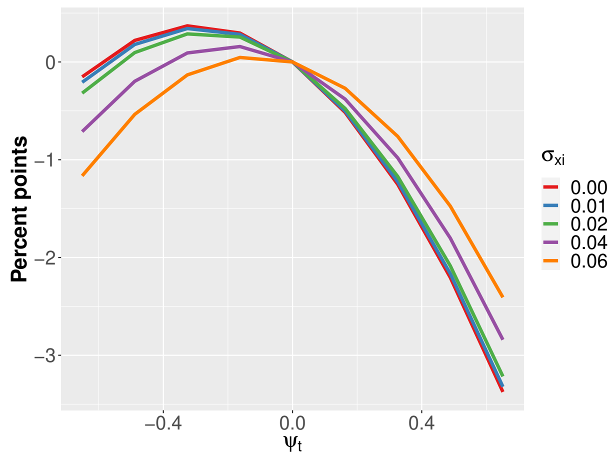
Note: This gure plots the volatility of the nominal yield, $y^{\$}_{t;}$, across two values of the expected growth rate of money, x_t^i, in the single-investor economy described in section (1.2). The parameters come from table (1.2). Low and high values of x_t^i are given by $0{:}5\overline{x}$ and $1{:}5\overline{x}$, respectively.



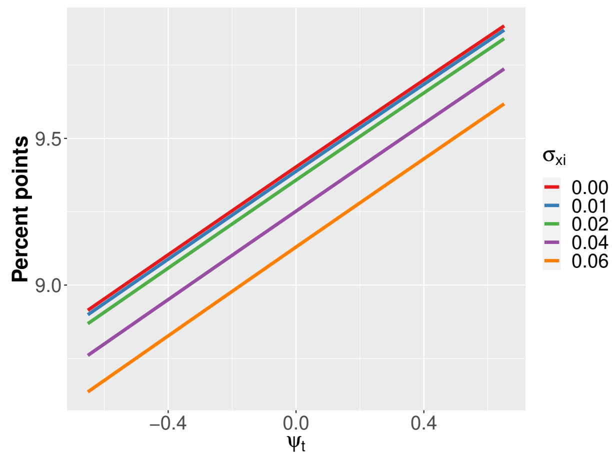
(a) Disagreement adjustment, $x_t^A; x_t^B;\ _t$; $f(\ _t) = 0{:}1$



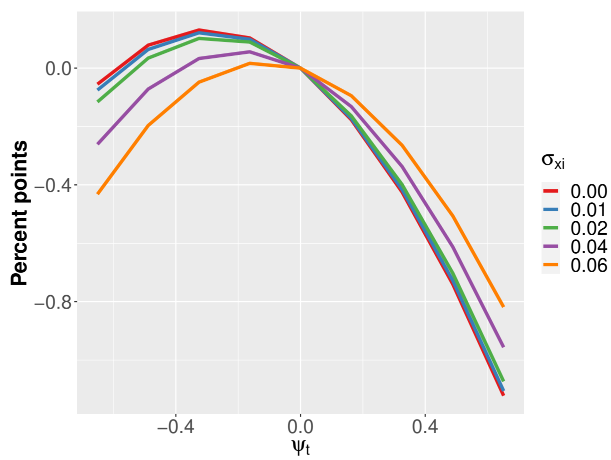
(b) Nominal short rate, R_t; $f(\ _t) = 0{:}1$



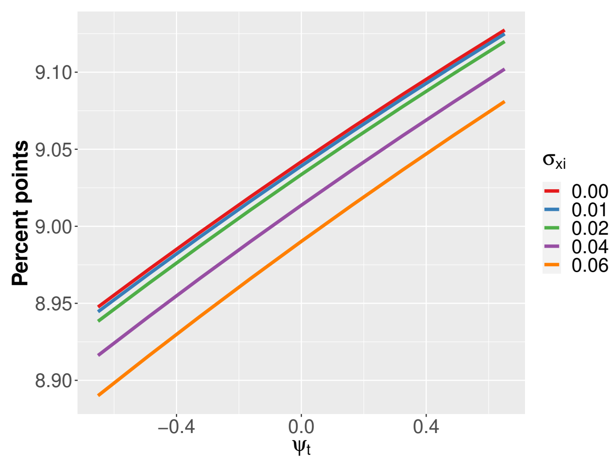
(c) Disagreement adjustment, $x_t^A; x_t^B;\ _t$; $f(\ _t) = 0{:}5$



(d) Nominal short rate, R_t; $f(\ _t) = 0{:}5$



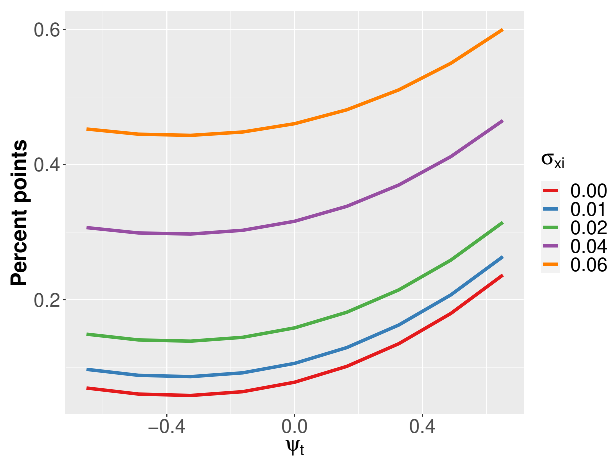
(e) Disagreement adjustment, $x_t^A; x_t^B;\ _t$; $f(\ _t) = 0{:}9$



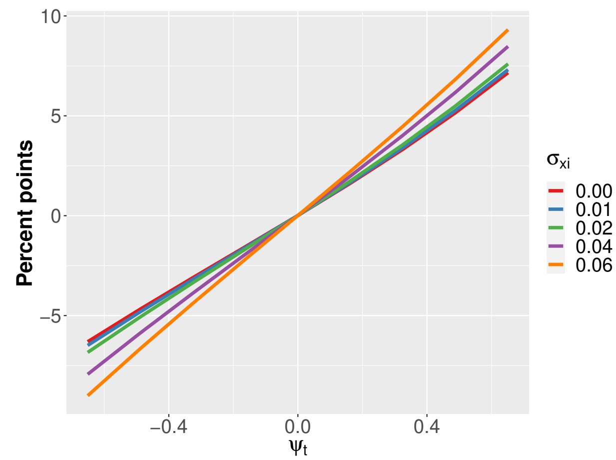
(f) Nominal short rate, R_t; $f(\ _t) = 0{:}9$

Figure 1.5: Impact of disagreement on $x_t^A; x_t^B;\ _t$ and nominal short rate across multiple sharing rules

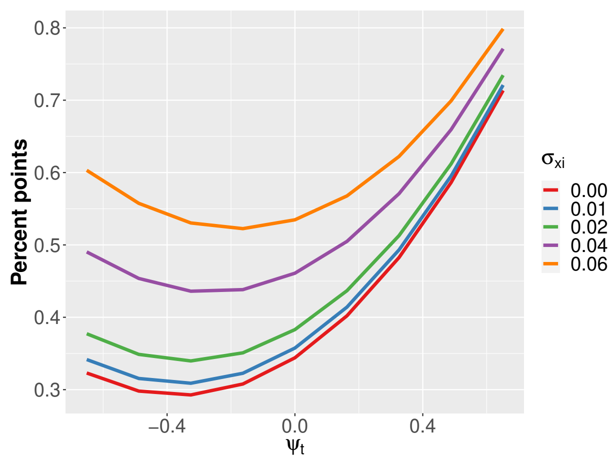
Note: This gure plots the disagreement adjustment, $x_t^A; x_t^B;\ _t$, and nominal short rate, R_t, across multiple beliefs spreads, $_t$, and sharing rules, $f(\ _t)$, in the disagreement economy described in section (1.3). The parameters come from table (1.2). x_t^A is xed at $0{:}9\overline{x}$. The grid of $_t$ is within two standard deviations below and above its unconditional mean, zero. See Eq. (1.3.7).



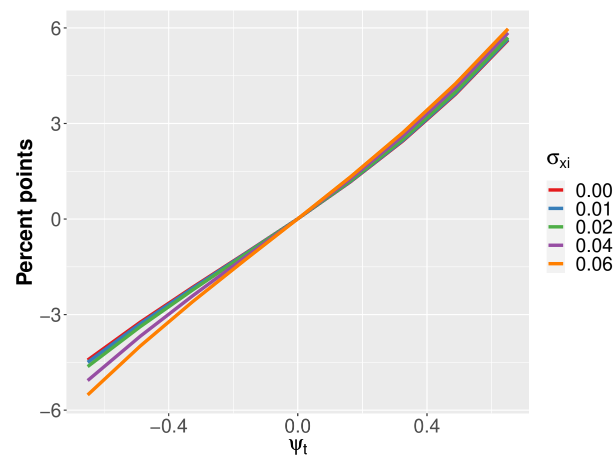
(a) Instant. volatility of R_t; $f(\psi_t) = 0.1$



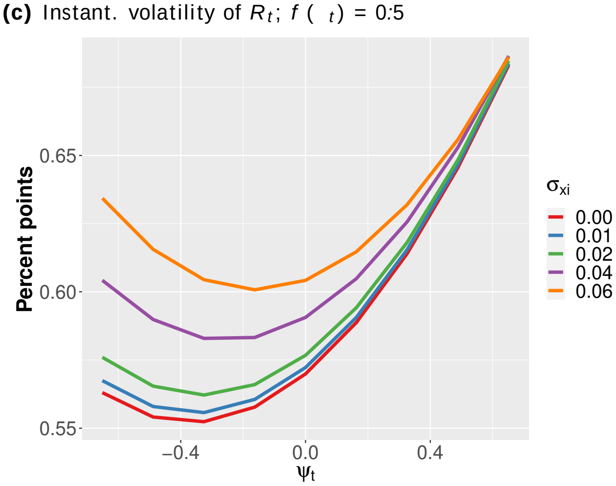
(b) Instant. inflation risk premium, IRP_t; $f(\psi_t) = 0.1$



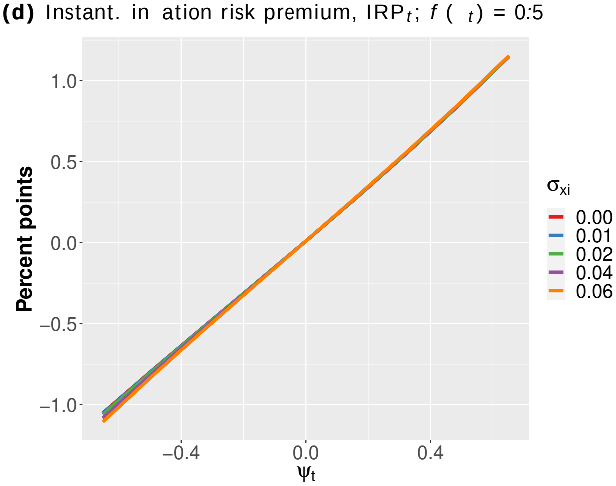
(c) Instant. volatility of R_t; $f(\psi_t) = 0.5$



(d) Instant. inflation risk premium, IRP_t; $f(\psi_t) = 0.5$



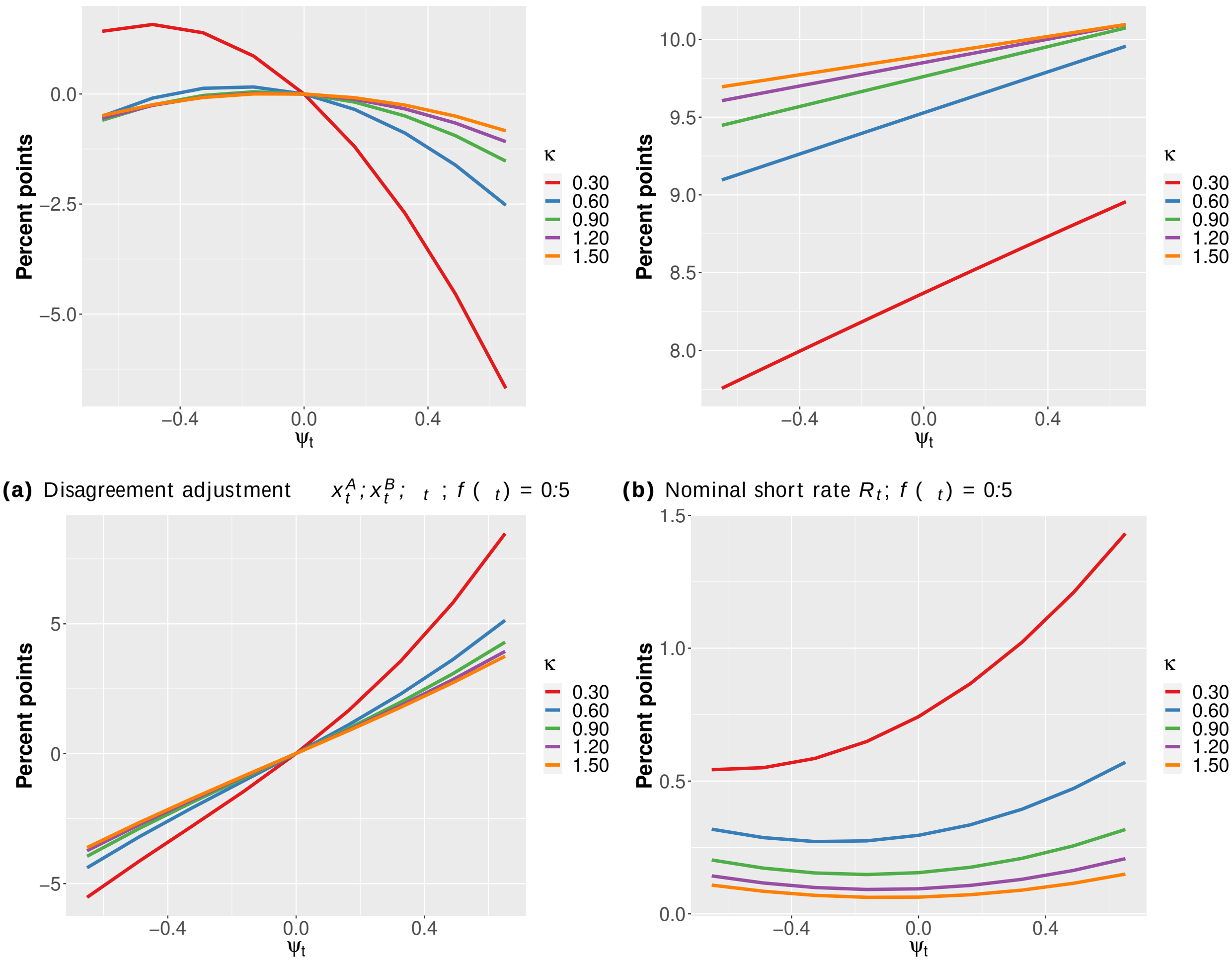
(e) Instant. volatility of R_t; $f(\psi_t) = 0.9$



(f) Instant. inflation risk premium, IRP_t; $f(\psi_t) = 0.9$

Figure 1.6: Impact of disagreement on instant. volatil. and inflat. risk premium across multiple sharing rules

Note: This figure plots the instantaneous volatility of the nominal short rate and inflation risk premium across multiple beliefs spreads, ψ_t, and sharing rules, $f(\psi_t)$, in the disagreement economy described in section (1.3). The parameters come from table (1.2). x_t^A is fixed at $0.9\overline{x}$. The grid of ψ_t is within two standard deviations below and above its unconditional mean, zero. See Eq. (1.3.7).



(a) Disagreement adjustment $x_t^A; x_t^B;\ _t$; $f(\ _t) = 0{:}5$

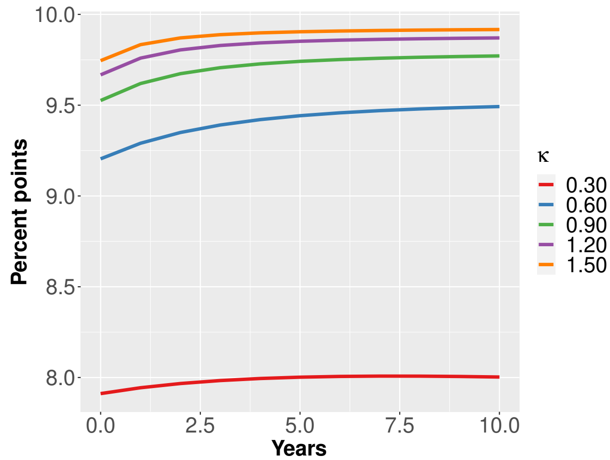
(b) Nominal short rate R_t; $f(\ _t) = 0{:}5$

(c) Instant. in ation risk premium; $f(\ _t) = 0{:}5$

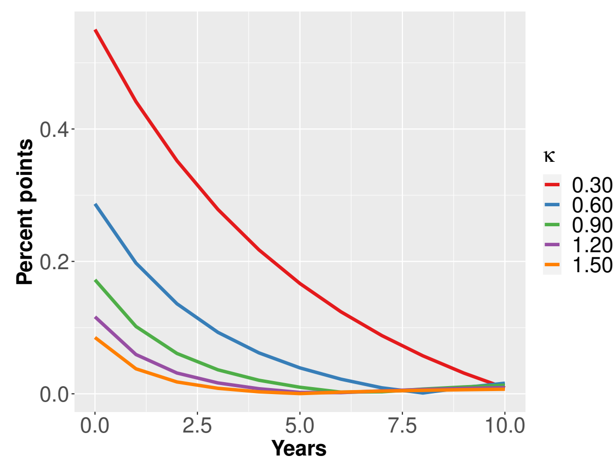
(d) Instant. volatility of R_t; $f(\ _t) = 0{:}5$

Figure 1.7: Impact of disagreement on $x_t^A; x_t^B;\ _t$ and in ation risk premium across multiple

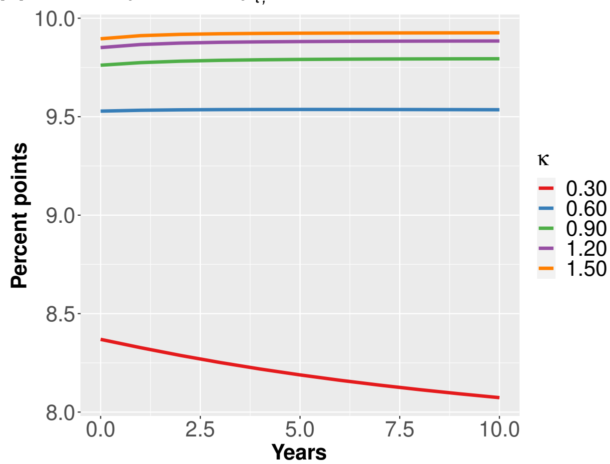
Note: This gure plots the disagreement adjustment, $x_t^A; x_t^B;\ _t$, and nominal short rate, R_t, across multiple beliefs spreads, $_t$, and sharing rules, $f(\ _t)$, in the disagreement economy described in section (1.3). The parameters come from table (1.2). x_t^A is xed at $0{:}9\overline{x}$. The grid of $_t$ is within two standard deviations below and above its unconditional mean, zero. See Eq. (1.3.7).



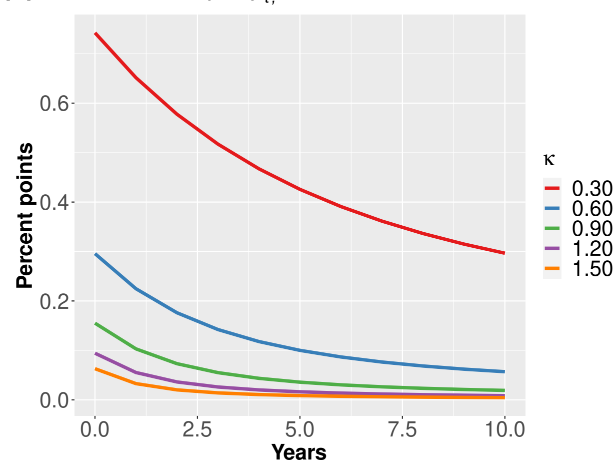
(a) Nominal yield curve $y^{\$}_{t;}$; low $_t$



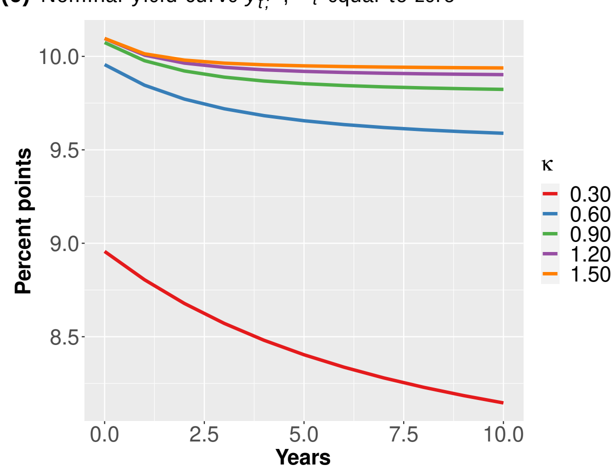
(b) Instant. volatility of $y^{\$}_{t;}$; low $_t$



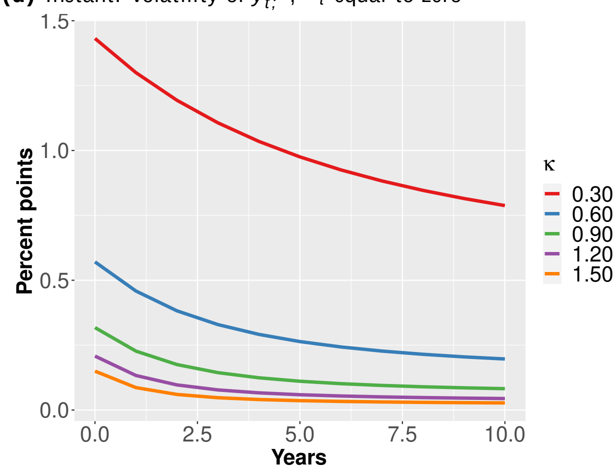
(c) Nominal yield curve $y^{\$}_{t;}$; $_t$ equal to zero



(d) Instant. volatility of $y^{\$}_{t;}$; $_t$ equal to zero



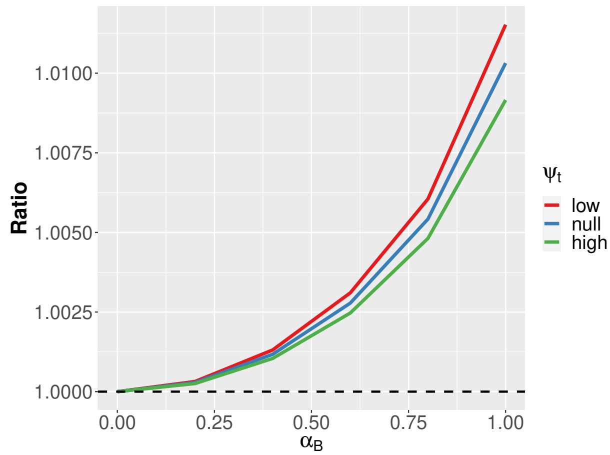
(e) Nominal yield curve $y^{\$}_{t;}$; high $_t$



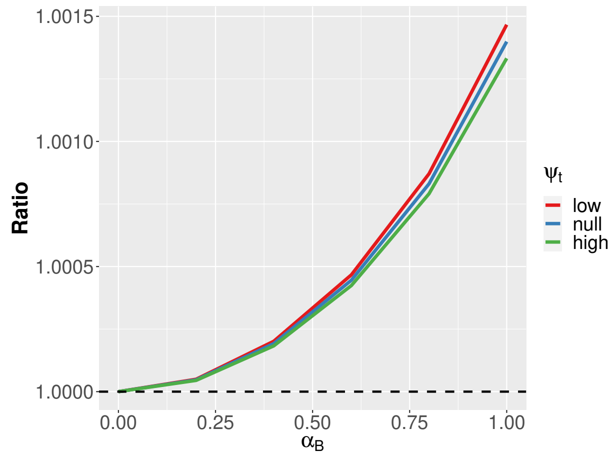
(f) Instant. volatility of $y^{\$}_{t;}$; high $_t$

Figure 1.8: Impact of disagreement on level and volatility of nominal yield curve across multiple

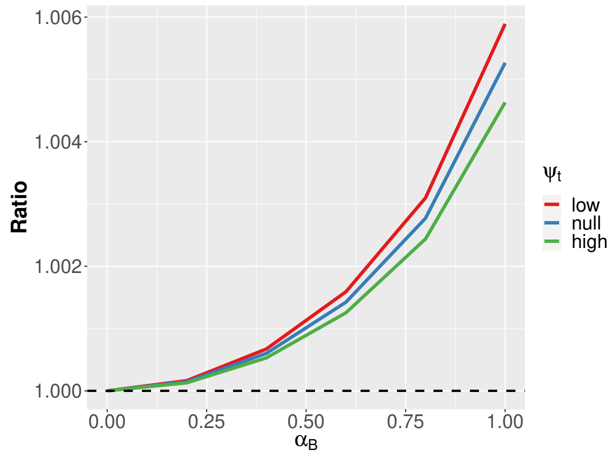
Note: This gure plots the level and instantaneous volatility of the nominal yield curve across multiple beliefs spreads, $_t$, and sharing rules, f ($_t$), in the disagreement economy described in section (1.3). The parameters come from table (1.2). x^A_t is xed at $0{:}9\overline{x}$. Low and high $_t$ correspond to, respectively, two standard deviations below and above its unconditional mean, zero. See Eq. (1.3.7).



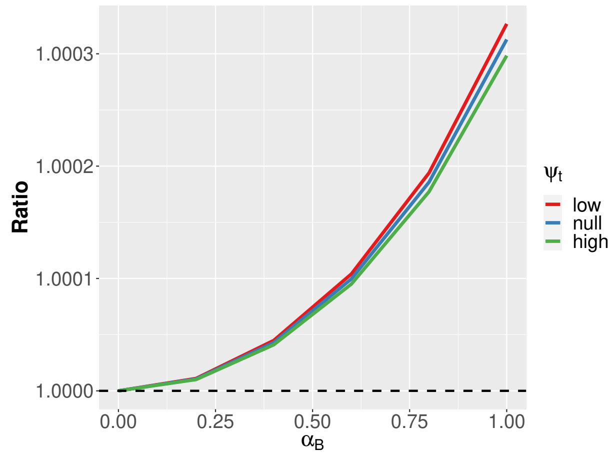
(a) Nominal short rate R_t; = 0:3



(b) Nominal short rate R_t; = 0:7



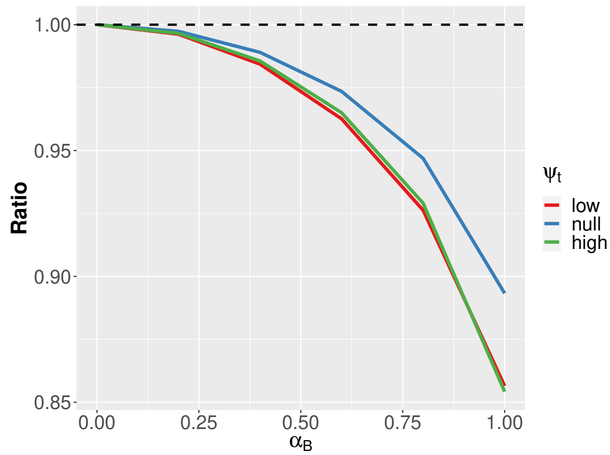
(c) 10-year nominal rate $y^{\$}_{t;t+10}$; = 0:3



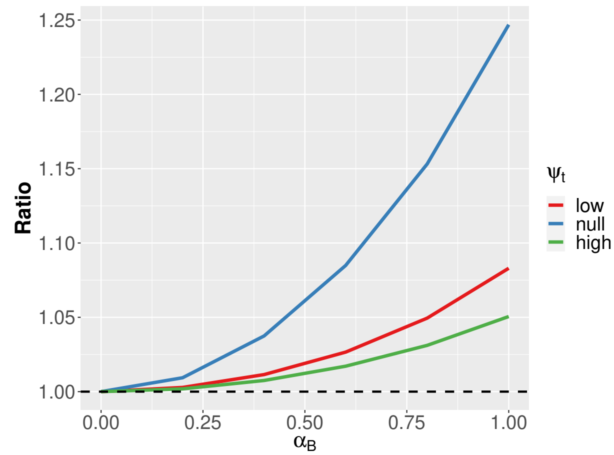
(d) 10-year nominal rate $y^{\$}_{t;t+10}$; = 0:7

Figure 1.9: Ratio of nominal short and 10-year rates in mixed to doubtful economies as $_B$ varies

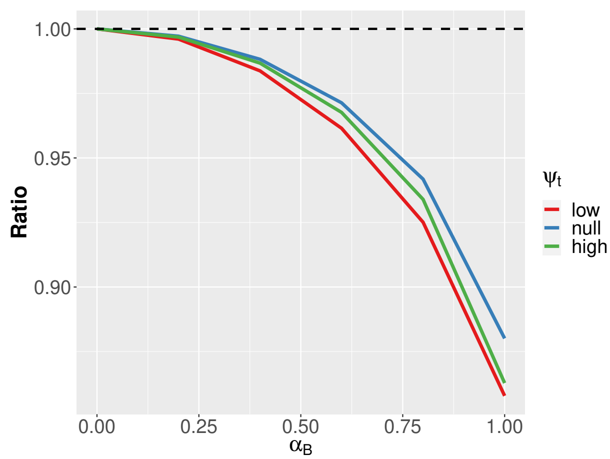
Note: This gure plots the ratio of selected variables in two economies: a mixed economy where $_B$ 2 (0; 1); $_B$ = 0, and a doubtful economy where $_A$ = $_B$ = 0. A ratio is calculated for every variable across three values of $_t$: low, null, and high. Low and high correspond to two standard deviations below and above zero, respectively. The ratios are calculated for a grid of $_B$ between zero and one. The selected variables are the nominal short rate and 10-year rate. The equilibrium corresponds to disagreement economy described in section (1.3). The parameters come from table (1.2).



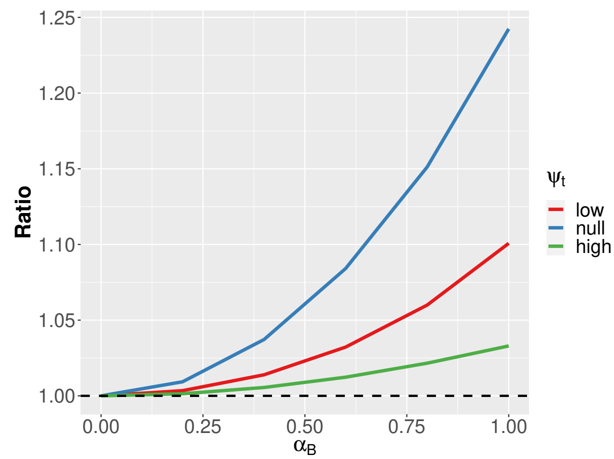
(a) Instant. volatility of R_t; = 0:3



(b) Instant. volatility of R_t; = 0:7



(c) Instant. volatility of 10-year nom. rate $y^{\$}_{t;t+10}$; = 0:3



(d) Instant. volatility of 10-year nom. rate $y^{\$}_{t;t+10}$; = 0:7

Figure 1.10: Ratio of volatility of nominal short and 10-year rates in mixed to doubtful economies as $_B$ varies

Note: This gure plots the ratio of selected variables in two economies: a mixed economy where $_B$ 2 (0; 1) ; $_B$ = 0, and a doubtful economy where $_A$ = $_B$ = 0. A ratio is calculated for every variable across three values of $_t$: low, null, and high. Low and high correspond to two standard deviations below and above zero, respectively. The ratios are calculated for a grid of $_B$ between zero and one. The selected variables are the volatilities of the nominal short rate and 10-year rate. The equilibrium corresponds to disagreement economy described in section (1.3). The parameters come from table (1.2).

1.6.2 Proofs and auxiliary results

Proof of Proposition 1.2.1

Price level: The dynamics of investor i's estimate of expected growth rate of money, x_t^i, follow a standard Ornstein–Uhlenbeck process. Thus, it has an explicit solution for $u \geq t$ of the form

$$x_u^i = e^{-\kappa(u-t)} x_t^i + \overline{x}\left(1 - e^{-\kappa(u-t)}\right) + \hat{\sigma}_m^i \int_t^u e^{-\kappa(u-v)} dz_{m,v}^i + \hat{\sigma}_s^i \int_t^u e^{-\kappa(u-v)} ds_v.$$

The conditional moments of x_u^i are given by

$$\mathrm{E}_t\left[x_u^i\right] = e^{-\kappa(u-t)} x_t^i + \overline{x}\left(1 - e^{-\kappa(u-t)}\right); \tag{1.6.1}$$

$$\mathrm{var}_t\left[x_u^i\right] = \frac{1}{2\kappa}\left[\left(\hat{\sigma}_m^i\right)^2 + \left(\hat{\sigma}_s^i\right)^2\right]\left(1 - e^{-2\kappa(u-t)}\right) = \frac{\left(\hat{\sigma}^i\right)^2}{2\kappa}\left(1 - e^{-2\kappa(u-t)}\right). \tag{1.6.2}$$

Furthermore, the dynamics of M_t have an explicit solution for $u \geq t$ given by

$$\ln(M_u) = \ln(M_t) + \int_t^u x_v dv - \frac{\sigma_m^2}{2}(u-t) + \sigma_m\left(z_{m,u} - z_{m,t}\right).$$

The conditional moments of $\ln(M_u)$ are computed using the Fubini's theorem for stochastic integrals (see Munk (2013)), and Eqs. (1.6.1) and (1.6.2). Thus

$$\mathrm{E}_t\left[\ln(M_u)\right] = \ln(M_t) + \frac{1}{\kappa}\left(1 - e^{-\kappa(u-t)}\right)\left(x_t - \overline{x}\right) + \left(\overline{x} - \frac{\sigma_m^2}{2}\right)(u-t); \tag{1.6.3}$$

$$\mathrm{var}_t\left[\ln(M_u)\right] = \frac{\left(\hat{\sigma}^i\right)^2}{4\kappa^2}\left(e^{-2\kappa(u-t)} - 1\right) + \left(\frac{\left(\hat{\sigma}^i\right)^2}{\kappa^2} + \sigma_m^2\right)(u-t). \tag{1.6.4}$$

Let $M\left(x_t^i; u-t\right) \equiv \mathrm{E}_t\left[e^{-\rho(u-t)}\left(\frac{M_u}{M_t}\right)^{-1}\right] = \exp\left\{-\rho(u-t) - \mathrm{E}_t\left[\ln\frac{M_u}{M_t}\right] + \frac{1}{2}\mathrm{var}_t\left[\ln\frac{M_u}{M_t}\right]\right\}$ where the last equality follows from log-normality. Inserting the conditional moments (1.6.3) and (1.6.4) into this expression, we obtain $M\left(x_t^i; u-t\right) = \exp\left(-A(u-t) - B(u-t)x_t^i\right)$ where $A(u-t)$ and $B(u-t)$ are given by Eqs. (1.2.18) and (1.2.17), respectively. Inserting $M\left(x_t^i; u-t\right)$ into Eq. (1.2.13), we conclude the proof of the first part of Proposition (1.2.1).

Nominal short rate The second part of the proposition follows from inserting the optimality condition (1.2.10) into the equilibrium price level in Eq. (1.2.13) and the definition of real money holdings, $m_t \equiv \frac{M_t}{P_t}$. Thus, we obtain that R_t moves is inversely proportional to $Q\left(x_t^i\right)$. That R_t is non-negative follows trivially from the definition of $Q\left(x_t^i\right)$.

Proof of Proposition 1.2.2

Dynamics of the price level: For convenience, let $q_t \quad \frac{1}{P_t}$ such that Eq. (1.2.13) becomes

$$q_t \;=\; \frac{1 \quad '}{\quad '} \frac{C_t}{M_t} Q \;\; x_t^i \;\; ;$$

or $q_t \quad q \;\; C_t; M_t; x_t^i$. Before computing the dynamics of q_t, note that

$$\frac{@M \;\; x_{t}^{i}; u \quad t}{@x_t^i} = \quad B \;\; (u \quad t)\, M \;\; x_t^i; u \quad t \;\; < 0; \;\; \frac{@^B M \;\; x_t^i; u \quad t}{@x_t^B} = B \;\; (u \quad t)^2\, M \;\; x_t^i; u \quad t \;\; > 0.$$

Moreover

$$\frac{@Q \;\; x_t^i}{@x_t^i} \;=\; \quad G \;\; x_t^i \;\; < 0;$$

where $G \;\; x_t^i \qquad \int_t^{1} B \;\; (u \quad t)\, M \;\; x_t^i; u \quad t \;\; du$. Therefore

$$\frac{@q_t}{@x_t^i} \;=\; \frac{1 \quad '}{\quad '} \;\; \frac{C_t}{M_t} \frac{@Q \;\; x_t^i}{@x_t^i} \;=\; \frac{G \;\; x_t^i}{Q \;\; x_t^i} q_t;$$

$$\frac{@^B q_t}{@ \;\; x_t^i \;\; ^2} \;=\; \frac{1 \quad '}{\quad '} \;\; \frac{C_t}{M_t} \frac{@Q \;\; x_t^i \;\; ^2}{@ \;\; x_t^i \;\; ^2} = \frac{H \;\; x_t^i}{Q \;\; x_t^i} q_t;$$

where $H \;\; x_t^i \;\; = \int_t^{1} B \;\; (u \quad t)^2\, M \;\; x_t^i; u \quad t \;\; du$.

Applying Ito's lemma to q_t and combining the resulting expression with the derivatives from above

$$\begin{aligned} \frac{dq_t}{q_t} &= \frac{dC_t}{C_t} \quad \frac{dM_t}{M_t} + \frac{@q_t}{@x_t^i} \;\; dx_t^i \;\; + \;\; \frac{dM_t}{M_t} \;\; ^2 + \frac{1}{2} \frac{@^B q_t}{@ \;\; x_t^i \;\; ^2} \;\; dx_t^i \;\; ^2 \\ &\quad \frac{dC_t}{C_t} \quad \frac{dM_t}{M_t} \;\; + \frac{@^B q_t}{@C_t @x_t^i} (dC_t) \;\; dx_t^i \;\; + \frac{@^B q_t}{@M_t @x_t^i} (dM_t) \;\; dx_t^i \\ &= \;\; {}^q_t dt + \;\; {}^q_c dz_{c;t} + \;\; {}^q_{m;t} dz^i_{m;t} + \;\; {}^q_{s;t} ds_t; \end{aligned}$$

where ${}^q_t \qquad {}_c + \;\; {}^2_m \quad x_t^i + \frac{G(x_t^i)}{Q(x_t^i)} \;\; \hat{}^i_x \qquad \overline{x} \quad x_t^i \quad + \frac{(\hat{}^i)^2}{2} \frac{H(x_t^i)}{Q(x_t^i)}$, ${}^q_c \qquad {}_c$, ${}^q_{m;t} \qquad {}_m \quad \hat{}^i_m \frac{G(x_t^i)}{Q(x_t^i)}$, and ${}^q_{s;t} \qquad \hat{}^i_s \frac{G(x_t^i)}{Q(x_t^i)}$.

Applying Ito's lemma to $P_t = \frac{1}{q_t}$ and using the dynamics for q_t computed above

$$\begin{aligned} \frac{dP_t}{P_t} &= \quad \frac{dq_t}{q_t} + \quad \frac{dq_t}{q_t} \;\; ^2 \\ &= \quad {}_t dt + \;\; {}^p_c dz_{c;t} + \;\; {}^p_{m;t} dz^i_{m;t} + \;\; {}^p_{s;t} ds_t; \end{aligned}$$

where

$${}_t \qquad {}^2_c \qquad {}_c + x_t^i + \frac{G \;\; x_t^i}{Q \;\; x_t^i} \qquad \overline{x} \quad x_t^i \quad + \hat{}^i_x \qquad \frac{\hat{}^i \;\; ^2}{2} \frac{H \;\; x_t^i}{Q \;\; x_t^i} + \;\; \hat{}^i \;\; ^2 \;\; \frac{G \;\; x_t^i}{Q \;\; x_t^i} \;\; !^2 ; \tag{1.6.5}$$

$${}^p_c \qquad {}_c; \tag{1.6.6}$$

$${}^p_{m;t} \qquad {}_m + \hat{}^i_m \frac{G \;\; x_t^i}{Q \;\; x_t^i}; \tag{1.6.7}$$

$${}^p_{s;t} \qquad \hat{}^i_s \frac{G \;\; x_t^i}{Q \;\; x_t^i}. \tag{1.6.8}$$

Dynamics of the short nominal rate: Applying Ito's lemma to Eq. (1.2.15) and using the results from above when computing the dynamics of q_t

$$\begin{aligned} dR_t &= R_t \left(\frac{\partial Q\left(x_t^i\right)}{\partial x_t^i} dx_t^i + \frac{1}{2} \frac{\partial^B Q\left(x_t^i\right)}{\partial \left(x_t^i\right)^2} \left(dx_t^i\right)^2 \right) + R_t \left(\frac{\partial Q\left(x_t^i\right)}{\partial x_t^i} dx_t^i \right)^2 \\ &= {}^R_t dt + {}^R_{m;t} dz^i_{m;t} + {}^R_{s;t} ds_t; \end{aligned}$$

where

$$^R_t \quad R_t \left[{}^2_m + R_t \quad x_t^i + \hat{}^i_x \frac{G\left(x_t^i\right)}{Q\left(x_t^i\right)} + \left(\hat{}^i_m\right)^2 \left(\frac{G\left(x_t^i\right)}{Q\left(x_t^i\right)}\right)^2 \right]; \tag{1.6.9}$$

$$^R_{m;t} \quad \hat{}^i_m \frac{G\left(x_t^i\right)}{Q\left(x_t^i\right)} R_t; \tag{1.6.10}$$

$$^R_{s;t} \quad \hat{}^i_s \frac{G\left(x_t^i\right)}{Q\left(x_t^i\right)} R_t. \tag{1.6.11}$$

Dynamics of the state-price densities: We first compute the dynamics of the real state-price density. By definition

$$\begin{aligned} {}_t &= e^{\quad t} u_c\left(C_t; m_t\right) \\ &= {}' e^{\quad t} C_t^{\quad 1}. \end{aligned}$$

The dynamics are given by

$$\begin{aligned} \frac{d_{\ t}}{_t} &= \quad dt \quad \frac{dC_t}{C_t} + \quad \left(\frac{dC_t}{C_t}\right)^2 \\ &= \quad r_t dt \quad {}_{c;t} dz_{c;t}; \end{aligned}$$

where

$$r_t = \quad + \quad {}_c \quad {}^2_c; \tag{1.6.12}$$

is the instantaneous real rate and ${}_{c;t} \quad {}_c$ is the real price of risk which only depends on real variables.

The dynamics of the nominal state-price density are computed from

$$^\$_t = {}_t q_t.$$

By Ito's lemma

$$\begin{aligned} \frac{d^\$_t}{^\$_t} &= \frac{d_{\ t}}{_t} + \frac{dq_t}{q_t} + \frac{d_{\ t}}{_t} \frac{dq_t}{q_t} \\ &= R_t dt \quad {}^\$_{m;t} dz^i_{m;t} \quad {}^\$_{s;t} ds_t; \end{aligned}$$

where the nominal short rate is given by

$$R_t \quad = \quad r_t \qquad {}_{q;t} + \quad {}_{c;t} \, {}^{q}_{c;t}; \tag{1.6.13}$$

and the vector of nominal prices of risk ${}^{\$}_{t} \quad \left[{}^{\$}_{m;t}; \; {}^{\$}_{s;t} \right]^{0}$ is

$$ {}^{\$}_{m;t} \quad = \quad {}_{m} + \hat{}^{i}_{m} \frac{G \quad x^i_t}{Q \quad x^i_t}; \tag{1.6.14}$$

$$ {}^{\$}_{s;t} \quad = \quad \hat{}^{i}_{s} \frac{G \quad x^i_t}{Q \quad x^i_t}: \tag{1.6.15}$$

Inserting the explicit solutions we found above for r_t, ${}_{q;t}$, ${}_{c;t}$, and ${}^{q}_{c;t}$ into R_t, we obtain an alternative expression for R_t

$$R_t \quad = \qquad {}^{2}_{m} + x^i_t + \frac{G \quad x^i_t}{Q \quad x^i_t} \qquad \overline{x} \quad x^i_t \qquad \hat{}^{i}_{x} \qquad \frac{\hat{}^{i}\,{}^{2}}{2} \frac{H \quad x^i_t}{Q \quad x^i_t}: \tag{1.6.16}$$

Proof of Proposition 1.2.3

In ation risk premium: After noting that ${}^{q}_{c;t}{}^{2} + {}^{q}_{m;t}{}^{2} + {}^{q}_{s;t}{}^{2} = \mathrm{var}_t \frac{\mathrm{d}q_t}{q_t} = \mathrm{var}_t \frac{\mathrm{d}P_t}{P_t}$, we can re-arrange Eq. (1.6.5) as follows

$$ {}^{q}_{t} = \quad {}_{t} + \mathrm{var}_t \quad \frac{\mathrm{d}P_t}{P_t} \quad :$$

By inserting this expression into Eq. (1.6.13) and re-arranging terms, we obtain

$$R_t \qquad {}_{t} + \mathrm{var}_t \quad \frac{\mathrm{d}P_t}{P_t} \qquad r_t \quad = \quad {}_{c;t} \, {}^{q}_{c;t}:$$

The di erence between the model implied real rate $R_t \qquad {}_{t} + \mathrm{var}_t \; \frac{\mathrm{d}P_t}{P_t}$ and r_t is the in ation risk premium. By letting IRP_t denote the instantaneous in ation risk premium, it follows that $\mathrm{IRP}_t = \; {}_{c;t} \, {}^{q}_{c;t} = \; {}^{2}_{c}$.

Real yield curve: The price of a real bond in t with maturity in can be computed from the expression $B_{t;} \; = \mathrm{E}_t \left[\frac{}{{}_{t}} \right] = \mathrm{E}_t \; e \;\; (\quad t) \; \frac{C}{C_t}$. From log-normality and Eq. (1.2.1)

$$B_{t;} \quad = \quad e^{\; (\; + \; c \quad {}^{2}_{c})(\quad t)}:$$

By letting the real yield curve between t and be equal to $y_{t;} \quad = \quad \frac{\log(B_{t;}\;)}{t}$ the result obtains.

Level of the nominal yield curve: The price of a nominal bond in t with maturity τ is given by $B^{\$}_{t,\tau} = E_t\left[\frac{\pi^{\$}_{t+\tau}}{\pi^{\$}_t}\right] = E_t\left[\frac{\pi_{t+\tau}}{\pi_t}\frac{P_t}{P_{t+\tau}}\right]$. We can manipulate the expression inside the conditional expectation using the price level from Eq. (1.2.13) and the real state-price density into this expression as follows

$$\frac{\pi^{\$}_{t+\tau}}{\pi^{\$}_t} = \frac{E_{t+\tau}\left[\int_{t+\tau}^{\infty} e^{-\rho u} M_u^{-1} du\right]}{E_t\left[\int_t^{\infty} e^{-\rho u} M_u^{-1} du\right]} = \frac{\int_{t+\tau}^{\infty} E_{t+\tau}\left[e^{-\rho(u-t)}\left(\frac{M_u}{M_t}\right)^{-1}\right] du}{Q\left(x^i_t\right)}.$$

Inserting the expression above into the price of a nominal bond with maturity $\tau - t$ at time t

$$\begin{aligned} B^{\$}_{t,\tau} &= E_t\left[\frac{\int_{t+\tau}^{\infty} E_{t+\tau}\left[e^{-\rho(u-t)}\left(\frac{M_u}{M_t}\right)^{-1}\right] du}{Q(x_t)}\right] \\ &= \frac{1}{Q(x_t)} \int_{t+\tau}^{\infty} E_t\left[e^{-\rho(u-t)}\left(\frac{M_u}{M_t}\right)^{-1}\right] du, \end{aligned}$$

where the last equality follows from the tower property of conditional expectations. Thus, the price of the nominal bond $B^{\$}_{t,\tau}$ has an explicit solution given by

$$B^{\$}_{t,\tau} = \frac{\int_{\tau}^{\infty} \mathcal{M}\left(x^i_{t;u-t}\right) du}{\int_t^{\infty} \mathcal{M}\left(x^i_{t;u-t}\right) du}.$$

The desired result obtains from the definition of the yield curve $y^{\$}_{t,\tau} = -\frac{\log\left(B^{\$}_{t,\tau}\right)}{\tau}$ and by letting $Q\left(x^i_{t,\tau}\right)$ be given by $Q\left(x^i_{t,\tau}\right) \equiv \int_{\tau}^{\infty} \mathcal{M}\left(x^i_{t;u-t}\right) du$ (with some abuse of notation).

Dynamics of nominal yield curve: From an application of Ito's lemma to $y^{\$}_{t,\tau}$ and by letting $G\left(x^i_{t,\tau}\right) \equiv \int_{\tau}^{\infty} \mathcal{B}(u-\tau)\, \mathcal{M}\left(x^i_{t;u-t}\right) du$, we obtain the diffusion terms. We omit the exact expression for the drift of the yield curve for the sake of briefness.

Proof of Proposition 1.3.1

Dynamics of stochastic weight: The Radon-Nikodym derivative of the model is the density function $\xi_u = \frac{dP^B}{dP^A}$. By Girsanov's theorem, ξ_u is given by the stochastic exponential

$$\xi_u = e^{w_u},$$

where $w_u \equiv \int_t^s \kappa_u dz^A_{m,u} - \frac{1}{2}\int_t^s \kappa_u^2 du$ is a standard Ito's process adapted to $z^A_{m,s}$. Equivalently

$$dw_t = -\frac{1}{2}\kappa_t^2 dt + \kappa_t dz^A_{m,t}.$$

Applying Ito's lemma to ξ_u and inserting dw_t into the resulting equation

$$\begin{aligned} d\xi_t &= \frac{\partial \xi_t}{\partial w_t}(dw_t) + \frac{1}{2}\frac{\partial^2 \xi_t}{\partial w_t^2}(dw_t)^2 \\ &= \kappa_t \xi_t dz^A_{m,t}. \end{aligned}$$

Equilibrium allocations: Inserting the optimality condition for investor i in Eq. (1.3.2) into the feasibility constraint for total consumption, along with the stochastic weights $_t$ that we identi ed in Eq. (1.3.4) and re-arranging terms

$$
\begin{aligned}
C_t &= C_{A;t} + C_{B;t} \\
=) \;\; C_t &= \;\; ' \;\; ^{1} e^{\;\, t} \;\; _{A;t} y_1 \quad ^{1} (1 + \;\; _t) :
\end{aligned}
$$

Using again Eq. (1.3.2) and re-arranging terms

$$
C_{A;t} = f(\;\, _t) C_t ;
$$

where $f(\;\, _t) \quad \frac{1}{1+ \;\, _t}$. Thus, we obtain Eqs. (1.3.8) and (1.3.9). Following a similar procedure for $m_t = \frac{M_t}{P_t}$ from the feasibility constraint (in real terms) $\frac{M_t}{P_t} = \frac{M_{A;t}}{P_t} + \frac{M_{B;t}}{P_t}$ and the optimality condition (1.3.3), we arrive at Eqs. (1.3.10) and (1.3.11). Finally, inserting the equilibrium consumption allocations into the individual optimality condition (1.3.2), we obtain the equilibrium real state-price densities

$$
_{A;t} = \; ' y_1^{\;1} e^{\quad t} \; ' f(\;\, _t) \;\; ^{1} C_t \;\; ^{1} :
$$

We set $y_1 = f(\;\, _0) \;\; ^{1} C_0 \;\; ^{1}$ w.l.o.g. for $_{A;0} = \; '$ and $_{B;0} = \; '$. Thus, the previous expression becomes

$$
\frac{_{A;t}}{_{A;0}} = e^{\quad t \; '} \quad \frac{f(\;\, _t)}{f(\;\, _0)} \quad ^{1} \quad \frac{C_t}{C_0} \quad ^{1} :
$$

Furthermore, from the de nition of the stochastic weight $_t$

$$
y_2 = \frac{f(\;\, _0) \;\; ^{1} C_0 \;\; ^{1}}{_0} :
$$

Inserting into the state-price density for investor B

$$
\frac{_{B;t}}{_{B;0}} = e^{\quad t} \quad \frac{1 \quad f(\;\, _t)}{1 \quad f(\;\, _0)} \quad ^{1} \quad \frac{C_t}{C_0} \quad ^{1} :
$$

Proof of Proposition 1.3.2

Price level: We start from the equation that determines the price level under the beliefs of investor A

$$
\frac{1}{P_t} = \frac{1 \quad '}{\quad '} \mathrm{E}_t^A \int_t^{1} e^{\quad (u \quad t)} \frac{C_{A;t}}{M_{1;u}} du \;\; :
$$

Inserting the sharing rules we derived in Proposition (1.3.1) into the previous expression

$$
\frac{1}{P_t} = \frac{1 \quad '}{\quad '} \mathrm{E}_t^A \int_t^{1} e^{\quad (u \quad t)} \frac{f(\;\, _t) C_t}{f(\;\, _u) M_u} du \;\; :
$$

Re-arranging terms and by Fubini's theorem

$$\frac{1}{P_t} = \frac{1-\varphi}{\varphi}\frac{f(\lambda_t)C_t}{M_t}\int_t^{\infty} e^{-\rho(u-t)}\left(\mathbb{E}_t^A\left[\left(\frac{M_u}{M_t}\right)^{-1}\right] + \mathbb{E}_t^A\left[\lambda_u\left(\frac{M_u}{M_t}\right)^{-1}\right]\right)\mathrm{d}u.$$

Changing the probability measure of the last term in the equation above to that of investor B by multiplying and dividing by λ_t, and recalling that $\lambda_u = \frac{y_1}{y_2}\frac{\mathrm{dP}^B}{\mathrm{dP}^A}$ and $\lambda_t = \frac{y_1}{y_2}$ for $u > t$, it follows that

$$\begin{aligned}\frac{1}{P_t} &= \frac{1-\varphi}{\varphi}\frac{f(\lambda_t)C_t}{M_t}\int_t^{\infty} e^{-\rho(u-t)}\left(\mathbb{E}_t^A\left[\left(\frac{M_u}{M_t}\right)^{-1}\right] + \lambda_t\mathbb{E}_t^A\left[\frac{\lambda_u}{\lambda_t}\left(\frac{M_u}{M_t}\right)^{-1}\right]\right)\mathrm{d}u \\ &= \frac{1-\varphi}{\varphi}\frac{C_t}{M_t}\int_t^{\infty}\left(\frac{M\left(x_t^A; u-t; \hat\sigma^A; \hat\sigma_x^A\right) + \lambda_t M\left(x_t^B; u-t; \hat\sigma^B; \hat\sigma_x^B\right)}{1+\lambda_t}\right)\mathrm{d}u;\end{aligned}$$

where $M\left(x_t^i; u-t; \hat\sigma^i; \hat\sigma_x^i\right) \equiv \mathbb{E}_t^i\left[e^{-\rho(u-t)}\left(\frac{M_u}{M_t}\right)^{-1}\right]$, for $i \in \{A; B\}$. Following the same steps from the proof Proposition (1.2.2), we can show that

$$M\left(x_t^i; u-t; \hat\sigma^i; \hat\sigma_x^i\right) = \exp\left(A\left(u-t; \hat\sigma^i; \hat\sigma_x^i\right) - B(u-t)\,x_t^i\right); \tag{1.6.17}$$

where $A\left(u-t; \hat\sigma^i; \hat\sigma_x^i\right)$ and $B(u-t)$ are defined as in Eqs. (1.2.18) and (1.2.17) from Proposition (1.2.1), respectively for $i \in \{A; B\}$. Observe that, after introducing disagreement, we make the dependence of $A\left(u-t; \hat\sigma^i; \hat\sigma_x^i\right)$ and $M\left(x_t^i; u-t; \hat\sigma^i; \hat\sigma_x^i\right)$ on $\hat\sigma^i$ and $\hat\sigma_x^i$ explicit to emphasize that investors A and B have different confidence regarding their estimates of x_t.

Nominal short rate: The proof of the nominal short rate is the same as the one in Proposition 1.2.1.

Proof of Proposition 1.3.3

Dynamics of the price level: Let $q_t \equiv \frac{1}{P_t}$ and $Q_t \equiv Q\left(x_t^A; x_t^B; \lambda_t\right)$ for convenience. Thus

$$q_t = q\left(x_t^A; x_t^B; \lambda_t\right) = \frac{\varphi}{1-\varphi}\frac{C_t}{M_t}Q\left(x_t^A; x_t^B; \lambda_t\right).$$

Before calculating the dynamics of q_t, we calculate the first and second derivatives of q_t with respect to its arguments. To this end, observe that

$$\frac{\partial Q\left(x_t^A; x_t^B; \lambda_t\right)}{\partial x_t^A} = -\frac{1}{1+\lambda_t}G\left(x_t^A\right); \quad \frac{\partial Q\left(x_t^A; x_t^B; \lambda_t\right)}{\partial x_t^B} = -\frac{\lambda_t}{1+\lambda_t}G\left(x_t^B\right);$$

where $G\left(x_t^i\right) \equiv \int_t^{\infty} B(u-t)\,M\left(x_t^i; u-t; \hat\sigma^i; \hat\sigma_x^i\right)\mathrm{d}u$. Additionally

$$\frac{\partial^2 Q\left(x_t^A; x_t^B; \lambda_t\right)}{\partial\left(x_t^A\right)^2} = \frac{1}{1+\lambda_t}H\left(x_t^A\right); \quad \frac{\partial^2 Q\left(x_t^A; x_t^B; \lambda_t\right)}{\partial\left(x_t^B\right)^2} = \frac{\lambda_t}{1+\lambda_t}H\left(x_t^B\right);$$

where $H\left(x_t^i\right) \quad \int_t^{1} B\ (u \quad t)^2\, M\left(x_t^i; u \quad t; \hat{\ }^i; \hat{\ }_x^i\right) \mathrm{d}u$. Thus

$$
\begin{aligned}
\frac{\partial q_t}{\partial x_t^A} &= \frac{f_t G\left(x_t^A\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} q_t; \quad \frac{\partial q_t}{\partial x_t^B} = \frac{(1 \quad f_t)\, G\left(x_t^B\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} q_t; \\
\frac{\partial^B q_t}{\partial\left(x_t^A\right)^2} &= \frac{f_t H\left(x_t^A\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} q_t; \quad \frac{\partial^B q_t}{\partial\left(x_t^B\right)^2} = \frac{(1 \quad f_t)\, H\left(x_t^B\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} q_t.
\end{aligned}
$$

Furthermore, the derivatives of Q_t with respect to the likelihood ratio are

$$
\frac{\partial Q\left(x_t^A; x_t^B; \ _t\right)}{\partial \ _t} = \left(\frac{1}{1 + \$ \ _t}\right)^2 D\left(x_t^A; x_t^B\right); \quad \frac{\partial^B Q\left(x_t^A; x_t^B; \ _t\right)}{\partial \ _t^B} = \frac{2}{(1 + \$ \ _t)^3} D\left(x_t^A; x_t^B\right);
$$

where

$$
D\left(x_t^A; x_t^B\right) \quad \int_t^{1} \left[M\left(x_t^B; u \quad t; \hat{\ }^B; \hat{\ }_x^B\right) \quad M\left(x_t^A; u \quad t; \hat{\ }^A; \hat{\ }_x^A\right) \right] \mathrm{d}u. \tag{1.6.18}
$$

Similarly, the cross-derivatives are given by

$$
\begin{aligned}
\frac{\partial^B q_t}{\partial \ _t \partial x_t^A} &= \quad f_t^B \frac{G\left(x_t^A\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} q_t; \quad \frac{\partial^B q_t}{\partial \ _t \partial x_t^B} = \quad f_t^B \frac{G\left(x_t^B\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} q_t; \\
\frac{\partial^B q_t}{\partial \ _t \partial M_t} &= \quad f_t^B \frac{D\left(x_t^A; x_t^B\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} \frac{q_t}{M_t}; \quad \frac{\partial^B q_t}{\partial \ _t \partial C_t} = f_t^B \frac{D\left(x_t^A; x_t^B\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} \frac{q_t}{C_t}; \\
\frac{\partial^B q_t}{\partial x_t^A \partial M_t} &= \frac{f_t G\left(x_t^A\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} \frac{q_t}{M_t}; \quad \frac{\partial^B q_t}{\partial x_t^B \partial M_t} = \frac{(1 \quad f_t)\, G\left(x_t^B\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} \frac{q_t}{M_t}; \\
\frac{\partial^B q_t}{\partial x_t^A \partial C_t} &= \quad \frac{f_t G\left(x_t^A\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} \frac{q_t}{C_t}; \quad \frac{\partial^B q_t}{\partial x_t^B \partial C_t} = \quad \frac{(1 \quad f_t)\, G\left(x_t^B\right)}{Q\left(x_t^A; x_t^B; \ _t\right)} \frac{q_t}{C_t}.
\end{aligned}
$$

In turn, the instantaneous covariances of the stochastic processes that characterize the economy are summarized in the following table

	$\frac{\mathrm{d}C_t}{C_t}$	$\frac{\mathrm{d}M_t}{M_t}$	$\mathrm{d}x_t^A$	$\mathrm{d}x_t^B$	$\frac{\mathrm{d}\ _t}{\ _t}$
$\frac{\mathrm{d}C_t}{C_t}$	^2_c				
$\frac{\mathrm{d}M}{M_t}$	0	^2_m			
$\mathrm{d}x_t^A$	0	$_m \hat{\ }_m^A$	$\left(\hat{\ }_m^A\right)^2 + \left(\hat{\ }_s^A\right)^2 = \left(\hat{\ }^A\right)^2$		
$\mathrm{d}x_t^B$	0	$_m \hat{\ }_m^B$	$\hat{\ }_m^A \hat{\ }_m^B + \hat{\ }_s^A \hat{\ }_s^B$	$\left(\hat{\ }_m^B\right)^2 + \left(\hat{\ }_s^B\right)^2 = \left(\hat{\ }^B\right)^2$	
$\frac{\mathrm{d}\ _t}{\ _t}$	0	$x_t^B \quad x_t^A$	$\hat{\ }_m^A \ _t$	$\hat{\ }_m^B \ _t$	^B_t

With these results we apply Ito's lemma to q_t

$$
\begin{aligned}
dq_t = {} & q_t \left(\frac{dC_t}{C_t} - \frac{dM_t}{M_t} \right) + \frac{\partial q_t}{\partial x_t^A}\, dx_t^A + \frac{\partial q_t}{\partial x_t^B}\, dx_t^B + \frac{\partial q_t}{\partial \ _t}(d \ _t) + q_t \left(\frac{dM_t}{M_t} \right)^2 + \\
& \frac{1}{2}\frac{\partial^2 q_t}{\partial \left(x_t^A\right)^2}\left(dx_t^A\right)^2 + \frac{1}{2}\frac{\partial^2 q_t}{\partial \left(x_t^B\right)^2}\left(dx_t^B\right)^2 + \frac{1}{2}\frac{\partial^2 q_t}{\partial \ _t^2}(d \ _t)^2 - q_t \left(\frac{dC_t}{C_t} \right)\left(\frac{dM_t}{M_t} \right) + \\
& \frac{\partial^2 q_t}{\partial C_t \partial x_t^A}(dC_t)\left(dx_t^A\right) + \frac{\partial^2 q_t}{\partial C_t \partial x_t^B}(dC_t)\left(dx_t^B\right) + \frac{\partial^2 q_t}{\partial M_t \partial x_t^A}(dM_t)\left(dx_t^A\right) + \frac{\partial^2 q_t}{\partial M_t \partial x_t^B}(dM_t)\left(dx_t^B\right) + \\
& \frac{\partial^2 q_t}{\partial x_t^A \partial x_t^B}\left(dx_t^A\right)\left(dx_t^B\right) + \frac{\partial^2 q_t}{\partial \ _t \partial C_t}(d \ _t)(dC_t) + \frac{\partial^2 q_t}{\partial \ _t \partial M_t}(d \ _t)(dM_t) + \\
& \frac{\partial^2 q_t}{\partial \ _t \partial x_t^A}(d \ _t)\left(dx_t^A\right) + \frac{\partial^2 q_t}{\partial \ _t \partial x_t^B}(d \ _t)\left(dx_t^B\right).
\end{aligned}
$$

After much algebra, we arrive at

$$
\frac{dq_t}{q_t} = \ _t^q dt + \ _c^q dz_{c,t} + \ _{m,t}^q dz_{m,t}^A + \ _{s,t}^q ds_t, \tag{1.6.19}
$$

where the drift terms are given by

$$
\begin{aligned}
\ _t^q \equiv {} & \ \left(x_t^A + \ _c + \ _m^2 \frac{G\left(x_t^A, x_t^B, \ _t\right)}{Q\left(x_t^A, x_t^B, \ _t\right)} + \frac{1}{2}\frac{H\left(x_t^A, x_t^B, \ _t, \hat{\ }\right)}{Q\left(x_t^A, x_t^B, \ _t\right)} + \right. \\
& \left. \left[(1 - f_t) \ _t \frac{G\left(x_t^A, x_t^B, \ _t, \hat{\ }_m\right)}{Q\left(x_t^A, x_t^B, \ _t\right)} - \ _m f_t \frac{D\left(x_t^A, x_t^B\right)}{Q\left(x_t^A, x_t^B, \ _t\right)} + \ \left(x_t^A, x_t^B, \ _t\right) \right] \right),
\end{aligned} \tag{1.6.20}
$$

where

$$
\begin{aligned}
\ \left(x_t^A, x_t^B, \ _t\right) & \equiv (1 - f_t) f_t \frac{D\left(x_t^A, x_t^B\right)}{Q\left(x_t^A, x_t^B, \ _t\right)} \ _t & (1.6.21) \\
G\left(x_t^A, x_t^B, \ _t\right) & \equiv f_t G\left(x_t^A\right)\left(\overline{x} - x_t^A - \hat{\ }_x^A\right) + (1 - f_t)\left(\overline{x} - x_t^B - \hat{\ }_x^B\right), & (1.6.22) \\
G\left(x_t^A, x_t^B, \ _t, \hat{\ }_l\right) & \equiv f_t G\left(x_t^A\right)\hat{\ }_l^A + (1 - f_t) G\left(x_t^B\right)\hat{\ }_l^B, \text{ for } l \in \{m, s\}, & (1.6.23) \\
H\left(x_t^A, x_t^B, \ _t, \hat{\ }\right) & \equiv f_t H\left(x_t^A\right)\left(\hat{\ }^A\right)^2 + (1 - f_t) H\left(x_t^B\right)\left(\hat{\ }^B\right)^2. & (1.6.24)
\end{aligned}
$$

Thus, $\ \left(x_t^A, x_t^B, \ _t\right)$ is a disagreement adjustment term, $G\left(x_t^A, x_t^B, \ _t\right)$ is an adjusted mean-reversion term, $G\left(x_t^A, x_t^B, \ _t, \hat{\ }_l\right)$ for $l = m, s$ is an uncertainty adjustment, and $H\left(x_t^A, x_t^B, \ _t, \hat{\ }\right)$ is a Jensen's inequality term.

Furthermore, the diffusion terms are

$$
\begin{aligned}
\ _c^q & = \ _c, & (1.6.25) \\
\ _{m,t}^q & = \ _m \left(\frac{G\left(x_t^A, x_t^B, \ _t, \hat{\ }_m\right)}{Q\left(x_t^A, x_t^B, \ _t\right)} + \ \left(x_t^A, x_t^B, \ _t\right) \right), & (1.6.26) \\
\ _{s,t}^q & = \frac{G\left(x_t^A, x_t^B, \ _t, \hat{\ }_s\right)}{Q\left(x_t^A, x_t^B, \ _t\right)}. & (1.6.27)
\end{aligned}
$$

Applying Ito's lemma to $P_t = \frac{1}{q_t}$, we obtain the dynamics of the price level

$$
\frac{dP_t}{P_t} = \ _t dt + \ _c^p dz_{c,t} + \ _{m,t}^p dz_{m,t}^A + \ _{s,t}^p ds_t, \tag{1.6.28}
$$

where the expected in ation (drift term) is given by

$$_t \qquad {}^q_t + \quad {}^q_{c;t}\;{}^2 + \quad {}^q_{m;t}\;{}^2 + \quad {}^q_{s;t}\;{}^2; \tag{1.6.29}$$

and the di usion terms by

$${}^p_c \qquad {}_c; \tag{1.6.30}$$

$${}^p_{m;t} \qquad {}_m + \frac{G\;\; x^A_t; x^B_t; \;\; {}_t; \hat{}_m}{Q\;\; x^A_t; x^B_t; \;\; {}_t} \qquad x^A_t; x^B_t; \;\; {}_t \;\; ; \tag{1.6.31}$$

$${}^p_{s;t} \qquad \frac{G\;\; x^A_t; x^B_t; \;\; {}_t; \hat{}_s}{Q\;\; x^A_t; x^B_t; \;\; {}_t}: \tag{1.6.32}$$

Dynamics of the short nominal rate: Making use of the relation between R_t and Q_t $Q\;\; x^A_t; x^B_t; \;\; {}_t$ from proposition (1.3.2) and Ito's lemma, we obtain

$$\mathrm{d}R_t \quad = \quad {}_{R;t}\mathrm{d}t + {}^R_{m;t}\mathrm{d}z^A_{m;t} + {}^R_{s;t}\mathrm{ds}_t; \tag{1.6.33}$$

with drift terms given by

$${}^R_t \qquad R_t \quad {}^Q_t \quad \Big[\quad {}^Q_{m;t}\;{}^2 + \quad {}^Q_{s;t}\;{}^2 \Big] \quad ; \tag{1.6.34}$$

and di usion terms by

$${}^R_{m;t} \qquad \frac{G\;\; x^A_t; x^B_t; \;\; {}_t; \hat{}_m}{Q\;\; x^A_t; x^B_t; \;\; {}_t} R_t \qquad x^A_t; x^B_t; \;\; {}_t \;\; R_t; \tag{1.6.35}$$

$${}^R_{s;t} \qquad \frac{G\;\; x^A_t; x^B_t; \;\; {}_t; \hat{}_s}{Q\;\; x^A_t; x^B_t; \;\; {}_t} R_t: \tag{1.6.36}$$

To arrive at the previous results, we rst calculated the dynamics of Q_t from Ito's lemma

$$\mathrm{d}Q_t \quad = \quad {}^Q_t\mathrm{d}t + {}^Q_{m;t}\mathrm{d}z^A_{m;t} + {}^Q_{s;t}\mathrm{ds}_t;$$

where

$$\begin{aligned}
{}^Q_t \qquad & \frac{f_t G\;\; x^A_t \quad \bar{x} \quad x^A_t \;\; + (1 \quad f_t)\, G\;\; x^B_t \quad \bar{x} \quad x^B_t}{Q\;\; x^A_t; x^B_t; \;\; {}_t} + \frac{1}{2} \frac{H\;\; x^A_t; x^B_t; \;\; {}_t; \hat{}}{Q\;\; x^A_t; x^B_t; \;\; {}_t} + \\
& (1 \quad f_t) \quad {}_t \left[\frac{G\;\; x^A_t; x^B_t; \;\; {}_t; \hat{}_m}{Q\;\; x^A_t; x^B_t; \;\; {}_t} \qquad x^A_t; x^B_t; \;\; {}_t \right] ; \\
{}^Q_{m;t} \qquad & \frac{G\;\; x^A_t; x^B_t; \;\; {}_t; \hat{}_m}{Q\;\; x^A_t; x^B_t; \;\; {}_t} + \qquad x^A_t; x^B_t; \;\; {}_t \;\; ; \\
{}^Q_{s;t} \qquad & \frac{G\;\; x^A_t; x^B_t; \;\; {}_t; \hat{}_s}{Q\;\; x^A_t; x^B_t; \;\; {}_t}:
\end{aligned}$$

Dynamics of the real state-price density: The real state-price density of investor A is given by

$$_{A;t} = y_1^{\ 1}{}'e \quad {}^t f\,(\quad _t) \ {}^1 C_t^{\ 1} = \quad y_1^{\ 1}{}'e \quad {}^t C_t^{\ 1}(1 + \quad _t):$$

By Ito's lemma

$$\frac{\mathrm{d} \ _{A;t}}{_{A;t}} = \quad \mathrm{d}t \quad \frac{\mathrm{d}C_t}{C_t} + \quad \frac{\mathrm{d}C_t}{C_t}^{2} + \quad \frac{_t}{1 + \quad _t} \quad \frac{\mathrm{d} \ _t}{_t}$$

$$= \quad r\mathrm{d}t \quad _c\mathrm{d}z_{c;t} \quad _{m;t}\mathrm{d}z^A_{m;t};$$

where r_t is the instantaneous real risk-free rate given by

$$r \quad + \ _c \quad {}^2_c; \tag{1.6.37}$$

and the prices of risk are

$$_c \quad _c; \tag{1.6.38}$$

$$_{m;t} \quad (1 \quad f_t) \ _t: \tag{1.6.39}$$

Dynamics of the nominal state-price density: The nominal state-price density is given by

$$^{\$}_{A;t} = q_t \ _{A;t}:$$

By Ito's lemma

$$\frac{\mathrm{d} \ ^{\$}_{A;t}}{^{\$}_{A;t}} = \frac{\mathrm{d} \ _{A;t}}{_{A;t}} + \frac{\mathrm{d}q_t}{q_t} + \frac{\mathrm{d} \ _{A;t}}{_{A;t}} \quad \frac{\mathrm{d}q_t}{q_t}$$

$$= R_t\mathrm{d}t \quad ^{\$}_{m;t}\mathrm{d}z^A_{m;t} \quad ^{\$}_{s;t}\mathrm{d}s_t;$$

where the nominal short rate R_t is

$$R_t = \quad + x^1_t \quad {}^2_m + \frac{G \ \ x^A_t; x^B_t; \ \ _t}{Q \ \ x^A_t; x^B_t; \ \ _t} \quad \frac{1}{2}\frac{H \ \ x^A_t; x^B_t; \ \ _t; \hat{}}{Q \ \ x^A_t; x^B_t; \ \ _t} \quad _m \ _{m;t} + \quad _m \quad x^A_t; x^B_t; \ \ _t \ : \tag{1.6.40}$$

In turn, the vector of nominal prices of risk, $^{\$}_t \quad \left[^{\$}_{m;t}; \ ^{\$}_{s;t} \right]_0$ is

$$^{\$}_{m;t} \quad _{m;t} \quad ^q_{m;t} = \quad _m + \frac{G \ \ x^A_t; x^B_t; \ \ _t; \hat{}_m}{Q \ \ x^A_t; x^B_t; \ \ _t} \quad x^A_t; x^B_t; \ \ _t \ + \quad _{m;t}; \tag{1.6.41}$$

$$^{\$}_{s;t} \quad ^q_{s;t} = \frac{G \ \ x^A_t; x^B_t; \ \ _t; \hat{}_s}{Q \ \ x^A_t; x^B_t; \ \ _t}; \tag{1.6.42}$$

and ^q_t, $_c$, $_{m;t}$, ^q_c, and $^q_{m;t}$ are given by Eqs. (1.6.20), (1.6.38), (1.6.39), (1.6.25), and (1.6.26), respectively.

Proof of Proposition 1.3.4

Instantaneous in ation risk premium: This proof is very similar to the one of proposition (1.2.3). The only di erence is that now we have to account for an additional term $_{m;t}\ ^{q}_{m;t}$ in Eq. (1.6.40). This additional term implies that the instantaneous in ation risk premium in the model with disagreement is given by $\text{IRP}_t \quad \left[R_t \quad _t + \text{var}_t \left(\frac{\text{d}P_t}{P_t} \right) \quad r_t \right]$. Thus

$$
\begin{aligned}
\text{IRP}_t &= \ _{c;t}\ ^{q}_{c;t} + \ _{m;t}\ ^{q}_{m;t} \\
&= \ ^{2}_{c} \quad (1 \quad f_t) \quad _t \qquad _m \quad \frac{G \quad x^A_t; x^B_t; \quad _t; \hat{}_m}{Q \quad x^A_t; x^B_t; \quad _t} + \qquad x^A_t; x^B_t; \quad _t \quad \Bigg] :
\end{aligned}
$$

Real yield curve: We begin with the de nition of the price in t of a real bond with maturity in period under the beliefs of investor A, that is $B_{t;} = \text{E}^A_t \left[\frac{_{A;}}{_{A;t}} \right] = \text{E}^A_t \ e^{\ (\quad t)} \ \frac{f(\quad)C}{f(\quad _t)C_t} \ ^{1}$, where the last equality follows from Eq. (1.3.12). After inserting the sharing rule and re-arranging terms, the bond price becomes

$$
B_{t;} = \frac{e^{\ (\quad t)}}{1 + \quad _t} \left(\text{E}^A_t \left[\frac{C}{C_t} \ ^{1} \right] + \ _t\text{E}^A_t \left[\frac{}{_t} \quad \frac{C}{C_t} \ ^{1} \right] \right) :
$$

Changing the probability measure of the last term in the equation above to that of investor B as we did when proving proposition (1.3.2), we obtain $\text{E}^A_t \ \frac{}{_t} \ \frac{C}{C_t} \ ^{1} = \text{E}^B_t \ \frac{C}{C_t} \ ^{1} = \text{E}^A_t \ \frac{C}{C_t} \ ^{1}$. The last equality follows from the assumption that information about the consumption process is complete. Therefore, from log-normality

$$
B_{t;} = e^{\ (\quad t)}\text{E}^A_t \left[\frac{C}{C_t} \ ^{1} \right] = e^{\ (+ \ _c \quad ^{2}_{c})(\quad t)} :
$$

The real yield curve follows directly from $y_{t;} = \frac{\log(B_{t;}\ )}{\ t}$.

Nominal yield curve: The price in t of a nominal bond with maturity in period under the beliefs of investor A is $B^{\$}_{t;} = \text{E}^A_t \ \frac{^{\$}_{A;}}{^{\$}_{A;t}}$. We can manipulate the ratio $\frac{^{\$}_{A;}}{^{\$}_{A;t}}$ by inserting Eqs. (1.3.12) and (1.3.16) as follows

$$
\begin{aligned}
\frac{^{\$}_{A;}}{^{\$}_{A;t}} &= \frac{\int^{1} e^{\ u} \ \text{E}^A \left[(M_u) \ ^{1} \right] + \text{E}^A \left[\ _u (M_u) \ ^{1} \right] \text{d}u}{\int^{1}_{t} e^{\ u} \ \text{E}^A_t \left[(M_u) \ ^{1} \right] + \ _t\text{E}^B_t \left[(M_u) \ ^{1} \right] \text{d}u} \\
&= \frac{\int^{1} \ \text{E}^A \ e^{\ (u \quad t)} \ \frac{M_u}{M_t} \ ^{1} + \text{E}^A \ e^{\ (u \quad t)} \ _u \ \frac{M_u}{M_t} \ ^{1} \quad \text{d}u}{\int^{1}_{t} \ M \quad x^A_t; u \quad t; \hat{}^A; \hat{}^A_x \ + \ _t M \quad x^B_t; u \quad t; \hat{}^B; \hat{}^B_x \quad \text{d}u} :
\end{aligned}
$$

Using the the previous result and the price of a nominal bond with maturity t at time t under the beliefs of investor A

$$B^{\$}_{t;} = E_t \left[\frac{\int^{\infty} E^A\left[e^{(u\ t)} \left(\frac{M_u}{M_t}\right)^{1} + E^A\left[e^{(u\ t)}\ {}_u \left(\frac{M_u}{M_t}\right)^{1}\right]\right] du}{\int_t^{\infty} M\left(x_t^A; u\ \ t; \hat{}^A; \hat{}^A_x\right) + {}_t M\left(x_t^B; u\ \ t; \hat{}^B; \hat{}^B_x\right) du} \right]$$

$$= \frac{\int^{\infty} E_t^A\left[E^A\left[e^{(u\ t)} \left(\frac{M_u}{M_t}\right)^{1} + E^A\left[e^{(u\ t)}\ {}_u \left(\frac{M_u}{M_t}\right)^{1}\right]\right]\right] du}{\int_t^{\infty} M\left(x_t^A; u\ \ t; \hat{}^A; \hat{}^A_x\right) + {}_t M\left(x_t^B; u\ \ t; \hat{}^B; \hat{}^B_x\right) du}:$$

By the tower property of conditional expectations

$$E_t^A\left(E^A\left[e^{(u\ t)}\left(\frac{M_u}{M_t}\right)^{1}\right] + E^A\left[e^{(u\ t)}\ {}_u\left(\frac{M_u}{M_t}\right)^{1}\right]\right) = E_t^A\left[e^{(u\ t)}\left(\frac{M_u}{M_t}\right)^{1}\right] + E_t^A\left[e^{(u\ t)}\ {}_u\left(\frac{M_u}{M_t}\right)^{1}\right]:$$

Furthermore, we can change the measure from investor A's beliefs to investor B's such that

$$E_t^A\left[e^{(u\ t)}\ {}_u\left(\frac{M_u}{M_t}\right)^{1}\right] = {}_t E_t^A\left[e^{(u\ t)}\frac{u}{t}\left(\frac{M_u}{M_t}\right)^{1}\right] = {}_t E_t^B\left[e^{(u\ t)}\left(\frac{M_u}{M_t}\right)^{1}\right]:$$

Therefore, the integrand in the numerator of $B^{\$}_{t;}$ becomes

$$E_t^A\left[e^{(u\ t)}\left(\frac{M_u}{M_t}\right)^{1}\right] + E_t^A\left[e^{(u\ t)}\ {}_u\left(\frac{M_u}{M_t}\right)^{1}\right] = E_t^A\left[e^{(u\ t)}\left(\frac{M_u}{M_t}\right)^{1}\right] + {}_t E_t^B\left[e^{(u\ t)}\left(\frac{M_u}{M_t}\right)^{1}\right]$$

$$= M\left(x_t^A; u\ \ t; \hat{}^A; \hat{}^A_x\right) + {}_t M\left(x_t^B; u\ \ t; \hat{}^B; \hat{}^B_x\right):$$

Hence, the price of the nominal bond is given by

$$B^{\$}_{t;} = \frac{\int^{\infty} M\left(x_t^A; u\ \ t; \hat{}^A; \hat{}^A_x\right) + {}_t M\left(x_t^B; u\ \ t; \hat{}^B; \hat{}^B_x\right) du}{\int_t^{\infty} M\left(x_t^A; u\ \ t; \hat{}^A; \hat{}^A_x\right) + {}_t M\left(x_t^B; u\ \ t; \hat{}^B; \hat{}^B_x\right) du}:$$

The nominal yield curve follows from $y^{\$}_{t;} = \frac{\log\left(B^{\$}_{t;}\right)}{t}$, the expression above and by letting $Q\left(x_t^A; x_t^B; \ {}_t;\right) \int^{\infty} \frac{M\left(x_t^A;u\ \ t;\hat{}^A;\hat{}^A_x\right) + {}_t M\left(x_t^B;u\ \ t;\hat{}^B;\hat{}^B_x\right)}{1 + {}_t} du$ for t with some abuse of notation.

Dynamics of the nominal yield curve: The di usion terms of the nominal yield curve follow from an application of Ito's lemma and by letting, for $l\ 2\ fm; sg$

$$G\left(x_t^A; x_t^B; \ {}_t; \ ; \hat{}_l\right) \quad \frac{f_t G\left(x_t^A; \ \ \hat{}_l^A\right) + (1 \quad f_t)\, G\left(x_t^B; \ \ \hat{}_l^B\right)}{Q\left(x_t^A; x_t^B; \ {}_t;\right)}; \tag{1.6.43}$$

$$D\left(x_t^A; x_t^B;\right) \quad \int_t^{\infty} M\left(x_t^B; u\ \ t; \hat{}^B; \hat{}^B_x\right) \quad M\left(x_t^A; u\ \ t; \hat{}^A; \hat{}^A_x\right) du: \tag{1.6.44}$$

Chapter 2

Quantifying Private Information about Fundamentals in Treasury Auctions

2.1 Introduction

The role of private information about fundamentals occupies a central position in the study of financial markets. It is directly related to key questions about market efficiency (Fama, 1970, 1991), market microstructure (Kyle, 1985) and information aggregation (Grossman and Stiglitz (1980); Diamond and Verrecchia (1981); Kyle (1989)). In the context of the primary market for sovereign debt, early work argued that the main incentive for bidders in treasury auctions (*i.e.*, the primary market) is to resell purchased bonds to investors in the secondary market (Cammack (1991); Spindt and Stolz (1992); Bikhchandani and Huang (1993)). These authors concluded that private information about the resale price (the fundamental value) should determine bidding strategies, whereas idiosyncratic differences in valuations can safely be assumed to be negligible. Nyborg et al. (2002) and Bjønnes (2001), among others, provide reduced-form empirical evidence consistent with private information. In contrast, Hortacsu and Kastl (2012) do not find statistical evidence of private information in Canadian treasury auctions. Hortacsu et al. (2018) argue that bidders

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in US Treasury auctions are sophisticated traders with access to overlapping data sources on the value of the bonds being offered, including information on secondary market prices right before the auction. Hence, even if reselling bonds is a chief motivation for bidders, it is unlikely that they are privately informed about resale prices. Instead, heterogeneity in bids is more likely attributable to differences in idiosyncratic valuations. More recently, however, Boyarchenko et al. (2020) study information sharing in US treasury auctions by estimating a model with private information about fundamentals and idiosyncratic valuations. In summary, the importance of private information in primary debt markets remains an open empirical question.

We contribute to this debate by proposing an empirical method to quantify the amount of private information in treasury auctions structured as divisible good uniform price auctions. The method is based on the estimation of the theoretical model presented in Vives (2010, 2011), where bidders are allowed to be privately informed about fundamentals but may also receive idiosyncratic valuation shocks. The relative importance of private information is determined by the correlation between bidders' valuations and the precision of their private signals. These are model parameters that, we show, are identified from data on bids and secondary market prices. Through an application to the Colombian sovereign debt market, we find that the importance of private information varies with market conditions. In particular, our estimates indicate that the information structure of the market under consideration is less characterized by private information after an exogenous positive shock to the demand for government bonds in the secondary market. Furthermore, we provide model-free empirical evidence to support this finding.

Bidders in treasury auctions have market power because access to the primary market is generally restricted to a few financial institutions. These bidders strategically submit bids below their marginal valuations, a result known in the auctions literature as *bid shading* or *demand reduction*. One key result of Vives (2010, 2011) is that private information about fundamentals raises market power above the full-information level. Consequently, the model predicts that holding all else constant, an observed reduction in private information would lower market power and bid shading, thereby reducing issuer borrowing costs. However, the model allows for decreasing marginal valuations, and, in equilibrium, bidders with steeper marginal value curves shade their bids more. The slope of the marginal value function, which is a parameter of the model, can be interpreted as the cost to the bidders of deviating from their target inventory, or inventory costs. The model implies that higher inventory costs also result in greater market power. This poses an identification challenge. The same level of bid shading can be rationalized by high private information and low inventory costs, or alternatively, high inventory costs and low private information. Nonetheless, we show that the model parameters that determine private information and inventory costs are identified separately from auction data (including bids) and a secondary price benchmark.

We apply our method to the Colombian sovereign debt market. In March 2014, J.P. Morgan announced that it would boost the weights of Colombian sovereign bonds on two local currency emerging market debt indices. Williams (2018) documents that this unanticipated rebalancing triggered a surge in the demand for Colombian sovereign debt, which in turn prompted banks in Colombia to reduce their holdings of government bonds. We provide supplementary empirical evidence that Colombian sovereign bonds were traded more actively after the shock. Furthermore, the secondary debt market became more liquid and experienced a significant decrease in volatility. We hypothesize that the primary market was likely also affected by the rebalancing. Consequently, we estimate the model separately before and after the shock. We find a substantial reduction in private information about fundamentals as a determinant of observed bids, but also a sizeable increase in inventory costs. These two findings have opposing effects on bid shading. The estimation results indicate that the change in revenue for the treasury is lower than two basis points (bp) of the par value of the bond offering. However, the latter masks the separate effects of private information and inventory costs. We decompose these two effects and find that the reduction in private information alone implies a drop in bid shading (increase in revenue) of 34–112bp, depending on the number of bidders. In contrast, the increase in inventory costs raises bid shading by 35–110bp. In Chapter III, we provide reduced-form evidence consistent with both channels.

Our proposed methodology is well suited to quantify the effects of private information and inventory costs on policies meant to lower treasuries' borrowing costs. We study two such policies: increasing the number of primary market participants and reducing the total bond offering per auction. After the rebalancing, the structural estimation results imply that increasing the number of bidders from 10 to 14 (the minimum and maximum observed participation in our sample) reduces bid shading by 4 bp. In contrast, in a hypothetical but empirically feasible scenario with high market power (high private information matching the pre-rebalancing estimates and high inventory costs matching the post-rebalancing result), bid shading drops by 80 bp when the number of bidders increases from 10 to 14. Analogously, we quantify the effect of reducing the total bond offering in a specific auction on market power. From the equilibrium of the model in Vives (2010, 2011), market power increases with bond supply. We find that, following a one standard deviation decrease in the total bond supply below its sample mean, market power would fall by at most 4 bp according to our post-rebalancing estimates. In the high market power scenario, the reduction in market power lies between 20 bp and 63 bp, depending on the number of bidders.

Debates on whether the aforementioned policies should be pursued require estimates of the prevailing level of market power in the primary market under consideration. Furthermore, more targeted policies intended to address either private information frictions or inventory costs related to market-making services would benefit from methodologies that can decompose market power into its determinants to discern which feature is more preponderant in the market. Our study contributes to both objectives.

We conclude this section with a discussion on related literature. The remainder of Chapter II is organized as follows. In Section (2.2) we summarize the model presented in Vives (2010, 2011). We then show that the model parameters are identified using only bidding data and prices from the secondary debt market in Section (2.3), and we outline our empirical strategy to estimate the parameters of interest. In Section (2.4), we describe the main features of the Colombian debt market along with the data we use to estimate the model. In Section (2.5), we present the results of our structural estimation, a quantitative assessment of the change in bidders' market power, and a counterfactual analysis to evaluate policies to reduce borrowing costs.

Related literature In the auctions literature, questions about the importance of private information have been formulated in terms of which theoretical model better describes the information structure that determines bidders' valuations. Private information is consistent with common value models. In such models, the ex post value of the asset after the auction is the same for all bidders (*e.g.*, its resale price in the secondary market), but it is uncertain to the bidders ex ante. Instead, each bidder observes a private signal of the common value that amounts to private information about fundamentals. On the other extreme, the absence of private information corresponds to a model with only private values. In that model, any common component in bidders' values is common knowledge; hence, all information about fundamentals is public, but bidders have different idiosyncratic valuations. The theoretical model in Vives (2010, 2011) encompasses both common and private values. The weight of each of these two components in bidders' valuations determines the importance of private information.[1]

Our study is closely related to Boyarchenko et al. (2020). They extend the uniform price common value model of Kyle (1989) and Wang and Zender (2002) by adding a private value component to bidders' valuations. However, they do not rely on an estimation procedure based on bidding data. In contrast to our work, they calibrate the model to auction results and post-auction secondary market prices. In their calibration, the relative importance of the common value component is not recovered from the data and is instead fixed. Hence, in their study, the importance of private information is imposed by assumption. To the best of our knowledge, ours is the first study to estimate the importance of private information in a divisible good uniform price auction with both private and common values.

Methodologically, our estimation approach complements Hortacsu and McAdams (2010) and Kastl (2011). Their estimation method is non-parametric and takes into account the fact that bids are discrete. However, it requires an assumption of independent private values. Hence, it cannot be used to recover the distribution of bidders' valuations in settings with private information (common

[1] Armantier and Sbaï (2006) propose a parametric estimation of a generalized version of the model in Wang and Zender (2002) where bidders have pure common values but are asymmetric. Recent work on non-parametric structural estimation of multiunit auctions under the private values paradigm includes Chapman et al. (2007), Kang and Puller (2008), Hortacsu and McAdams (2010), Kastl (2011), Hortacsu et al. (2018).

values). In contrast, we follow the model in Vives (2011), which has a unique equilibrium in linear demand schedules. We focus on this equilibrium, acknowledging that it imposes strong parametric assumptions, and treats bids as continuous linear functions. However, these assumptions allow us to derive a likelihood function and, consequently, estimate the model on bidding data and readily available secondary market pricing data alone. Furthermore, unlike in non-parametric estimation methods, we can decompose market power into private information and inventory costs.

The results of our application indicate that the information structure is not an immutable characteristic of treasury auctions. Hence, although we find evidence of private information, our findings are not necessarily inconsistent with those of Hortacsu and Kastl (2012). Using detailed bidding data from Canadian Treasury auctions, they do not reject the null hypothesis of private values in favor of an alternative common values hypothesis. However, their test could, in principle, reject the null hypothesis if applied to a different market or at a different point in time.[2]

Our study is also related to the empirical literature on the aggregation of private information in sovereign debt markets. Green (2004), Brandt and Kavajeckz (2004), Pasquariello and Vega (2007), Valseth (2013), and Czech et al. (2020) show that order-flow predicts future bond yields.[3] They argue that order-flow contributes to price discovery (*i.e.*, aggregation of asymmetric information) in secondary debt markets. While our main focus is on the primary market, we provide empirical evidence that private information is revealed to market participants through the auction's cutoff price, which in turn predicts future secondary market yields and prices. Furthermore, Green (2004) and Pasquariello and Vega (2007) document that the empirical relationship between order-flow and yields changes with macroeconomic announcements. This result resonates with our findings on the time-varying nature of private information in debt markets.

2.2 Model

We follow the uniform price auction model of Vives (2010, 2011) with private and common values. k_t units of an asset are sold at each auction $t = 1, \ldots, T$, and bidders are uncertain about the *ex post* value of the asset. For each bidder $i = 1, \ldots, n_t$, the marginal value of buying $x_{i,t}$ units is $v_{i,t}(x_{i,t}) = \theta_{i,t} - \lambda x_{i,t}$, with $\lambda > 0$. λ captures decreasing marginal valuation and can be interpreted as an opportunity cost. Bidders do not observe their valuations $\theta_{i,t}$, instead they receive private independent signals $s_{i,t} = \theta_{i,t} + \varepsilon_{i,t}$. The distributions of $\theta_{i,t}$ and $\varepsilon_{i,t}$ are common knowledge. $\varepsilon_{i,t} \sim N\left(0, \sigma_\varepsilon^2\right)$ for all i and all t, $\operatorname{cov}(\varepsilon_{i,t}, \varepsilon_{j,t}) = 0$ for $i \neq j$ and $\operatorname{cov}(\varepsilon_{is}, \varepsilon_{i,t}) = 0$ for $s \neq t$. $\theta_{i,t} \sim N\left(\bar{\theta}_t, \sigma_\theta^2\right)$ with $\bar{\theta}_t > 0$ and $\sigma_\theta^2 > 0$, $\operatorname{cov}(\theta_{i,t}, \theta_{j,t}) = \rho\sigma_\theta^2$ for $i \neq j$, $\operatorname{cov}(\theta_{is}, \theta_{i,t}) = 0$ for $s \neq t$,

[2] The test proposed by Hortacsu and Kastl (2012) relies on the econometrician observing how dealers change their bids after they learn their customers' bids. Therefore, we cannot apply it to our data because there are no customer bids in Colombian treasury auctions.

[3] Evans and Lyons (2005) and Menkhoff et al. (2016), among others, document a similar result for order-flow and future exchange rates.

and cov $(\ _{i;t};\ _{jt}) = 0$ for all i and j.

Notice that bidders' valuations can be decomposed into a common value and idiosyncratic independent components, as follows. Let $_{i;t} = \tilde{}_t + \ _{i;t}$, where $\tilde{}_t \quad N \left(\bar{}_t; \ ^2\right)$, $_{i;t} \quad N \left(0; (1 \quad) \ ^2\right)$, cov $\left(\tilde{}_t; \ _{i;t}\right) = 0$, and cov $(\ _{i;t};\ _{jt}) = 0$ for $i \neq j$. $\tilde{}_t$ is a common value, as traditionally de ned in the auctions literature. It is common to all bidders and, crucially, for some values of the parameters, bidders are imperfectly and privately informed about its value. This is precisely the feature of the model that captures private information about fundamentals.

The model encompasses extreme cases traditionally considered in the auctions literature. When $\ = 1$ and $\ ^2 > 0$, the model reduces to a pure common value auction as the asset has the same uncertain *ex post* value for all bidders, with probability one. In such a case, bidders are privately informed about the unknown common value of the asset. In contrast, if $\ < 1$ and $\ ^2 = 0$, bidders have pure private values because they are no longer uncertain about their private *ex post* values. Individual valuations still include a common component, but no bidder has private information that is value relevant for other bidders who already know their valuations with certainty, hence private information about fundamentals (about the common component) is irrelevant. Finally, when $\ = 0$ and $\ ^2 > 0$, there is no uncertain common value, the common component $\bar{}_t$ is commonly known, but bidders are not perfectly informed about their *ex post* idiosyncratic values, moreover, their signals are independent conditional on $\bar{}_t$.

In summary, bidders in the model have private information about fundamentals if and only if $\ > 0$ and $\ ^2 > 0$. In such a case, there is a common component $\tilde{}_t$ in bidders valuations that is not perfectly known ($\ > 0$), and bidders receive di erent private signals about its value ($\ ^2 > 0$). More broadly, the importance of private information in bidders valuations is determined by and .

2.2.1 Auction equilibrium

After observing their private signals, bidders submit linear demand schedules of the form $X(s_{i;t}; p) = b_t + a_t s_{i;t} \quad c_t p$. The auctioneer determines the auction cuto price that clears the market, that is, the price $\hat{p}_t$ that satis es $\sum_{i=1}^{n_t} X(s_{i;t}; \hat{p}_t) = k_t$. If there is no such price for a given realization of signals, the auction is canceled and no units of the asset are sold. Following Vives (2011), we only consider symmetric Bayesian Nash equilibria in linear demand schedules. He proves the existence of a unique such equilibrium. We reproduce his Proposition 1 below for convenience.

Proposition 2.2.1. *Let* $\frac{\ ^2}{\ ^2} < 1$. *Then there is a unique symmetric linear Bayesian demand function equilibrium if and only if* $n \quad 2 \quad M > 0$, *where*

$$M \quad \frac{\ ^2 n}{(1 \quad) \ ^2 + \ ^2 (1 + (n \quad 1) \ )}: \tag{2.2.1}$$

This equilibrium is given by

$$X(s_i;p) = (E[\theta_i|s_i;p] - p)/(d+\lambda) = b + as_i - cp; \tag{2.2.2}$$

where $c = \frac{n-2-M}{\lambda(n-1)(1+M)}$, $d = \frac{1}{(n-1)c}$, $a = \frac{(1-\rho)\sigma_\theta^2}{(1-\rho)\sigma_\theta^2+\sigma_\varepsilon^2}(d+\lambda)^{-1}$ *and*

$$b = \frac{1}{1+M}\left(-\frac{k}{n}M + \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + (1+(n-1)\rho)\sigma_\theta^2}(d+\lambda)^{-1}\bar{\theta}\right). \tag{2.2.3}$$

In equilibrium, $1/(\lambda(1+M)) > c > 0; a > 0$*, c decreases with M and with* λ*, and d is decreasing in n.*

2.2.2 Bid shading, private information and decreasing marginal valuations

In equilibrium, bidders submit bids below their marginal valuations. This is a feature of several auction models, commonly refer to as bid shading. In fact, the demand function of a price-taking bidder is $X^{pt}(s_i;p) = (E[\theta_i|s_i;p] - p)/\lambda$, which is above the equilibrium demand schedule in 2.2.1 because $d > 0$. Moreover, it follows from Proposition 2.2.1 and the market clearing condition that the equilibrium cutoff price is $\hat{p} = \frac{1}{n}\sum_{i=1}^{n} E[\theta_i|s] - (d+\lambda)k/n$. It follows that the difference between the price that bidders pay for the asset and their average marginal valuation is dk/n. Therefore, as discussed in Vives (2010), this quantity measures the amount of bid shading or, alternatively, market power.

In the model, market power is thus explained by three factors: imperfect competition, private information and decreasing marginal valuations. Specifically, d depends on the number of bidders n, the parameters determining private information (ρ, σ_θ^2, and σ_ε^2), and the slope of the marginal valuation function λ.

We will first describe the effect of private information on bid shading. The equilibrium in Proposition 2.2.1 is privately revealing in the sense of Allen (1981). In particular, $E[\theta_i|s_i;p] = E[\theta_i|s]$, where $s = (s_1;\ldots;s_n)$ is the vector of all signals. That is, the equilibrium cutoff price conveys the same information to bidder i about its own valuation θ_i than the signals of all other bidders. This provides incentives for bid shading analogous to the winner's course in common value auctions. A low cutoff price reveals to bidder i that other bidders received low signals, hence decreasing its ex-post valuation of the asset. Rational bidders protect themselves against this bad news by increasing bid shading.

If $\rho = 0$ or $\sigma_\varepsilon^2 = 0$, the equilibrium in Proposition 2.2.1 coincides with a full-information equilibrium where all bidders observe their own signals and the signals of all other bidders. In such a case, $E[\theta_i|s_i] = E[\theta_i|s]$, hence the cutoff price no longer contains additional information about bidders valuations. There is still bid shading in the full-information equilibrium (which we denote using a superscript f, following Vives (2010)), but it is lower than in the equilibrium with private information ($\rho > 0$ and $\sigma_\varepsilon^2 > 0$). Notice that $M > M^f = 0$, therefore $c^f > c > 0$ and $0 < d^f < d$.

All things equal, it follows from equation (2.2.1) that larger values of M reduce c, therefore bidders submit less elastic demand schedules, and their market power, as measured by d, increases.

In the private information equilibrium, M determines the effect of private information on bid shading and market power, through its effect on the slope c. However, the latter is also determined by λ, that is, by how steep the bidders' decreasing marginal valuations are. Steeper marginal valuations (higher λ) imply steeper equilibrium demand schedules (higher c), which in turn results in higher bid shading (higher d).

Finally, as stated in Proposition 2.2.1, d (therefore bid shading as well) is monotonically decreasing in the number of bidders. Moreover, as $n \to \infty$, $d \to 0$, hence the equilibrium with private information converges to the full-information equilibrium.

2.3 Identification and estimation

We establish sufficient conditions under which the model in Section 2.2 is identified. For any auction t, we assume the econometrician observes the number of bidders n_t, the total number of units offered k_t, all bids submitted, the auction cutoff price p_t^c and the secondary market price p_t. We further assume that the bids correspond to the unique symmetric linear Bayesian equilibrium from Section 2.2, for a given realization of private signals $s_t = (s_{1t}, \ldots, s_{n_t t})$. Hence, bids are linear demand schedules of the form $X_{i,t}(p) = \alpha_{i,t} - c_t p$, where $\alpha_{i,t} = b_t + a_t s_{i,t}$. Equivalently, bids can be described as inverse demand schedules: $P_{i,t}(x) = p_{i,t}^0 - \frac{1}{c_t} x$, where $p_{i,t}^0 = \frac{\alpha_{i,t}}{c_t}$. For a given auction t, all the information contained in the bids can be summarized in a vector of intercepts, either $\alpha_t = (\alpha_{1t}, \ldots, \alpha_{n_t t})$ or $p_t^0 = \left(p_{1t}^0, \ldots, p_{n_t t}^0\right)$, and a slope c_t that is common to all bidders in a symmetric equilibrium.

We assume the econometrician has data from T independent auctions labeled $t = 1, \ldots, T$. Pooling bidding data from several different auctions is a common practice in the empirical study of treasury auctions, since the number of potential bidders tends to be low (e.g, Hortacsu and McAdams (2010), Armantier and Sbaï (2006), Kastl (2011), and Hortacsu and Kastl (2012)).

In the model, the mean of the bidders' values $\bar{\mu}_t$ is common knowledge for the bidders, but it is unlikely that the econometrician will know it with certainty. This introduces unobserved auction heterogeneity posing a challenge to the identification strategy, since we are combining data from several different auctions. To address this challenge we allow the econometrician to have partial knowledge of $\bar{\mu}_t$, captured by a proxy p_t. Secondary market prices, which are widely available for sovereign bond markets around the world, are likely good proxies of average valuations. Consequently, besides the assumptions of the model in Section 2.2, we add an assumption stating that secondary market prices are proxies of average valuations.

Assumption 2.3.1. *Let p_t be the observable secondary market price of the asset. Then $\bar{\mu}_t = p_t + \eta_t$, where $\eta_t \sim N\left(\mu_\eta, \sigma_\eta^2\right)$ with unknown mean and variance.*

Our goal now is to establish the identi cation of all time-invariant parameters in the model, i.e., (; ; ;). We will also discuss the identi cation of and ^{2}, although these are are not main parameters of interest and are added to the model only to address unobserved heterogeneity. We exploit variation in the number of bidders to prove identi cation. Speci cally, we assume there are arbitrarily many auctions with at least two di erent numbers of bidders.

To establish identi cation, we start by noticing that in equilibrium the sensitivity of the intercepts to the signals, a_t, and the slope of the linear bids, c_t, both depend on the number of bidders, n_t, but do not depend on total supply, k_t. Hence, across multiple auctions all with the same number of bidders, a_t and c_t remain constant. The same is true of the variables M and d in Proposition 2.2.1. In what follows, we will denote the dependency of this variables on n_t by explicitly expressing them as functions of n_t. That is, we will use the notation $a(n_t); c(n_t)$, $M(n_t)$ and $d(n_t)$.

Proposition 2.3.2. *For all i and t, the intercepts of the inverse demand schedules can be expressed as*

$$p^0_{i;t} = p_t + \frac{M(n_t)}{c(n_t)\, n_t (1 + M(n_t))} k_t + \ _t + \frac{a(n_t)}{c(n_t)} (\ _{i;t} + \ _{i;t}) ; \tag{2.3.1}$$

where $_{i;t} = \ _{i;t} \ \ _t$ is normally distributed with mean zero and variance ^{2}.

The proof of Proposition (2.3.2) can be found in the Appendix. Therefore, conditional on n_t, k_t and p_t, p^0_t follows a normal distribution with

$$E \ p^0_{i;t} j n_t; k_t; p_t \ = p_t + \frac{M(n_t)}{c(n_t)\, n_t (1 + M(n_t))} k_t + \ \ ; \tag{2.3.2}$$

and

$$\text{Var} \ p^0_{i;t} j n_t; k_t; p_t \ = \frac{a(n_t)^2}{c(n_t)^2} \ \ ^2 + \ ^2 \ + \ ^2; \tag{2.3.3}$$

for bidder i, and

$$\text{Cov} \ p^0_{i;t}; p^0_{j;t} j n_t; k_t; p_t \ = \frac{a(n_t)^2}{c(n_t)^2} \ \ ^2 + \ ^2; \tag{2.3.4}$$

for $i \neq j$.

The conditional variance and covariances are constant across all auctions with $n_t = n$, as well as the slopes of the linear demand schedules $c(n)$, and the coe cients of the linear conditional expectation: $\frac{M(n)}{c(n) n (1+M(n))}$ and . It follows immediately that the parameter , and the constants $M(n)$ and $d(n)$ are separately identi ed. Moreover, since $c(n) = \frac{n \ \ 2 \ \ M(n)}{(n \ \ 1)(1+M(n))}$, then the parameter is also identi ed. Variation in the number of bidders allows us to identify all other parameters of the model. We establish this result in Proposition 2.3.3. The proof is left to the Appendix.

Proposition 2.3.3. *Given a dataset of T independent auctions $D = \ n_t; k_t; p_t; p^0_t; c_t \ ^T_{t=1}$, if p^0_t and c_t correspond to the equilibrium linear demand schedules of model 2.2 for each t, and assumption 2.3.1 holds, then , , , , and ^{2} are identi ed when there are arbitrarily many auctions with the same number of bidders $n > 2$, for at least two di erent values of n denoted n and $\hat{n}$.*

The identification result above assumes bids are linear (inverse) demand schedules. It shows that the model in Vives (2010, 2011) is identified. Moreover, a symmetric linear Bayesian equilibrium is a common feature of several other theoretical and empirical studies of treasury auctions including Wang and Zender (2002), Armantier and Sbaï (2006), and Boyarchenko et al. (2020). However, in most countries, treasury auction rules require bidders to submit discrete bids consisting of a finite number of price-quantity pairs, as those described in Section 2.4.1. For such cases, we assume the model equilibrium from Section 3 is still a good approximation of the observed bids. More precisely, we assume that the distribution of the highest price submitted by each bidder coincides with the distribution of the intercepts of the inverse linear demand schedules predicted by the model, and that the observed cutoff price is equal to the model's equilibrium market cutoff price $\hat{p}_t$ plus a normal error due to misspecification. We formally state these two assumptions below and then establish identification of the model parameters under these assumptions.

Assumption 2.3.4. *Let $p^0_{i;t}$ denote the highest price submitted by bidder i. Conditional on the number of bidders n_t, total supply k_t and the mean value $\;_t$, $p^0_{i;t}$ follows a normal distribution with mean*

$$E\left[p^0_{i;t} | n_t; k_t; \;_t\right] = \;_t + \frac{M(n_t)}{c(n_t)\, n_t\,(1 + M(n_t))} k_t; \tag{2.3.5}$$

and variance

$$\text{Var}\left[p^0_{i;t} | n_t; k_t; \;_t\right] = \frac{a(n_t)^2}{c(n_t)^2}\left(\;^2 + \;^2\right): \tag{2.3.6}$$

Moreover, for any two bidders $i \neq j$

$$\text{Cov}\left[p^0_{i;t}; p^0_{j;t} | n_t; k_t; \;_t\right] = \frac{a(n_t)^2}{c(n_t)^2}\;^2; \tag{2.3.7}$$

Assumption 2.3.5. $p^c_t = \hat{p}_t + \;_t$*, and* $\;_t \sim N\left(0; \;^2\right)$

Proposition 2.3.6. *Under assumptions 2.3.4 and 2.3.5, and given a dataset of T independent auctions $\left\{n_t; k_t; p_t; p^0_t; p^c_t\right\}_{t=1}^T$, if there are arbitrarily many auctions with the same number of bidders $n > 2$, for at least two different values of n (denoted n and $\hat{n}$) the parameters $(\;;\;;\;;\;;\;;\;)$ are identified.*

In the model, the market cutoff price is $\hat{p}_t = \frac{1}{n_t}\sum_{i=1}^{n_t} p^0_{i;t} - \frac{k_t}{n_t c(n_t)}$, hence assumption 2.3.5 implies that conditional on n_t, k_t, and the vector of price bids p^c_t, the observed cutoff price is normally distributed with mean

$$E\left[p^c_t | n_t; k_t; p^0_t\right] = \frac{1}{n_t}\sum_{i=1}^{n_t} p^0_{i;t} - \frac{k_t}{n_t c(n_t)}: \tag{2.3.8}$$

It follows that $c(n)$ and $c(\hat{n})$ are identified, even if they are not directly observable. The rest of the proof of Proposition 2.3.6 is identical to that of Proposition 2.3.3.

The parametric assumptions of the model (2.2) allow us to easily estimate its parameters by maximum likelihood. The vector p_t^0 is normally distributed conditional on n_t, k_t and p_t. Moreover, the observed cutoff price p_t^c is also normally distributed conditional on n_t, k_t, p_t and p_t^0. The respective conditional densities can be used to derived the likelihood function and estimate the model parameters, even in the presence of unobserved auction heterogeneity. The likelihood function is included in the Appendix.

2.4 Data

We apply our method to the Colombian sovereign debt market. As with many other sovereign fixed-income markets, the Colombian debt market is organized around a set of primary dealers (PDs). They are large financial institutions that directly participate in the primary market, where bonds are offered by the issuer, namely, the sovereign government. PDs are expected to participate in the primary market by purchasing securities that they can hold in their balance sheets or resell to other dealers or customers in the secondary market. There are 14 PDs in our sample. All of them are commercial banks, financial corporations, or broker-dealers. For more details on the PD program in Colombia, see Cardozo (2013) and Williams (2018).

2.4.1 Primary market

Colombian government bonds (TES) are offered at biweekly prescheduled auctions. TES are coupon-paying government securities issued in the local currency with a maturity of more than one year. The primary market for each bond is an auction conducted by the Central Bank of Colombia on behalf of the Colombian Treasury. Only PDs (which we indistinctly call bidders) designated by the Colombian Treasury can participate in the primary market. Before the auction is held, the Central Bank announces the par value of the bond offering, that is, the bond supply. When an auction opens, every PD may submit a sealed demand schedule that consists of a set of yield-quantity pairs. Equivalently, each demand schedule can be characterized as price-quantity pairs from the submitted yields. A single price-quantity pair within a given bid is a bid-step. When the auction closes, the Central Bank sorts all the individual bid-steps from the highest to the lowest prices. Then, it adds up all the submitted quantities cumulatively. The market-clearing or cutoff price is set at the point where aggregate demand (the cumulative sum of submitted quantities) meets the bond supply (the total amount offered at the auction). Thus, all pairs above or at the cutoff price are winning pairs. The auctions follow a uniform price mechanism, meaning that PDs pay exactly the cutoff price for all their winning price-quantity pairs, regardless of the actual prices (yields) they submitted. The auction results are announced on the same day as the offering, which is also when PDs receive their share of won securities.

The Colombian government has restricted the number of different bond issues it offers to a few benchmarks annually. Thus, rather than auctioning off a new five-year bond every two weeks, the government usually reopens the offering of an existing bond with a maturity of approximately five years on auction days. Except for newly issued bonds, an identical TES bond is traded simultaneously in the secondary debt market before and after the auction. Reopening existing bond offerings is a method frequently employed by treasuries in Sweden, Norway, and Italy, among others (Nyborg et al. (2002); Valseth (2013); Sigaux (2018)). Moreover, some US treasury bills and notes are frequently reopened (Fleming (2002); Lou et al. (2013)). Given that all PDs observe trading activity in the secondary market for a to-be-offered bond prior to entering a reopening auction, bidders may use the secondary market price to extract information about the security's fundamental value. In Section (2.3), we use a bond's secondary market price as a commonly known signal of its ex post value when estimating the model.

Bids

Our sample spans all TES auctions between 2013 and 2015. It contains 154 auctions in total, for nine different bonds that were either auctioned off for the first time or reopened. For a given auction, we observe the complete set of yield-quantity pairs submitted by each PD. We convert the yields to prices using the bond's face value (100), coupon payments, time to maturity, and day-count convention (NL/365), consistent with the guidelines of the Colombian treasury. Thus, for bidder i in auction t, we have H price-quantity pairs, denoted by $\{p_{i;t;h}; x_{i;t;h}\}_{h=0}^{H-1}$.

Table (2.1) presents some summary statistics regarding bidding data. On average, 12.4 bidders participate in an auction and each bidder submits around four price-quantity pairs.

Number of bidders per auction

As described in Section (2.3), we take advantage of the variation in the number of bidders that participate in a given auction both before and after the rebalancing announcement on March 19, 2014.[4] Table (2.2) presents the total number of bond auctions with a given number of bidders both before and after the rebalancing announcement. Up to 14 bidders may participate in any given auction. This value corresponds to the number of PDs designated by the Colombian government in our sample. It is immediately seen from table (2.2) that more than ten bidders participate in the vast majority of auctions before the rebalancing announcement and in all of them afterwards. Thus, at least 71% of all possible PDs submit bids to the primary market for most bond offerings..

[4] See Section (2.5) for more details on the rebalancing announcement.

Table 2.1: Summary statistics

Panel A: All sample

Variable	Mean	Std. dev.
Highest bid over cuto price (bp)	53.4	34.2
Number of participants	12.4	1.2
Number of price-quantity pairs	4.1	1.4
Total supply (bill. hundr. COP)	1.9	1.0

Panel B: Pre-rebalancing sample

Variable	Mean	Std. dev.
Highest bid over cuto price (bp)	49.0	40.7
Number of participants	12.2	1.3
Number of price-quantity pairs	3.9	1.6
Total supply (bill. hundr. COP)	1.8	1.2

Panel C: Post-rebalancing sample

Variable	Mean	Std. dev.
Highest bid over cuto price (bp)	57.2	27.1
Number of participants	12.6	1.0
Number of price-quantity pairs	4.4	1.0
Total supply (bill. hundr. COP)	1.9	0.7

Our sample spans all Colombian treasury auctions of all sovereign bonds in locally-denominated currency (COP) between 2013 and 2015. The pre-rebalancing sample corresponds to all auctions held before the J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014. "Highest bid over cuto price" corresponds to the highest submitted bid by a bidder over the auction's cuto or market-clearing price in bp. "Number of participants" corresponds to the e ective number of bidders who submitted demand schedules in an auction. "Number of price-quantity pairs" is the total number of price-quantity pairs submitted in an auction. "Total bond supply" is the pre-announced par value of the bond to be issued in the auction in hundreds of billions of COP. The average nominal spot exchange rate in March 2014 was 2,000 COP per 1.00 USD.

Table 2.2: Number of auctions per number of bidders

Number of bidders	Pre-rebalancing (1)	Post-rebalancing (2)
8	2	0
9	0	0
10	7	3
11	6	7
12	27	22
13	19	34
14	11	16
Total	72	82

This table presents the total number of auctions with a given number of bidders before and after the J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014. At most, up to fourteen bidders (also known as designated Primary Dealers) may participate in any given auction. Our sample spans all Colombian sovereign bond o erings between 2013 and 2015.

2.4.2 Secondary market

Primary dealers in the Colombian secondary debt market act as intermediaries between the bond issuer (treasury) and investors (either end-investors or other dealers), although they usually also keep a fraction of the purchased security in their own books.[5]

PDs may trade in the secondary market with di erent sets of customers, which can potentially give rise to di erences in clients' order- ow across bidders. In turn, heterogeneous clients' order- ow may result in dealers having private information about fundamental values in Treasury markets, as argued in Boyarchenko et al. (2020).[6]

We have access to all transactions in the main interdealer broker (IDB) market for TES bonds between 2013 and 2015.[7] A transaction from the IDB market is recorded as a vector that contains

[5] TES holders are mainly pension funds, commercial banks, brokerage rms, foreign funds, and public institutions.

[6] Green (2004), Brandt and Kavajeckz (2004), Pasquariello and Vega (2007), and Czech et al. (2020) investigate the relationship between order- ow imbalances and Treasury returns.

[7] There are three main di erent trading and registering systems to transact TES bonds in our sample: the Electronic Negotiation System (known as SEN in Colombia), the Colombian Electronic Market (MEC), and the Colombian Electronic Market Register Module (MEC-R). As discussed in Leon et al. (2014), SEN, a system managed and operated by the Central Bank, is the principal interdealer trading and negotiation platform for TES bonds in Colombia. PDs have exclusive access to this platform, where they can trade and settle anonymously without counterparty limits. More than 50 percent of total trading in TES bonds occurs within this system Survey (2011) . The Electronic Negotiation System is the main IDB that we refer to above in the text. See Leon et al. (2014) for more details on di erent trading systems in which PDs can transact TES bonds.

the transaction's clearing time stamped to the nearest second, its nominal value, its price, and an (anonymized) identifier for the dealers involved in the transaction.

We also gather the prices of the bonds offered from Bloomberg. Specifically, we collect Bloomberg's daily mid-quote for the opening- and end-of-day prices for all nine bonds offered in our sample. These prices are merely suggestive, as they are not necessarily the prices at which dealers or investors may actually transact. Nevertheless, Bloomberg's prices for TES bonds are calculated using information from different platforms and trading facilities in the Colombian secondary debt market; therefore, we include them in our information set as well.

We use all transactions in the main IDB market from the start of the trading day until immediately before an auction opens to compute the quantity-weighted price of the bond to-be-offered. We interpret this price as a proxy for the commonly known signal about the bond's fundamental value. We also perform robustness tests using prices from Bloomberg as an alternative secondary market price benchmark. We exclude bids from three auctions in which we do not observe a secondary market price because the bond is auctioned off for the first time, which reduces our effective sample to 151 auctions.

2.5 Application: Rebalancing

On March 19, 2014, J.P. Morgan announced that it would rebalance the weights in its JPM GBI-EM Global and JPM GBI-EM Global Diversified debt indexes. These indexes are composites of locally denominated government debt instruments issued in emerging markets and tracked by funds that managed approximately $200 billion (see Hong and Molinski, 2014). Colombia's weight in the JPM GBI-EM index was boosted by the inclusion of five additional Colombian sovereign bonds.[8] The rebalancing increased Colombia's total weight in the index from 3.24 to 7.69 percent. According to Arslanalp and Tsuda (2015), approximately 97.7 percent of the weight's increase was a consequence of the inclusion of new TES bonds, whereas the remaining portion was due to valuation effects. Romero et al. (2020) estimate that 80% of all offshore investors with significant dealings in Colombia use this index as a benchmark.

Williams (2018) documents that the share of debt held by foreigners in the Colombian debt market rose from 8.5 percent in March 2014 at the time of the announcement to 19 percent in September 2014. Similarly, Arslanalp and Tsuda (2015) estimate that the net foreign purchases of Colombian government bonds were USD $7.36 billion from March to September 2014, a much larger figure than in previous years. Moreover, Williams (2018) provides evidence that the rebalancing significantly impacted the balance sheet of Colombian PDs. After the rebalancing, PDs reduced their domestic sovereign debt holdings by 7.8 percentage points, as a fraction of their total assets,

[8] The five new Colombian bonds that were included in the JPM GBI-EM index had maturity in 2016, 2018, 2022, 2024 and 2028. All five bonds were offered during our sample.

relative to non-PD commercial banks.

In summary, the J.P. Morgan rebalancing implied a positive demand shock for Colombian sovereign debt, mostly driven by foreign funds tracking the indexes. This shock may have altered the information structure of the primary market for Colombian government bonds by redistributing the share of customers' order flows to which different dealers and PDs have access. In Section (2.5.1), we provide reduced-form empirical evidence of changes in the information structure of the Colombian government debt market based on observed patterns in secondary bond markets around the rebalancing.

2.5.1 Changes in the information structure around the rebalancing: evidence from the secondary market

In this section, we document changes in trading conditions in the Colombian secondary debt market after the rebalancing announcement. We study potential shifts in three separate, but related dimensions: total trading activity, liquidity, and price volatility. Below, we discuss why changes along these dimensions are consistent with variations in the market's information structure.

We begin by fitting the following panel data model

$$\text{trad}_{l,t} = \alpha_l + \beta_r \text{rebal}_t + \gamma' \text{controls}_{l,t} + \varepsilon_{l,t}, \tag{2.5.1}$$

where $\text{trad}_{l,t}$ is a secondary market measure for bond l on day t of the total trading activity, liquidity, or volatility. rebal_t is equal to one after the rebalancing date and zero otherwise. We add bond fixed-effects and control for time-varying bond characteristics (duration and convexity), market-wide conditions (10-year CDS on the Colombian sovereign government, and aggregate volatility proxied by the VIX). Only bonds auctioned off both before and after the rebalancing announcement are included in the panel.

Table (2.3) presents the estimation results. We use the daily total number of transactions (column 1) and total traded volume at par value (column 2) as measures of total trading activity. For liquidity, we calculate Amihud (2002)'s measure using transaction data from the main interdealer market (column 3), and the daily absolute bid-ask spread using Bloomberg's end-of-day bid and ask prices (column 4).[9] Finally, we construct two measures of daily price volatility. The first corresponds to realized volatility, that is, the sum of squared intraday returns using adjacent transaction prices from the main interdealer market (column 5). The second is equal to the fitted values of the variance equation of a GARCH (1,1) model on bond daily returns using the end-of-day midquotes from Bloomberg (column 6). Our sample spans the period between 2013 and 2015.

[9] These variables have been frequently used in the literature to measure bond market liquidity (see Schestag et al. (2016)) Amihud's measure is based on changes in trading prices from adjacent transactions, relative to the size of the transaction. We restrict our attention to these two liquidity proxies because our data do not include the sign of any transaction.

The results in column (1) of table (2.3) indicate a signi cant increase in the daily number of transactions in the main interdealer market after the rebalancing announcement for any of the o ered bonds. Column (2) shows similar results for the daily traded volume. Relative to the sample mean, there was an increase of approximately 40% in the daily total trade for any bond regardless of the chosen indicator.

We further investigate whether this increase in total trade is associated with a change in overall market liquidity.[10] Column (3) in table (2.3) shows a signi cant reduction in average Amihud's illiquidity measure after the rebalancing (approximately 30% relative to the sample mean) for any traded bond. The result is similar when we use the absolute bid-ask spread from Bloomberg in column (4), both in terms of direction and magnitude with respect to its own sample mean.

Finally, we explore whether there was a concurrent change in the price volatility of the secondary market after the rebalancing announcement. The estimation results in columns (5) and (6) of table (2.3) show that, regardless of the proxy, secondary market volatility signi cantly decreased after the rebalancing announcement. This result is particularly sharp for the daily realized volatility.

In summary, following the shock, Colombian sovereign bonds are more actively traded, in a more liquid secondary interdealer market, with lower price volatility. Previous works in the market microstructure literature have argued that one prominent motive for dealers to trade with each other is to share their inventory risk (Hansch et al. (1998) and Reiss and Werner (1998)). That is, dealers manage their inventory imbalances resulting from recent transactions with customers by trading in the interdealer market. Therefore, a substantial rise in interdealer trading suggests, at least indirectly, more trading between PDs and their customers.[11] Consequently, price discovery in the secondary market may be enhanced with the increase in trading volumes that followed the rebalancing announcement (see, for example, Barclay and Hendershott (2015)). Moreover, increased liquidity may lead to more informationally e cient markets that aggregate private information more rapidly (Admati and P eiderer, 1988; Chordia et al., 2008). Secondary market volatility may also a ect the market information structure. For instance, Pasquariello and Vega (2007) argue that the correlation between unanticipated order- ow and future bond yields is higher in periods of high volatility. In their setting, order- ow reveals private information, which is then impounded into market prices.

Hence, the trading patterns documented in this section are consistent with changes in the information structure. As private information is not directly observable, we estimate the model in Section 2.2 separately before and after the rebalancing to test for changes in the importance of

[10]Pandol and Williams (2019) argue that increases in capital in ows arising from mechanical changes in the composition of sovereign debt indexes bring about liquidity improvements in secondary markets of emerging economies. Similarly, (Hegde and McDermott, 2003) nd a sustained increase in the liquidity of stocks added to the S&P 500.

[11]We cannot provide direct evidence of changes in dealer-customer trading because we do not observe customer orders in our data.

private information about fundamentals in both environments.[12]

Table 2.3: Secondary market changes after rebalancing announcement

	Dependent variable:					
	Number of trans.	Total volume	Amihud's meas.	Abs. bid-ask	Volat. (IDB)	Volat. (Bl.)
	(1)	(2)	(3)	(4)	(5)	(6)
Post-rebal	18.088	0.728	36.099	12.406	0.082	0.099
	(7.544)	(0.335)	(12.952)	(3.203)	(0.034)	(0.029)
Duration	14.176	0.508	94.605	4.823	0.003	0.089
	(7.515)	(0.340)	(15.982)	(2.389)	(0.037)	(0.027)
Convexity	2.886	0.108	8.720	0.253	0.009	0.001
	(0.724)	(0.032)	(2.956)	(0.279)	(0.006)	(0.005)
CDS (10-year)	0.345	0.015	0.792	0.033	0.001	0.001
	(0.061)	(0.003)	(0.150)	(0.023)	(0.0004)	(0.0003)
VIX	0.080	0.004	2.687	0.170	0.009	0.001
	(0.560)	(0.023)	(0.915)	(0.156)	(0.002)	(0.001)
Sample mean	59.94	2.53	121.94	36.87	0.12	0.32
Units	Scalar	HBC	bps/HBC	bps	Perc.	Perc
Bond FE	Yes	Yes	Yes	Yes	Yes	Yes
Observations	5,002	5,002	3,384	4,815	4,103	4,456
R^2	0.209	0.193	0.372	0.493	0.122	0.590

Note: (*) Signi cant at 10% level; (**) signi cant at 5% level; (***) signi cant at 1% level. HBC stands for hundreds of billions of COP. This table displays the estimated coe cients of model (2.5.1). Dependent variables are secondary market measures. All dependent variables are at the daily frequency and at the bond level. Dependent variables are: total number of transactions in the main inter-dealer (IDB) market (column (1)); total traded volume in IDB market (column (2)); Amihud (2002)'s liquidity measure (column (3)); absolute bid-ask spread from Bloomberg end-of-day quotes (column (4)); realized volatility calculated as sum of squared of intraday returns using transaction prices from main IDB market (column (5)); tted values of a GARCH (1,1) model on bond daily returns using the end-of-day midquotes from Bloomberg (column (6)). Post-rebal. is dummy variable equal to one after J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014, and zero otherwise. Duration and convexity are bond-speci c at the daily frequency. CDS (10-year) is the 10-year CDS on the Colombian sovereign government. VIX is Chicago Board Options Exchange's CBOE Volatility Index. CDS and VIX are at the daily frequency and they take the same value for all bonds in the panel on a given trading day. Standard errors are clustered at the bond level.

[12] We do not claim to provide empirical evidence that the rebalancing announcement was the main cause of the fundamental changes in the Colombian secondary debt market documented here. Instead, we use the announcement date to test the ability of our empirical methodology to detect changes in the information structure of the market, given the reduced-form evidence suggesting that such changes at least concurred with the rebalancing announcement.

2.5.2 Maximum-likelihood estimation results

We estimate the model separately using bidding data from Colombian treasury auctions before and after the JPM GBI-EM index rebalancing announcement on March 19, 2014. The pre-rebalancing sample starts on January 1, 2013, and ends on March 19, 2014. The post-rebalancing sample spans March 20, 2014, through December 31, 2015. There are 72 and 82 bond offerings in the pre- and post-rebalancing subsamples, respectively. However, for some auctions, there is no secondary market price benchmark available, which reduces our sample sizes to 66 and 81 auctions (799 and 1024 bids) before and after the rebalancing, respectively.[13] For each parameter in the model, we test the null hypothesis that it would not change after the rebalancing. The estimation results are presented in table (2.4). All parameters are statistically different in both subsamples at the 1% level, except for .

, the correlation parameter in bidders' valuations conditional on their commonly known mean $_t$, falls substantially after the rebalancing, from 0.67 to 0.22. In the model, all else being equal, a lower implies that there is more precise public information about fundamentals (or, equivalently, lower variance of the unknown common component in bidders' valuations). Moreover, the standard deviation of the distribution of private signals, , also falls from 0.86 to 0.49 following the rebalancing. Therefore, bidders are less uncertain about their valuations. This implies that, from the standpoint of any bidder i, other bidders are less privately informed about its valuation. As the variance of bidders' valuations, ^{2}, remains roughly unaltered (relative to the change in), both the decline in and imply that private information about fundamentals is less relevant as a determinant of bids after the shock.

Table (2.4) also displays the estimates of the other four parameters of the model that are not related to private information. The estimated value of , which captures a bidder's valuation of one additional unit of the asset, increases from 0.18 to 2.28. This implies that bidders have much steeper decreasing marginal valuations after the rebalancing.

The estimation results also indicate that P_t is a biased but otherwise precise proxy of $_t$ because, even though is positive, is very close to zero, and not statistically significant, in both subsamples.[14] Finally, , the standard deviation of the difference between the observed and model-implied auction cutoff price due to potential misspecification, drops by 8 bp after the rebalancing.

---

[13] Some auctions correspond to first-time bond offerings. Bonds with maturities in 2018, 2019, 2028, and 2030 were offered for the first time in February 2013, January 2014, January 2013, and January 2015, respectively. We do not use these auctions to estimate the model because there are no secondary markets for these bonds prior to auctions.

[14] By assumption, the mean of the bidder's private valuation, $_t$, is common knowledge to all bidders but is unknown to the econometrician who instead only observes a noisy proxy P_t, as described in Section (2.3).

Table 2.4: Estimation results before and after rebalancing

	Pre-rebalancing	Post-rebalancing	Di erence in sub-samples
	(1)	(2)	(3)
	0.67***	0.22***	0.44***
	(0.04)	(0.07)	(0.08)
	0.8***	0.38***	0.41***
	(0.06)	(0.13)	(0.14)
	0.49***	0.57***	-0.08***
	(0.02)	(0.03)	(0.04)
	0.18***	2.28***	-2.1***
	(0.03)	(0.65)	(0.65)
	0.37***	0.68***	-0.32***
	(0.05)	(0.1)	(0.11)
	< 0:01	< 0:01	< 0:01
	(0.15)	(0.18)	(0.24)
	0.34***	0.26***	0.08***
	(0.03)	(0.02)	(0.04)
Number of auctions	66	81	147
Number of obs.	799	1024	1823

Note: (*) Signi cant at 10% level; (**) signi cant at 5% level; (***) signi cant at 1% level. This table presents the results of the estimation of the model before and after the J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014. Columns (1) and (2) report the maximum likelihood estimated parameters of the model described in 2.3. Standard errors reported in parentheses below estimated parameters. Standard errors are calculated using the numerical Hessian from the maximum likelihood estimation in Section (2.3). Statistical signi cance for every parameter was established using the reported estimated value along with its associated standard error through a t-test with degrees of freedom given by the number of auctions in the last row of the table. Column (3) reports the estimated di erence between the parameters before and after the rebalancing. To compute this di erence, we construct an augmented likelihood function that spans both sub-samples where, we split the set of parameters to be estimated through the augmented likelihood in two groups. The rst group corresponds to the vector of parameters in column (1) of table (2.4) for all auctions before the rebalancing. For every auction after the rebalancing, there is a transformed group of parameters de ned by the di erence in every parameter before and after the rebalancing. We estimate changes in every parameter in both sub-samples using the transformed vector of di erences. Standard errors reported in parentheses below estimated parameters. Standard errors constructed using the Hessian of the likelihood function evaluated at the estimated vector of parameters. Number of auctions corresponds to the number of bond o erings with available secondary market pricing information. Number of observations corresponds to the sum of all bidders that participated in every auction with available secondary market pricing information.

2.5.3 Private information, market power and number of bidders

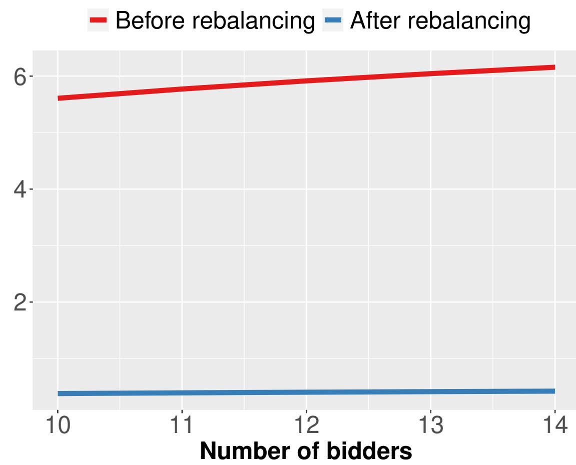
In this section, we explore the behavior of the endogenous equilibrium variables in the model before and after the rebalancing. Specifically, based on the maximum likelihood estimates from table (2.4), we calculate $M(n)$, $c(n)$, and $d(n)=n$ for multiple bidders n in these two scenarios. The results of this exercise are plotted in figure 2.1.

Panel (a) plots $M(n)$ as a function of n. As discussed in Section 2.2.2, $M(n)$ captures the amount of bid shading attributable to private information. The results indicate that $M(n)$ is higher before the rebalancing announcement than afterward for any n. This result is expected considering that $M(n)$ is a decreasing function of and the ratio = , both of which decrease after the rebalancing. All else being equal, the estimated change in private information thereby implies a reduction in the bidders' market power in the Colombian market.

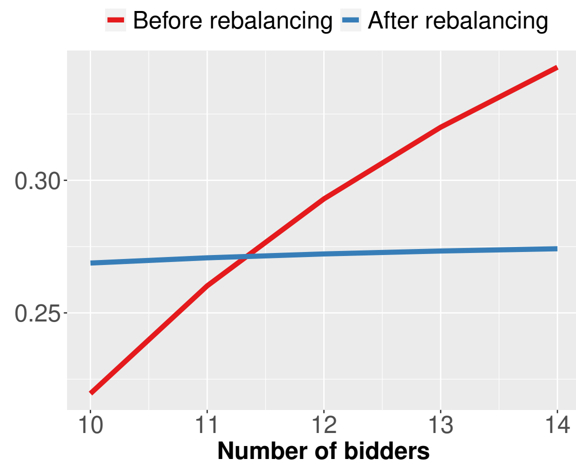
Panel (b) of figure 2.1 depicts $c(n)$, the slope (in absolute value) of the equilibrium demand schedule in Eq. (2.2.1) as a function of the number of bidders. Higher values of $c(n)$ imply more elastic demand schedules. For low values of n, the bidders submit more inelastic demand schedules before the rebalancing. However, for n 12, the opposite result is obtained. We observe a similar pattern for $\frac{d(n)}{n}k$ in panel (d). As discussed in Section (2.2.2), $\frac{d(n)}{n}k$ is a measure of bid shading or, equivalently, market power. Our results indicate that bid shading is slightly larger before the announcement for low n. However, this situation reverses when the number of bidders increases. Moreover, the absolute estimated change in bid shading is never larger than two bp.[15] Therefore, according to our estimates, bidders' market power remains mostly unaltered after the rebalancing.

At first glance, the results for $c(n)$ and $\frac{d(n)}{n}k$ might seem at odds with our discussion above regarding the change in $M(n)$. Given the evidence of less private information after the rebalancing, we might have expected that bidders would submit more elastic demand curves, thereby reducing their market power. The reason for this apparent discrepancy is that we also observe a substantial increase in , which is the slope of the decreasing marginal valuations. When n is large, the increase in more than offsets the impact of lower values of $M(n)$ on the elasticity of the demand schedule and, thus, on bid shading.

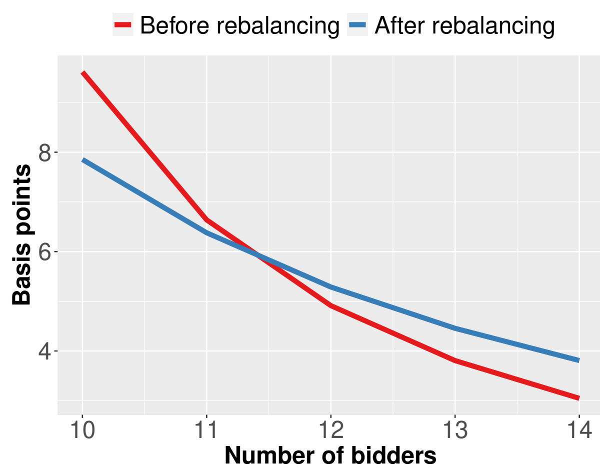
[15] Bid shading is expressed in basis points (bp) of the par value of the issue. It can be interpreted as a \discount" relative to the par amount.



(a) $M(n_t)$



(b) $c(n_t)$



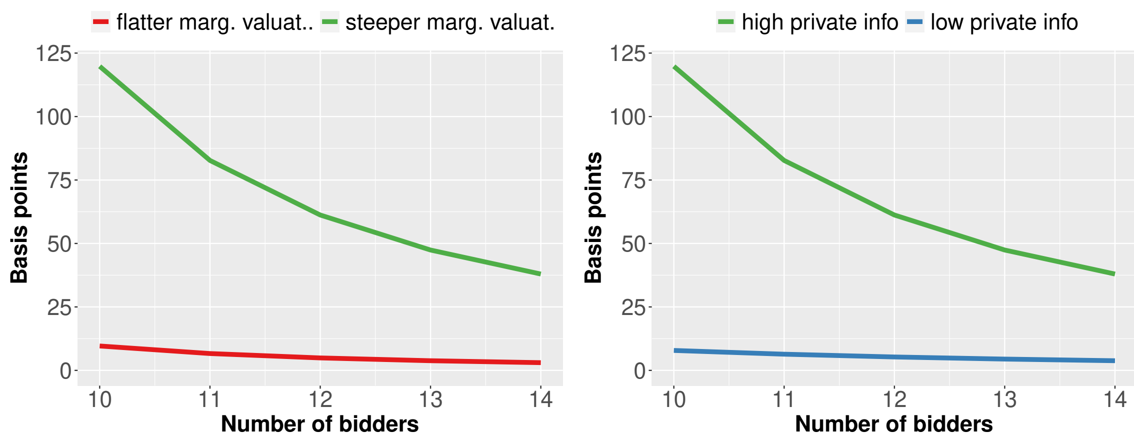
(c) Bid shading: $\frac{d(n_t)}{n_t} k_t$

Figure 2.1: Model equilibrium as a function of the number of bidders and parameter estimates

This gure plots the equilibrium of the model across multiple n, the number of bidders who participate in a given auction. The choice of n in all panels is guided by table (2.2). $M(n)$ and $c(n)$ are calculated according to Eq. (2.2.1) for a given n. Bid shading equals $d(n){=}nk$ for a given k, and $d(n)$ is calculated according to Eq. (2.2.1). k is evaluated at the sample mean of total supply from table (2.1). The parameters , , , and come from our estimates in table (2.4) before and after the J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014. The blue line corresponds to the $M(n)$, $c(n)$, and bid shading with the estimates from column (1) of table (2.4). The red line depicts 2.3. The red line depicts the same variables with the estimates from column (2) of table (2.4).

To further illustrate the economic impact of these countervailing forces, in gure (2.2), we plot the total bid shading, $d(n)k{=}n$, for di erent combinations of the estimates in table (2.4). In panel (a), we hold the private information parameters constant at their pre-rebalancing estimated values and plot $d(n)k{=}n$ for two di erent values of : its pre- and post-rebalancing estimated values, respectively. Clearly, the change in alone implies a substantial increase in bid shading, ranging from approximately 37 bp to 120 bp, depending on the number of bidders. For comparison, the average absolute bid-ask spread from Bloomberg for all o ered bonds in our sample is approximately 36 bp (see table (2.3)).

In panel (b) of gure (2.2), we keep constant at its post-rebalancing value but let the private information parameters drop from their pre-rebalancing to their post-rebalancing estimates. The reduction in private information has the opposite e ect on bid shading compared with the increase in . The combined e ect is shown in panel (c) of gure 2.1.



(a) Change in (high private info) (b) Change in private info (high)

Figure 2.2: Decomposition of bid shading into estimates of private information and slope of marginal valuation ( )

This gure plots the estimated equilibrium bid shading of the model as a function of the number of bidders. Bid shading is expressed in bp of the par value of the bond o ering. Bid shading equals $d(n)$=nk. n, the number of bidders who participate in a given auction, comes from table (2.2). k is evaluated at the sample mean of total bond supply from table (2.1). $d(n)$ is calculated according to Eq. (2.2.1) for a given n and combinations of the estimated parameters from table (2.4). High and low private information correspond to scenarios where the amount of private information is set at the estimates of , , and from the pre- and post-rebalancing samples, respectively. Flatter and steeper marginal valuation correspond to scenarios where is xed at pre- and post-rebalancing estimates, respectively.

2.5.4 Policy implications

Arguably, an auctioneer's goal is to increase revenue by sustaining high auction prices. Bidders' market power hinders this goal. The model in Section (2.2) predicts an unambiguous reduction in bid shading as the number of bidders increases. Hence, the auctioneer may want to consider a policy that fosters primary market participation. Moreover, the structural estimation in Section 2.3 allows us to quantify the change in bid shading that results from increased participation. However, increasing the maximum number of bidders is likely costly or not even feasible, as they expect compensation from their role as market makers. Thus, quantifying the bene ts to the bond issuer from increasing competition through lower bidders' market power is necessary to determine the optimal number of bidders.[16]

To illustrate this use of the model estimation in policy analysis, we consider two cases. The first is the estimated post-rebalancing scenario, characterized by low private information and high inventory costs. The second is a hypothetical but empirically feasible high market power scenario with high levels of private information matching the pre-rebalancing estimates and the same inventory costs from the post-rebalancing scenario. They are depicted in figure (2.3).

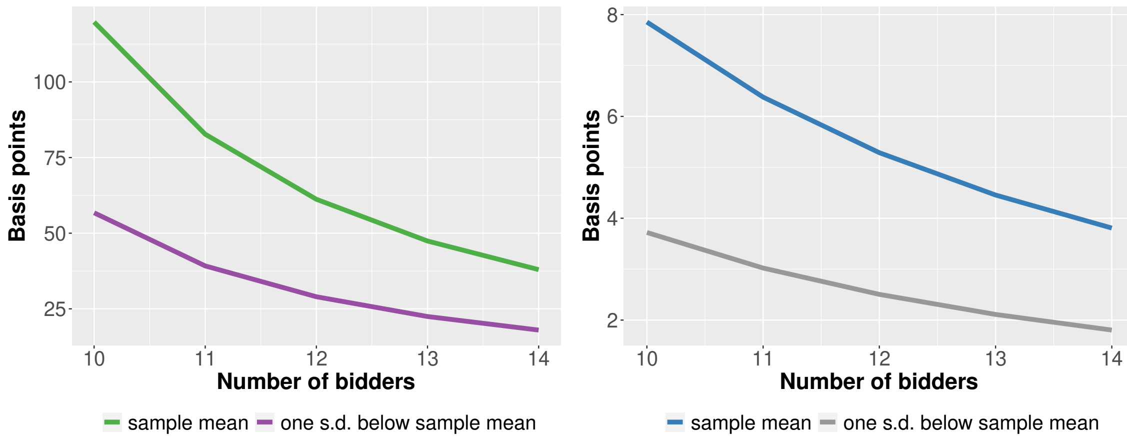
We first focus on bid shading evaluated at the sample mean of bond offering per auction. In the high market power scenario in panel (a), even though bid shading is never lower than 30 bp for any number of participants, it would fall by more than 80 bp if the number of bidders increases from 10 to 14. Therefore, with high market power, fostering market participation is a policy worth considering. In contrast, in the post-rebalancing scenario in panel (b), bid shading falls by only 4 bp in the same situation. In such situations, designating a small number of PDs would not result in significantly higher government borrowing costs.

Another implication of Vives (2010, 2011) is that the auctioneer should monitor current market conditions when deciding the size of each bond offering (per auction).[17] As shown in Figure 2.3, reducing the bond offering in the high market power scenario would result in a substantial decrease in the total bid shading for any number of participants. In particular, a one standard deviation decrease in total bond supply over its sample mean would lower bid shading by 20 bp with 14 bidders and by 63 bp with ten.[18] In contrast, in the post-rebalancing scenario, bid shading does not fall by more than 4 bp for any number of bidders between 10 and 14, following a one standard deviation decrease in bond supply over its sample mean.

[16] As of 2022, Norway, Sweden, and Luxembourg have, respectively, designated only four, eight, and nine financial institutions as PDs. In contrast, there are 44 PDs in Germany's debt market. See https://www.esma.europa.eu/sites/default/files/library/list_of_market_makers_and_primary_dealers.pdf.

[17] In the long run, total bond issuance is a fiscal policy decision. The policy evaluated here focuses on the size of each bond offering at a specific auction given a predetermined fiscal policy.

[18] A one standard deviation in total bond supply per auction corresponds to roughly 100 hundred billion Colombian pesos (approximately 50 billion USD as of March 2014). See panel A of table (2.1).



(a) High market power (b) Post-rebalancing estimates

Figure 2.3: Bid shading evaluated at different levels of total bond supply

This figure plots the estimated equilibrium bid shading of the model as a function of the number of bidders. Bid shading is expressed in bp of the par value of the bond offering. Bid shading equals $d(n)=nk$. n, the number of bidders who participate in a given auction, comes from table (2.2). k is evaluated at the sample mean and one standard deviation (s.d.) below the sample mean of the total bond supply per auction, respectively. The sample means come from table (2.1). $d(n)$ is calculated according to Eq. (2.2.1) for a given n and combinations of the estimated parameters from table (2.4). The high market power scenario corresponds to scenarios where the estimates of , , and are from the pre-rebalancing sample, and is set at the post-rebalancing estimate.

2.6 Final remarks

We propose a method to empirically quantify the importance of private information in divisible good uniform price auctions based on the estimation of a model presented in Vives (2010, 2011). We apply our method to the Colombian sovereign debt market after the rebalancing of two emerging market debt indices that triggered an increase in the demand for Colombian sovereign bonds. We find that these bonds were traded more actively after the shock, and the secondary market became more liquid and experienced a significant decrease in volatility. We estimate the model separately before and after the shock.

We find a substantial reduction in private information about fundamentals as a determinant of observed bids but also a sizeable increase in inventory costs. These two findings have opposing effects on market power. Our estimation results indicate that the change in revenue for the treasury is less than two bp of the par value of the bond. However, the latter masks the separate effects of private information and inventory costs. We decompose these two effects and find that a reduction in private information implies an increase in revenue of 34–112 bp. In contrast, the increase in inventory costs lowers revenue by 35–110 bp.

We quantify the effects of private information and inventory costs on policies aimed at lowering treasuries' borrowing costs. After the rebalancing, our structural estimates imply that increasing the number of bidders from 10 to 14 reduces bid shading by four bp. In contrast, in a hypothetical high private information and inventory scenario, market power falls signi cantly when the number of bidders increases from 10 to 14.

2.7 Appendix

Proof of Proposition 2.3.2

It can be shown from the equilibrium of the model in proposition (2.2.1) that

$$c(n_t) \quad a(n_t) \;=\; \frac{1}{1+M(n_t)} \quad \frac{\frac{2}{2}}{\frac{2}{2} + (1+(n_t \quad 1)\;)} (d(n_t) + \;)^{\;1} :$$

Inserting this result into the equilibrium intercept in proposition (2.2.1) and re-arranging terms

$$b(n_t) \;=\; \frac{M(n_t)}{n_t(1+M(n_t))} k_t + \quad \frac{1}{1+M(n_t)} \quad \frac{\frac{2}{2}}{\frac{2}{2} + (1+(n_t \quad 1)\;)} (d(n_t) + \;)^{\;1} \bar{\;}_t$$

$$= \frac{M(n_t)}{n_t(1+M(n_t))} k_t + (c(n_t) \quad a(n_t)) \bar{\;}_t :$$

The intercepts of the individual demand schedules for auction t and bidder $i \;2\; f1; : : : ; n_t g$ are given by $\;_{i;t} = b(n_t) + a(n_t)\, s_{i;t}$. In turn, the individual signals $s_{i;t}$ are $s_{i;t} = \;_{i;t} + \;_{i;t}$. Letting $\;_{i;t}$ be given by $\;_{i;t} = \;_{i;t} \quad \bar{\;}_t$ and inserting the signals into the individual intercepts, we obtain

$$_{i;t} \;=\; b(n_t) + a(n_t) \quad \bar{\;}_t + \;_{i;t} + \;_{i;t} \;:$$

Inserting the equation for $b(n_t)$ from before into the equation above and re-arranging terms

$$_{i;t} \;=\; b(n_t) + a(n_t) \quad \bar{\;}_t + \;_{i;t} + \;_{i;t}$$

$$= \frac{M(n_t)}{n_t(1+M(n_t))} k_t + c(n_t) \bar{\;}_t + a(n_t)(\;_{i;t} + \;_{i;t}) :$$

Inserting $\bar{\;}_t = \bar{p}_t + \;_t$ from assumption (2.3.5) into the equation above and dividing the resulting expression by $c(n_t)$, we obtain the desired expression for $p^0_{i;t} \quad \frac{\;_{i;t}}{c(n_t)}$, the inverse intercept of the demand schedule

$$p^0_{i;t} \;=\; \bar{p}_t + \frac{M(n_t)}{c(n_t)\, n_t(1+M(n_t))} k_t + \;_t + \frac{a(n_t)}{c(n_t)} (\;_{i;t} + \;_{i;t}) :$$

Proof of Proposition 2.3.3

Let us assume that there are in nitely large sets of auctions, each with a xed number of bidders per auction n and $\hat{n}$, respectively, where $n > 2$, $\hat{n} > 2$ and $\hat{n} \neq \hat{n}$. It follows directly from equation (2.3.2) that is identi ed and $M(n)$ is identi ed for $n \;2\; n; \hat{n}$. Hence, and $d(n)$ and are also identi ed. Now let,

$$K(n) = \frac{M(n)}{n(1+M(n))} = \frac{2}{(1+(n \quad 1)\;)\quad{}^2 + \;{}^2(1 \quad\;)} : \tag{2.7.1}$$

It follows that both $K(n)$ and $K(\hat{n})$ are identi ed. Moreover

$$\frac{K(n)}{K(\hat{n})} = \frac{1 + (\hat{n} \quad 1)}{1 + (n \quad 1)}; \tag{2.7.2}$$

hence is identi ed.

The di erence between the conditional variance and covariance of the intercepts can be expressed as

$$\begin{aligned} \text{Var}\ p^0_{i;t} jn; k_t; p_t \quad \text{Cov}\ p^0_{i;t}; p^0_{j t} jn; k_t; p_t &= \frac{a(n)^2}{c(n)^2} \quad {}^2 + \quad {}^2 \quad \frac{a(n)^2}{c(n)^2} \quad {}^2 \\ &= \frac{1 + (1 \quad) \frac{2}{2} \quad (1 + (n \quad 1) \quad)^2 \frac{2}{2} \quad {}^2}{1 + (1 + (n \quad 1) \quad) \frac{2}{2} \quad {}^2} : \end{aligned} \tag{2.7.3}$$

Moreover, from Proposition 2.2.1,

$$M(n) = \frac{{}^2 n}{(1 \quad) \quad {}^2 + (1 + (n \quad 1) \quad) \quad {}^2} : \tag{2.7.4}$$

Let us now consider two cases. If $M(n) = M(\hat{n}) = 0$, it follows from the de nition of M that $^2 = 0$. In such case, ^{2} is also identi ed from 2.7.3.

If instead, $M(n) > 0$ and $M(\hat{n}) > 0$, then $^2 > 0$ and

$$\frac{n}{(1 \quad) M(n)} \quad 1 \quad (1 + (n \quad 1) \quad) \quad {}^1 = \frac{2}{2}; \tag{2.7.5}$$

hence the ratio $\frac{2}{2}$ is identi ed. It then follows from 2.7.3 that ^{2} and ^{2} are separately identi ed.

From Proposition 2.2.1, $a(n) = \frac{(1 \quad) \quad {}^2}{(1 \quad) \quad {}^2 + \quad {}^2} (d(n) + \quad) \quad {}^1$, hence $a(n)$ is also identi ed, and the parameter ^{2} is then also identi ed from the conditional variance of the intercepts. This completes the proof that the model parameters $(\; ; \; ; \; ; \;)$ and the incidental parameters $\; ; \; {}^2$ are all identi ed.

Likelihood function

For each auction t the probability density function of p^0_t is given by

$$f^0_t \quad p^0_t jn_{t;} k_t; p_t; \quad = \frac{1}{\sqrt{(2 \quad)^{n_t} j \quad {}_t j}} \exp \quad \frac{1}{2} \quad p^0_t \quad {}_t \quad {}^T \quad {}_t^{\ 1} \quad p^0_t \quad {}_t \quad ; \tag{2.7.6}$$

where $= \; ; \; ; \; ; \; ; \; ; \; {}^2$, ${}_t$ is the vector of conditional means $E \left[p^0_{i;t} jn_{t;}; k_t; p_t \right]$ and ${}_t$ is the variance-covariance matrix with diagonal elements $\text{Var} \left[p^0_{i;t} jn_t; k_t; p_t \right] = \frac{a(n_t)^2}{c(n_t)^2} \quad {}^2 + \quad {}^2 + \quad {}^2$, for all i, and o -diagonal elements equal to $\text{Cov} \left[p^0_{i;t}; p^0_{j t} jn_t; k_t; p_t \right] = \frac{a(n_t)^2}{c(n_t)^2} \quad {}^2 + \quad {}^2$ for all $i \neq j$. Similarly, the conditional density of p^c_t is

$$f^c_t \quad p^c_t jn_{t;} k_t; p_t; p^0_t; \quad = \frac{1}{\sqrt{2 \quad {}^2}} \exp \quad \frac{1}{2} \frac{p^0_t \quad {}^c_t \quad {}^2}{2} \; ! \; ; \tag{2.7.7}$$

where ${}^{c}_{t}$ is the conditional mean $E\left(p^c_t | n_t, k_t, p^0_t\right)$. For a sample of T independent auctions $\left\{n_t, k_t, p_t, p^0_t, p^c_t\right\}^{T}_{t=1}$, the log-likelihood function is

$$L(\ ) = \sum_{t=1}^{T} \log f^0_t\left(p^0_t | n_t, k_t, p_t;\ \right) + \log f_t\left(p^c_t | n_t, k_t, p_t, p^0_t;\ \right). \tag{2.7.8}$$

Chapter 3

Validation of estimation results

3.1 Introduction

In Chapter II, we propose a method to measure the importance of private information about fundamentals as a determinant of market power in treasury auctions. This method is based on the theoretical model in Vives (2010, 2011), which has a unique equilibrium in linear demand schedules. We acknowledge that the linear equilibrium imposes restrictive parametric assumptions. Moreover, bids are assumed to be continuous linear functions. However, these assumptions allows us to estimate the auction using bidding data and readily available secondary market pricing data alone. In addition, the model allows us to decompose market power into private information and inventory costs, which would be unfeasible with non-parametric methods.[1] Furthermore, we exploit variation in the number of bidders across auctions to identify and estimate the model parameters. Such variation is assumed to be exogenous and known by all bidders in our empirical setting. In this chapter, we adopt a reduced-form approach to support the structural estimation results and the main assumptions of our proposed methodology.

According to our findings from Chapter II, there was a substantial reduction in private information about fundamentals as a determinant of observed bids in the Colombian primary market after the rebalancing announcement. Our structural results also show a considerable increase in inventory adjustment costs. In this chapter, we provide model-free evidence that supports both findings. First, auction outcomes such as market-clearing (cutoff) prices aggregate information about bidders' valuations. If private information is prevalent in this market, then the auction cutoff price would reveal bidders' private signals about fundamentals that were not already captured by observable market prices before the auction. Thus, upon observing a realization of the auction price, market participants would revise their valuation of the offered bond. We show that a high auction cutoff price relative to a pre-auction price benchmark predicts *subsequent* high secondary market

[1] See literature review in Chapter II for additional details on the discussion between parametric and non-parametric methods.

prices. Consistent with an information-revelation channel, we show that this result is significant and persistent, at least through a week after the auction. After the rebalancing, however, this effect either shrinks or vanishes altogether, which supports our view of the decline in the relative prevalence of private information.

Second, our structural results also show a large increase in inventory adjustment costs, namely, the slope of the bidders' marginal value curves. This implies that bidders should be more reluctant to deviate from their inventory target levels after the rebalancing. To test this implication, we estimate a reduced-form model of short-term (overnight) changes in inventory based on the literature on inventory risk (Madhavan and Smidt (1993); Friewald and Nagler (2016); Schultz (2017), among others). We find higher one-day mean reversion in inventories after the rebalancing, which implies lower inventory half-lives. On average, it takes bidders 11.7 days to offload half of their inventories before the rebalancing, but only 7.5 days after the shock.

Our identification and estimation strategies rely on exploiting the variation of the actual number of bidders who participate in any auction. In the model of Vives (2010, 2011), the number of participants is exogenous and known by all bidders. An implicit assumption when estimating the model is that bidders' decision to participate in a given auction is unrelated to the signal they receive regarding their valuation. In this chapter, we provide further reduced-form evidence that is consistent with this assumption. First, using pre-auction inventory as a proxy for bidders' valuation, we show that a bidder with a long (short) pre-auction inventory submits less (more) aggressive bids.[2] Then, we show that a bidder's pre-auction inventory does not predict whether such bidder will participate in a given auction. Therefore, our evidence indicates that our proxy for valuation is unrelated to the individual decision to enter an auction.

Another assumption regarding primary market participation is that bidders know the actual number of rivals they are competing against in any auction, although we do not have direct evidence to support it. We take an indirect approach to investigate how reasonable this assumption is. If bidders do not know with certainty how many competitors they are bidding against, then their equilibrium strategies cannot be a function of the actual number of auction participants. Accordingly, we split all auctions by the number of bidders. Then, for every group, we construct empirical measures of the demand slope based on the market-clearing condition and the linearity of the equilibrium demand schedules. We show that the resulting empirical demand slopes vary with changes in the number of bidders. Moreover, the empirical slopes are similar to the slopes derived directly from our structural estimates. Therefore, despite the restrictive parametric assumptions imposed by the model, the equilibrium strategies appear to respond to auction participation in a way that is consistent with the predictions of the model. We interpret this finding as suggestive

[2] (Nyborg and Strebulaev, 2004) show theoretically that differences in pre-auction position lead to heterogeneous valuations in uniform price auctions. Using data from Canadian treasury auctions, (Rydqvist and Wu, 2016) provide evidence consistent with this result.

evidence supporting the assumption that the number of bidders in a given auction is common knowledge.

The remainder of this chapter is organized as follows. In Section (3.2), we provide reduced-form evidence that is consistent with the main structural estimation results. Then, we discuss the validity of the assumptions regarding bidders' participation in Section (3.3).

3.2 Reduced-form evidence about private information and inventory costs

In this section, we provide model-free empirical evidence consistent with the main results of the structural estimation. First, we find sizable evidence of the existence of private information in the primary market before the rebalancing announcement, but significantly less so afterward. Second, we document significant changes in bidders' inventory policy that accord well with the increase in the estimates of λ, the slope of the marginal valuation function.

Lower private information: Auction's cutoff price and secondary market prices

A high realization of the auction cutoff price conveys to all market participants, including dealers in the secondary market who do not participate in the auction, that bidders received high private signals on average. Moreover, in the equilibrium with private information, the auction cutoff price aggregates information about the bond fundamental value contained in the private signals.

We investigate whether the new information aggregated by the auction predicts future secondary market prices. We measure new pricing information as the difference between the auction t cutoff price, p_t^c, and the bond secondary market price benchmark immediately before the auction, $\bar{p}_t$. Recall that $\bar{p}_t$ is the volume-weighted average price of all transactions of the bond at issue between the market opening time (8 a.m.) and the moment when the auction closes (10 a.m.). Therefore, $p_t^c - \bar{p}_t$ should mostly be driven by new information about bidders' valuations aggregated by the auction's cutoff price, which is not already impounded into observable secondary market prices.

We test whether $p_t^c - \bar{p}_t$ predicts $p_t(\tau) - \bar{p}_t$ for $\tau \in \{1, \ldots, 5\}$, where $p_t(\tau)$ is the bond's secondary market price τ days after auction t[3]. If the auction's cutoff price indeed reveals private information, then a high value of $p_t^c - \bar{p}_t$ predicts a positive $p_t(\tau) - \bar{p}_t$. Moreover, we would expect the effect on future prices to persist over time (Brandt and Kavajeckz (2004)).

To test this hypothesis, we pool all auctions in our sample and fit the following model

$$p_t(\tau) - \bar{p}_t = \beta_r(\tau)\text{rebal}_t + \beta_p(\tau)(p_t^c - \bar{p}_t) + \beta_{p,r}(\tau)\text{rebal}_t \times (p_t^c - \bar{p}_t) + \beta_n(\tau)n_t + \beta_k(\tau)k_t + \gamma(\tau)'\text{controls}_t + \varepsilon_t, \quad (3.2.1)$$

[3] For consistency, we use volume-weighted daily average transaction prices from the main Colombian interdealer market τ days after the auction as measures of secondary market prices.

for $\tau \in \{1, \dots, 5\}$, where t is a bond auction, rebal_t is a post-rebalancing dummy, n_t is the number of participants in auction t, k_t is auction's t total offering, and controls_t are bond-specific (bond duration and convexity) as well as market-wide (10-year CDS on Colombian bonds, VIX, and 1-year rate). Model (3.2.1) also includes bond fixed effects.

The results are presented in table (3.1). They indicate that the impact of $p_t^c - \overline{p}_t$ on $p_t(\tau) - \overline{p}_t$ is positive and statistically significant for all $\tau \in \{1, \dots, 5\}$ before the rebalancing. That is, a high cutoff auction price relative to the benchmark predicts high secondary market prices several days after the auction.[4]

As pointed out by Brandt and Kavajeckz (2004), when private information in opaque markets is revealed to market participants through signals like order-flow, then any resulting pricing effect should be persistent. Intuitively, in the presence of private information, market participants are prompted to revise their beliefs about fundamentals upon the observation of large trade imbalances or, as in our case, large pricing news from the auction's cutoff price. The persistence of the results in the first row of table (3.1) accords well with this intuition.

The interaction coefficient $\beta_{p,r}$ in Eq. (3.2.1) measures the change in the predictability of future prices from the auction price following the rebalancing announcement. $\hat{\beta}_{p,r}(\tau)$ is negative across all τ and statistically significant for $\tau \in \{3, 4, 5\}$. The lower persistence of the post-rebalancing effect is consistent with the auction's cutoff price, aggregating less private information about fundamentals after the rebalancing, thus validating the results from the structural estimation.[5]

Moreover, we estimate a similar model to the one in Eq. (3.2.1) where the dependent variable is the bond yield $\tau \in \{1, \dots, 5\}$ days after auction t. Furthermore, we augment Eq. (3.2.1) by controlling for the auction's bid-to-cover ratio, which is defined as the sum of quantity-bids submitted by all bidders in any given auction divided by total bond supply. Policy makers and market participants usually interpret this variable as a measure of an auction's success.[6] As shown by tables (3.4) and (3.5) in the Appendix, the results remain unaffected in both empirical models.

Higher inventory costs

The results of the structural estimation in Section 2.5.2 also show a significant increase in λ following the rebalancing, implying steeper decreasing marginal valuations. In Vives (2011), marginal valuations can also be expressed as a function of the deviation from their target inventory level $\hat{x}_{i,t}$, as $v(x_{i,t}) = \mu_{i,t} - \lambda(x_{i,t} - \hat{x}_{i,t})$. Hence, λ can be interpreted as an inventory adjustment cost. This

[4] Given that auctions occur every two weeks, by fixing τ between one and five trading days after the auction, we guarantee that no two adjacent auctions of the same bond overlap.

[5] We also estimate a similar model to the one in Eq. (3.2.1), where the dependent variable is the bond yield $\tau \in \{1, \dots, 5\}$ days after auction t. The results remain unaffected. Table (3.4) in the Appendix displays the estimated coefficients.

[6] See Beetsma et al. (2018) for an academic work on the ability of the bid-to-cover ratio to predict future yields.

Table 3.1: Auction price and secondary market prices

	Dependent variable:				
	Second. market price ms. benchm. -days after auction				
	= 1	= 2	= 3	= 4	= 5
	(1)	(2)	(3)	(4)	(5)
Cuto price ms. benchm.	1.552	1.765	1.637	2.536	2.857
	(0.318)	(0.359)	(0.344)	(0.539)	(0.839)
Post-rebal. dummy	0.218	0.009	0.018	0.396	0.082
	(0.273)	(0.292)	(0.288)	(0.317)	(0.462)
Cuto price ms. benchm. (post-rebal.)	0.637	0.727	1.018	1.866	1.737
	(0.365)	(0.451)	(0.422)	(0.572)	(0.819)
Number of bidders	0.113	0.189	0.137	0.186	0.135
	(0.056)	(0.081)	(0.076)	(0.091)	(0.121)
Bond o ering	0.046	0.050	0.035	0.030	0.126
	(0.088)	(0.107)	(0.100)	(0.111)	(0.129)
Bond FE	Yes	Yes	Yes	Yes	Yes
Bond controls	Yes	Yes	Yes	Yes	Yes
Macro controls	Yes	Yes	Yes	Yes	Yes
Observations	147	147	147	147	147
R^2	0.381	0.373	0.366	0.436	0.373

Note: (*) Signi cant at 10% level; (**) signi cant at 5% level; (***) signi cant at 1% level. Every observation corresponds to the auction of a given bond. Dependent variable is average secondary market price of o ered bond -days after auction minus price benchmark of to-be-o ered bond before auction. Average secondary market price is calculated as volume-weighted average price of all transactions in the main inter-dealer market involving the bond -days after it was o ered in the primary market. Price benchmark of to-be-o ered bond before auction is calculated as the volume-weighted average price of all transactions on an auction-day involving the to-be-o ered bond between the market opening time and the auction's closing time. If there are no transactions in the inter-dealer market in that interval, the opening price from Bloomberg is used instead. Post-rebal. dummy is equal to one after J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014, and zero otherwise. Cuto price ms. benchm. is equal to the auction cuto price minus price benchmark of to-be-o ered bond. Cuto price ms. benchm. (post-rebal.) is an interacted term between Cuto price ms. benchm and Post-rebal. dummy. Bond controls include duration and convexity calculated from prices of auction-day. Macro controls include the 10-year CDS on the Colombian sovereign government, Chicago Board Options Exchange's CBOE Volatility Index (VIX), and the 1-year yield on locally-denominated Colombian government debt calculated by the Central Bank of Colombia. Bond and macro controls are from auction-day. Standard errors are robust.

suggests that bidders became less willing to deviate from their inventory target levels after the rebalancing.

To test this implication, we use data from the Colombian Central Bank on bidders' daily bond inventory. These data are not used to estimate the auction model. Instead, we use it to examine how dealers control their overnight bond inventory, using the following model

$$\text{inv}_{i;t} - \text{inv}_{i;t-1} = \alpha_i + \beta_r \text{rebal}_t + \beta_{inv}\text{inv}_{i;t-1} + \beta_{inv;r}\text{rebal}_t \times \text{inv}_{i;t-1} + \varepsilon_{i;t}; \qquad (3.2.2)$$

where $\text{inv}_{i;t}$ is bidder i's (standardized) aggregate bond inventory on day t and rebal_t is a post-rebalancing dummy. We also add bidder fixed-effects α_i.

The above model is based on the inventory risk literature (see, for example, Madhavan and Smidt (1993); Friewald and Nagler (2016); Schultz (2017)). β_{inv} shows how the average daily changes in inventory between $t-1$ and t are affected by the average inventory level in $t-1$. Negative values of β_{inv} imply that the bidders' inventory exhibits mean-reversion. The coefficient $\beta_{inv;r}$ captures the changes in the bidders' inventory policy following the rebalancing announcement. Based on the estimates $\hat{\beta}_{inv}$ and $\hat{\beta}_{inv;r}$, we also estimate inventory half-life in the pre- and post-rebalancing samples as $\log(2)/(1 - \hat{\beta}_{inv})$ and $\log(2)/(1 - \hat{\beta}_{inv} - \hat{\beta}_{inv;r})$, respectively. According to Madhavan and Smidt (1993); Friewald and Nagler (2016); Schultz (2017), this measures the average number of days a dealer takes to offload half of their inventory position.

The estimates of Eq. (3.2.2) are reported in panel A of table (3.2). The estimated inventory half-lives before and after the rebalancing, calculated using the coefficients from panel A, are shown in panel B of the same table. Before the rebalancing, the inventory mean-reversion is equals either -0.061 or -0.058, respectively. This means that, on average, 6.1% or 5.8% of the aggregate bidders' inventory position is eliminated overnight. This implies that bidders take 11.7 days on average to offload half of their inventory holdings. Interestingly, after the rebalancing, there is a significant increase in the speed of the mean-reversion parameter in columns (1) and (2) in panel A, thus resulting in a concomitant reduction of the inventory half-life to 7.5 days. On average, it takes bidders fewer than four days to unwind half of their aggregate inventory position after the rebalancing, which corresponds to a time reduction of approximately 36%.

This change in bidders' inventory management is consistent with a shift from a principal market-making model toward an agency model.[7] In the former, bidders act as market makers who stand ready to accommodate asynchronous trading needs by temporarily warehousing bonds in their books until offsetting orders arrive. They are compensated through the bid-ask spread for the inventory risk they incur as liquidity providers. In contrast, in an agency model, bidders act as brokers who take on less inventory risk by attempting to match buyers and sellers directly. In

[7] See Duffie (2012); Adrian et al. (2017); Bessembinder et al. (2018), among others, for a related discussion in the aftermath of the adoption of the Volcker rule.

this scenario, the bid-ask spread would be lower, which is indeed the case in the post-rebalancing sample, as documented in Section 2.5.1.

Table 3.2: Inventory mean-reversion and half-life after rebalancing announcement

Panel A: Inventory mean-reversion		
	Dependent variable:	
	Daily change in inventory	
	(1)	(2)
Mean-reversion: $\hat{\ }_{inv}$	0.061	0.058
	(0.007)	(0.008)
Post-rebal. dummy: $\hat{\ }_{r}$	0.008	0.009
	(0.006)	(0.009)
Mean-reversion (post-rebal.): $\hat{\ }_{inv;r}$	0.036	0.041
	(0.011)	(0.013)
Panel B: Inventory half-life (in days)		
	(1)	(2)
Before rebalancing: $\frac{\log(2)}{(1\ \hat{\ }_{inv})}$	11:7	11:7
After rebalancing: $\frac{\log(2)}{(1\ \hat{\ }_{inv}\ \hat{\ }_{inv;r})}$	7:5	7:5
Bidder FE	No	Yes
Observations	7,264	7,264
R^2	0.044	0.044

Note: (*) Significant at 10% level; (**) significant at 5% level; (***) significant at 1% level. This table presents the estimated coefficients of model (3.2.2) (panel A), and estimated inventory half-life based on these coefficients (panel B). Aggregate daily inventory corresponds to end-of-day holdings of all TES bonds at the individual level for ten out of fourteen bidders. Aggregate inventories are standardized at the bidder level by subtracting the sample mean and then dividing every resulting observation by the sample standard deviation. Post-rebal. dummy is equal to one after J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014, and zero otherwise. The sample ranges from 2013 to 2015. Standard errors are clustered at the bidder level.

3.3 Assumptions about market participation

A crucial assumption of the theoretical model in Vives (2010, 2011) is that the number of bidders, n_t, is exogenous and known by all bidders. The theoretical model does not address the source of fluctuations in the number of bidders who decide to participate in any given auction. We acknowledge that this is a restrictive assumption. In this section, we discuss each aspect of this assumption separately.

Exogenous number of participants in any auction

Implicit in the theoretical model is the assumption that any bidder's decision to participate in a given auction is independent of his or her own signal regarding the *ex post* value of the object. To investigate whether this is actually the case in our setting, we use an individual bidder's inventory prior to entering the auction as a proxy for his or her ex ante valuation of the bond. Intuitively, if a bidder holds a long position on the to-be-offered bond, then the idiosyncratic component in her valuation will be lower, which would encourage the bidder to submit a conservative bid. Conversely, if the bidder holds a short position on the bond, it might be prompted to submit a more competitive bid to cover his or her position. With this idea in mind, we first empirically explore whether pre-auction inventory predicts bidding behavior. After establishing the significance of this empirical relationship, we investigate whether pre-auction inventory predicts bidder participation.

An important caveat needs to be addressed before testing this hypothesis. Bidders' inventory is idiosyncratic and not necessarily related to private information on bond fundamentals. Unfortunately, we do not have good proxies for the bidder's individual valuation prior to entering the auction, which is directly related to the amount of private information possessed by each bidder. In fact, one of the main contributions of our study is its proposal of an econometric methodology to quantify the importance of private information about fundamentals because such variables are difficult to come by. Notwithstanding this caveat, our structural estimation results indicate that, in both subsamples, an important idiosyncratic component exists in the bidders' valuation.

We formally explore whether bidders' inventory prior to entering the auction affects bidding behavior. To test this, we fit the following reduced-form model based on a modified version of Eq. (2.3.1)

$$\text{markdown}_{i,t} = \alpha_i + \beta_{inv}\text{inventory}_{i,t} + \beta_k k_t + \varepsilon_{i,t}, \qquad (3.3.1)$$

where bidder i's markdown in auction t is given by $\text{markdown}_{i,t} = p^0_{i,t} - \bar{p}_t$, where $p^0_{i,t}$ is the highest bid submitted by bidder i in auction t, $\bar{p}_t$ is the secondary market price benchmark in auction t, k_t is the total bond supply offered, and $\text{inventory}_{i,t}$ is bidder i's (detrended and standardized) inventory on bond t one day before auction t.[8] Intuitively, a higher markdown captures a more

[8]Unfortunately, we only have daily aggregate inventory data at the bidder level for all Colombian government

competitive or aggressive bid because $p^0_{i;t}$ is high relative to the price benchmark. $_i$ are bidder xed-e ects.

The model results are reported in column (1) of table (3.3). A one standard-deviation increase in aggregate inventory reduces bidders' markdown by almost 10 percent. Thus, when a bidder is, on average, long Colombian bonds, it will be prompted to submit less competitive bids, which is consistent with a lower idiosyncratic valuation. Now that we have shown that pre-auction inventory predicts bidding behavior, we test whether it also predicts bidder participation by tting the following linear probability model

$$\text{partic}_{i;t} = \ _i + \ _{inv}\text{inventory}_{i;t} + \ _k k_t + "_{i;t}; \qquad (3.3.2)$$

where $\text{partic}_{i;t}$ takes the value of one if bidder i participates in auction t and zero otherwise. $_i$ are bidder xed-e ects. The results are reported in column (2) of table (3.3). From the rst row, we cannot reject the null hypothesis that $_{inv}$ is signi cantly di erent from zero.[9]

We cannot categorically conclude from these two empirical exercises that n_t is exogenous to bidders' private signals because, as mentioned earlier, individual inventory is a predictor of idiosyncratic valuation that is not necessarily related to private information about fundamentals, which is important according to our estimates. However, to the extent that pre-auction inventory is a good proxy for their signals, bidders' decision to participate in the auction would appear to be unrelated to their valuation.

Number of participants in any auction is common knowledge

Next, we focus on how realistic it is to assume that the number of bidders who participate in auction t, n_t, is known to all auction participants. By regulation, only 14 PDs are allowed to participate in any Colombian treasury auction. However, not all 14 PDs participate in every auction. This raises the question of whether PDs know with certainty the number of auction participants they will be e ectively competing against. We do not have direct evidence to conclusively answer this question.[10] Instead, we test whether the bidding data are consistent with this assumption.

If the bidders did not know the e ective number of participants, they are bidding against n_t, but only the maximum number of PDs allowed to participate in any auction N, and their equilibrium strategies cannot be a function of n_t. In particular, the slope of the equilibrium demand schedules

bonds denominated in local currency. Thus, if, for instance, auctions for bonds with maturity in 2024 and 2028, for instance, are held simultaneously, the inventory measure will be the same in both auctions. Furthermore, we do not have inventory measures for four bidders at any point in time because these four bidders are not commercial banks and, as such, are not required to report their inventory holdings.

[9] We also tted a logistic model (not reported in this paper) where we could not reject the null hypothesis either.

[10] There is survey evidence suggesting that even though bidders do not necessarily disclose to one another whether they will participate in any given auction, some of them are able to infer who will do so based on conversations with their clients (Cardozo, 2013).

Table 3.3: Bidder's markdown and auction participation

	Dependent variable:	
	Bidder's markdown	Bidder participation
	(1)	(2)
Inventory before auction	0.061	0.002
	(0.023)	(0.011)
Auction FE	Yes	Yes
Observations	1,174	1,563
R^2	0.299	0.089

Note: (*) Signi cant at 10% level; (**) signi cant at 5% level; (***) signi cant at 1% level. This table displays the estimated coe cients of the linear models (3.3.1) in column (1), and (3.3.2) in column (2). Dependent variables are bidder's markdown (column (1)) and bidder participation (column (2)), respectively. Bidder's markdown is calculated as the di erence between a bidder's highest submitted price in a given auction minus price benchmark. Price benchmark of a bond is calculated as the volume-weighted average price of all transactions on an auction-day involving the to-be-o ered bond between the market opening time (8 am) and the moment when the auction closes up (10 am). If there are no transactions in the inter-dealer market in that two-hour interval for a given to-be-o ered bond on an auction-day, the opening price from Bloomberg is used instead. Bidder participation takes the value of one when a bidder participates in an auction and zero otherwise. Inventory before auction is the (detrended and standardized) bidder's aggregate inventory on all Colombian sovereign bonds on the day before the auction denominated in COP. Standard errors are clustered at the auction level.

should not change with n_t, but only with N, because the former would be unknown to bidders. Therefore, we can test whether bidders know n_t by examining whether c varies with n_t.

We do not directly observe the equilibrium slope of the demand curves, but we can recover it from the bidding data and auction cutoff price $\hat{p}_t$. As we show in Section 2.3, the model implies a direct relationship between $\hat{p}_t$ and slope c, given by $\hat{p}_t = \frac{1}{n_t}\sum_{i=1}^{n_t} p^0_{it} - \frac{k_t}{n_t c}$. In fact, this is the market-clearing condition for any symmetric linear equilibrium with uniform cutoff prices. It has the appealing feature that it is solely a function of observed variables such as the average bidders' intercept, total bond supply, number of bidders, and the auction cutoff price. As in Section (2.3), we assume that the observed cutoff price p^c_t is only a proxy for the model-implied price $\hat{p}_t$ such that $p^c_t = \hat{p}_t + \epsilon_t$. Thus, we can estimate c using the following equation

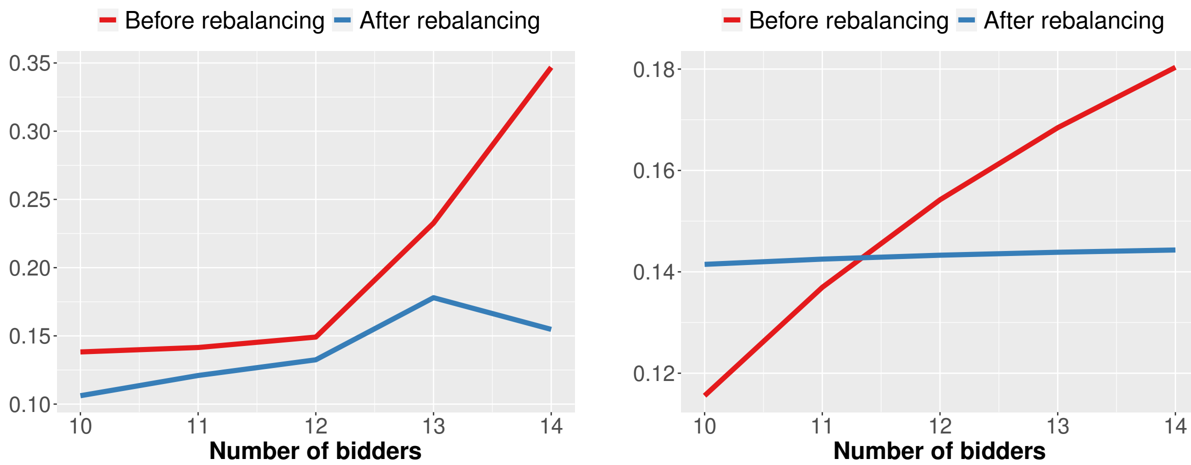
$$p^c_t - \frac{1}{n_t}\sum_{i=1}^{n_t} p^0_{i;t} = \beta_k(n_t)\, k_t + \epsilon_t; \tag{3.3.3}$$

where $\beta_k(n_t) = -\frac{1}{n_t c}$.

We estimate $\beta_k(n_t)$ for different numbers of observed bidders, n_t. We then calculate the implied slopes $\hat{c}(n_t) = -\frac{1}{n_t \hat{\beta}_k(n_t)}$. Once again, if the bidders did not know n_t when submitting their bids, then $\hat{c}(n_t)$ should be constant in n_t because optimal strategies depend only on the bidders' information sets.

In figure (3.1), panel (a) shows $\hat{c}(n_t)$ for all the observed values of n_t. Here, we express $\hat{c}(n_t)$ as a fraction of the average bond supply per auction, k_t. When n_t increases from ten to fourteen, $\hat{c}(n_t)/k_t$ jumps from 0.14 to 0.35 before the rebalancing, and from 0.11 to 0.16 after the shock. Thus, we provide empirical evidence that bidder strategies respond to changes in n_t, which suggests that they are part of their information sets.

Under the assumption that bidders know n_t, the model predicts that $c(n_t)$ is increasing in n_t. In panel (b) of figure (3.1), we plot the values of $c(n_t)$ implied by our structural estimation results. Again, the model-implied $c(n_t)$ is increasing but flatter after the rebalancing. We observe a similar pattern for "empirical" slopes $\hat{c}(n_t)$. Hence, we interpret figure (3.1) as suggestive evidence that the data support the model's predictions regarding the relationship between the number of bidders and the slope of the demand curve.



(a) Empirical slopes

(b) Model slopes

Figure 3.1: $c(n_t)$=k_t: model-implied vs empirical demand schedule slope standardized by average supply per auction

This gure plots $c(n_t)$, the demand schedule's slope, across multiple n_t, the number of bidders who participate in a given auction. $c(n_t)$ are standardized by the average total bond supply per auction k_t. Choice of n_t in both panels is guided by table (2.2). $c(n_t)$ in panel (a) are calculated from $_k(n_t) = \frac{1}{n_t c(n_t)}$, using the estimates of $_k(n_t)$ from the linear model in Eq. (3.3.3) for all n_t. Red line in panel (a) depicts $c(n_t)$ using estimated $_k$ for all n_t before the J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014. Blue line in panel (a) depicts $c(n_t)$ using estimated $_k$ from the post-rebalancing sample for all n_t. $c(n_t)$ in panel (a) are calculated according to Eq. (2.2.1), where value of parameters , , , and come from estimates in table (2.4). Red line in panel (b) depicts $c(n_t)$ as a function of n_t using the estimates in column (1) of table (2.4). Blue line in panel (b) depicts $c(n_t)$ as a function of n_t using the estimates in column (2) of table (2.4).

Final remarks

We provide model-free evidence supporting the reduction in the importance of private information and the increase in inventory adjustment costs. If private information about fundamentals is prevalent in this market, then an auction's cuto price would re ect bidders' private signals about fundamentals that were not captured by observable market prices prior to the auction. We show that auctions with high cuto prices relative to a pre-auction benchmark predict high subsequent secondary market prices. After the rebalancing, this e ect either shrinks or vanishes altogether, which is consistent with a decline in the importance of private information. Furthermore, we nd a faster overnight mean-reversion in inventories after the rebalancing. On average, it takes bidders 11.7 days to o oad half of their inventories before the rebalancing, but only 7.5 days after the shock.

Finally, we provide empirical evidence consistent with the model assumptions regarding primary market participation. First, we show that while a bidder's pre-auction inventory predicts bidding behavior, it does not predict auction participation. To the extent that pre-auction inventory is related to bidders' valuation signals, then the individual decision to participate in a given auction is unrelated to the individual signal. Second, for groups of auctions with a given number of participants, we construct empirical measures of the demand curve slope. We show that these empirical demand slopes respond to changes in the number of bidders, which should not be the case if bidders do not know with certainty the true number of auction competitors. Furthermore, the empirical slopes are similar to those directly derived from the model equilibrium and our structural estimates. Therefore, despite the parametric model, equilibrium strategies appear to respond to auction participation in a way that is consistent with the predictions of the model.

3.4 Appendix

Additional reduced form evidence

Auction's cuto price and secondary market yields

Table 3.4: Auction price and secondary market yields

	Dependent variable:				
	Second. market yields ms. benchm. -days after auction				
	= 1	= 2	= 3	= 4	= 5
	(1)	(2)	(3)	(4)	(5)
Cuto price ms. benchm.	0.486	0.504	0.491	0.608	0.650
	(0.096)	(0.091)	(0.091)	(0.080)	(0.089)
Post-rebal. dummy	1.383	1.345	1.341	1.402	1.363
	(0.118)	(0.115)	(0.116)	(0.114)	(0.111)
Cuto price ms. benchm. (post-rebal.)	0.813	0.809	0.845	0.960	0.945
	(0.188)	(0.187)	(0.185)	(0.178)	(0.180)
Number of bidders	0.068	0.059	0.066	0.061	0.067
	(0.024)	(0.025)	(0.025)	(0.026)	(0.027)
Bond o ering	0.074	0.076	0.088	0.078	0.064
	(0.054)	(0.054)	(0.056)	(0.055)	(0.054)
Bond FE	Yes	Yes	Yes	Yes	Yes
Bond controls	Yes	Yes	Yes	Yes	Yes
Macro controls	Yes	Yes	Yes	Yes	Yes
Observations	147	147	147	147	147
R^2	0.916	0.917	0.915	0.918	0.918

Note: (*) Signi cant at 10% level; (**) signi cant at 5% level; (***) signi cant at 1% level. This table displays the estimated coe cients of a model analogous to that of Eq. (3.2.1) with the bond yield as dependent variable. Every observation corresponds to the auction of a given bond. Dependent variable is average secondary market yield of o ered bond -days. Average secondary market yield is converted from volume-weighted average price of all transactions in the main inter-dealer market involving the o ered bond -days after the bond was o ered in the primary market. Post-rebal. dummy is equal to one after J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014, and zero otherwise. Cuto price ms. benchm. is equal to the auction cuto price minus price benchmark of to-be-o ered bond. Cuto price ms. benchm. (post-rebal.) is an interacted term between Cuto price ms. benchm and Post-rebal. dummy. Bond controls include duration and convexity calculated from prices of auction-day. Macro controls include the 10-year CDS on the Colombian sovereign government, Chicago Board Options Exchange's CBOE Volatility Index (VIX), and the 1-year yield on locally-denominated Colombian government debt calculated by the Central Bank of Colombia. Bond and macro controls are from auction-day. Standard errors are robust.

Auction's cuto price and secondary market prices: controlling for bid-to-cover ratio

Table 3.5: Auction price and secondary market prices

	Dependent variable:				
	Second. market price ms. benchm. -days after auction				
	= 1	= 2	= 3	= 4	= 5
	(1)	(2)	(3)	(4)	(5)
Cuto price ms. benchm.	1.629	1.920	1.763	2.730	3.097
	(0.356)	(0.395)	(0.373)	(0.581)	(0.916)
Post-rebal. dummy	0.185	0.075	0.072	0.313	0.021
	(0.271)	(0.290)	(0.291)	(0.312)	(0.446)
Cuto price ms. benchm. (post-rebal.)	0.630	0.713	1.006	1.848	1.715
	(0.361)	(0.442)	(0.414)	(0.556)	(0.802)
Number of bidders	0.103	0.167	0.119	0.159	0.102
	(0.055)	(0.076)	(0.074)	(0.088)	(0.114)
Bond o ering	0.013	0.016	0.088	0.052	0.023
	(0.094)	(0.115)	(0.111)	(0.126)	(0.153)
Bid-to-cover ratio	0.042	0.084	0.069	0.106	0.131
	(0.045)	(0.061)	(0.050)	(0.063)	(0.087)
Bond FE	Yes	Yes	Yes	Yes	Yes
Bond controls	Yes	Yes	Yes	Yes	Yes
Macro controls	Yes	Yes	Yes	Yes	Yes
Observations	147	147	147	147	147
R^2	0.384	0.382	0.373	0.445	0.382

Note: (*) Signi cant at 10% level; (**) signi cant at 5% level; (***) signi cant at 1% level. Every observation corresponds to the auction of a given bond. Dependent variable is average secondary market price of o ered bond -days after auction minus price benchmark of to-be-o ered bond before auction. Average secondary market price is calculated as volume-weighted average price of all transactions in the main inter-dealer market involving the bond -days after it was o ered in the primary market. Price benchmark of to-be-o ered bond before auction is calculated as the volume-weighted average price of all transactions on an auction-day involving the to-be-o ered bond between the market opening time and the auction's closing time. If there are no transactions in the inter-dealer market in that interval, the opening price from Bloomberg is used instead. Post-rebal. dummy is equal to one after J.P. Morgan GBI-EM index rebalancing announcement on March 19, 2014, and zero otherwise. Cuto price ms. benchm. is equal to the auction cuto price minus price benchmark of to-be-o ered bond. Cuto price ms. benchm. (post-rebal.) is an interacted term between Cuto price ms. benchm and Post-rebal. dummy. Bond controls include duration and convexity calculated from prices of auction-day. Macro controls include the 10-year CDS on the Colombian sovereign government, Chicago Board Options Exchange's CBOE Volatility Index (VIX), and the 1-year yield on locally-denominated Colombian government debt calculated by the Central Bank of Colombia. Bond and macro controls are from auction-day. Bid-to-cover ratio is calculated as the sum of all submitted quantity-bids over total bond supply (par amount). Standard errors are robust.

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