Two views of a robust Ramsey planner

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November 9, 2011

1 Introduction

The two views in the title refer to alternative assumptions about what a Ramsey planner thinks private agents believe. Hansen and Sargent (2008, ch. 16) described a discrete-time setting in which a Ramsey planner or Stackelberg leader shares a common approximating model with private agents. The private agents trust an approximating model and take as given a history-dependent government policy when choosing their actions. Private agents' Euler equations are implementability constraints that confront the government when at time 0 it once-and-for-all chooses a history dependent strategy. In Hansen and Sargent (2008, ch. 16), the government (or Stackelberg leader) confronts model uncertainty, represented in terms of the 'multiplier preferences' of Hansen and Sargent (2001). Hansen and Sargent construct a Ramsey plan that is robust to alternative specifications of the government's model.

This paper deepens and extends our earlier analysis by (a) formulating continuous time models by shrinking the discrete time increment to zero, and (b) comparing two alternative assumptions about what the government knows. A continuous time formulation allows us to make precise some approximations that Hansen and Sargent (2008, ch. 16) justified informally by alluding to continuous time counterparts.

As for (b), we construct two models distinguished by what the government believes about what the private sector believes.

1. In model 1, the government believes that the private sector *knows* a ('correct') probability model that is distinct from the government's approximating model, but un-

*We thank Anmol Bhandari for extraordinary research assistance and comments.

known to the government. The government also knows that correct model is absolutely continuous over finite intervals with respect to the government's approximating model and that its discounted relative entropy is limited. This structure leaves the space of models that are unknown to the government but known to the private sector so vast that the government cannot the private sector's probability model from a history of observed outcomes. Therefore, the government constructs a robust Ramsey problem by solving what Hansen and Sargent (2001, 2008) call a multiplier problem.[1]

2. Model 2 resembles a continuous time version of a related model of Hansen and Sargent (2008, ch. 16) in which the government believes that the private sector completely trusts the approximating model that it shares with the government. But the government distrusts it, again believing that the model that actually governs outcomes differs from the approximating model. We seek a robust Ramsey plan under this set of assumptions about beliefs.[2] [3]

Several papers have proposed and implemented Stackelberg solutions to policy problems in which the government seeks to be robust. See, for instance, Walsh (2004), Giordani and Soderlind (2004), Dennis (2008), Leitemo and Soderstrom (2008), and our own flawed paper Hansen and Sargent (2003) and its subsequent refinement Hansen and Sargent (2008, ch. 16). While we may not be representative readers, we find it difficult to keep track of the beliefs imputed to private agents in this most of this work. A problem is that a single conditional expectations operator is often asked to do too much work. To help us understand this literature better, it is pedagogically useful formally to represent what private firms and the government believe and how their beliefs are related to their common approximating model. We use two unit mean nonnegative martingales, $\hat{z}_t$ and $\tilde{z}_t$, as likelihood ratios relative to their common approximating model to represent the firm's model and the government's model, respectively. The martingales appear in the firm's first order conditions and the government's decision problems in ways that help us to be precise about what the government believes about what the firm knows and believes.

[1] In his actual calculations, Dennis (2008, pages 2071-2072) features a model of this type, though his verbal description says it is an equilibrium of the second type. In his words "the Stackelberg leader believes that the followers will use the approximating model to form expectations and formulates policy accordingly."

[2] We load a lot into the term "resembles" here. Our formulation in this paper corrects an important "sign error" in the model of Hansen and Sargent (2008, ch. 16).

[3] Karantounias (2011) studies a related but different problem, namely, that faced by a Ramsey planner who does not fear model misspecification while knowing that private agents do.

Our original model in Hansen and Sargent (2008, ch. 16) featured a monopolist and competitive fringe, both facing quadratic costs of adjusting output. Here we adopt a model with a similar but simpler structure, in particular, a streamlined New Keynesian model formulated by Woodford (2010). The endogenous dynamics are governed by a New Keynesian Phillips curve. Our wish to study versions of models 1 and 2 above impels us to begin with a price-setting firm's decision problem that gives rise to a New Keynesian Phillips curve. We use our two types of models to reinterpret some of the existing robust Stackelberg solutions proposed in Walsh (2004), Giordani and Soderlind (2004), Dennis (2008), and Leitemo and Soderstrom (2008).

2 Firm behavior

Let p_t be the log of a nominal price level, y_t be the log of aggregate output, c_t be a stochastic cost-push shock. We follow Woodford (2010) and regard nature as setting the cost-push shock, the government as choosing a strategy that controls y_t, and a representative firm as choosing $p_{t+1} - p_t$ in a way that can be described by a New Keynesian Phillips curve. Because we focus on how the private firm's beliefs influence its decisions, we begin by stating an optimization problem that can be viewed as underlying Woodford's New Keynesian Phillips curve.

2.1 No model distrust

With no model distrust, under the approximating model the representative firm solves:

$$\max_u E\left[\sum_{t=0}^{\infty} \beta^t \left\{-\frac{(u_t)^2}{2} + p_{t+1}(\kappa y_t + c_t + c^*)\right\} |\mathcal{F}_0\right]$$

subject to:

$$\begin{aligned} p_{t+1} - p_t &= u_t \\ x_{t+1} - x_t &= Ax_t + C(w_{t+1} - w_t) \end{aligned}$$

where $w_{t+1} - w_t$ is an iid sequence of standard normally distributed random variables. We write this in first-difference form because we eventually consider a continuous-time version of this model in which w is a standard Brownian motion. The firm treats y and c as processes that are exogenous to its decisions. In equilibrium these will depend linearly on a state vector x, so $c_t = H_c \cdot x_t$ and $y_t = H_y \cdot x_t$. The cost shock process c is

$$c_{t+1} - c_t = \nu_c c_t + \sigma_c (w_{t+1} - w_t). \tag{1}$$

We will focus on the case in which $\nu_c < 0$.

2.2 No distrust, but altered measure

We induce a change of measure by multiplying the original joint density over outcomes by a positive martingale $\hat{z}$ with unit expectation. Multiplicative increments in $\hat{z}$ evidently satisfy

$$E\left[\left(\frac{\hat{z}_{t+1}}{\hat{z}_t}\right)|\mathcal{F}_t\right] = 1$$

for all t, where $\hat{z}_0 = 1$. We use this martingale as a likelihood ratio to represent beliefs that are altered relative to a baseline approximating model. When it has beliefs indexed by $\hat{z}$, the firm wants to maximize:

$$E\left[\sum_{t=0}^{\infty} \beta^t \hat{z}_t \left\{-\frac{(u_t)^2}{2} + p_{t+1}(\kappa y_t + c_t + c^*)\right\} \Big| \mathcal{F}_0\right].$$

Form the Lagrangian:

$$\begin{aligned}
- \quad & E\left(\sum_{t=0}^{\infty} \beta^t \hat{z}_t \left[\frac{(u_t)^2}{2} - p_{t+1}(\kappa y_t + c_t + c^*)\right] |\mathcal{F}_0\right) \\
- \quad & E\left(\sum_{t=0}^{\infty} \beta^t \hat{z}_t \lambda_t (p_{t+1} - p_t - u_t) |\mathcal{F}_0\right) \\
- \quad & E\left(\sum_{t=0}^{\infty} \beta^{t+1} \hat{z}_{t+1} \varphi_{t+1} \cdot [x_{t+1} - x_t - A x_t - C(w_{t+1} - w_t)] \, |\mathcal{F}_0\right)
\end{aligned}$$

and solve the associated saddle point problem. The first-order conditions for u, p, and x, respectively, are[4]

$$
\begin{aligned}
u_t &= \lambda_t && (2)\\
\lambda_t &= (\kappa y_t + c_t + c^*) + \beta E\left[\left(\frac{\hat{z}_{t+1}}{\hat{z}_t}\right)\lambda_{t+1}|\mathcal{F}_t\right] && (3)\\
\varphi_t &= (H_c + \kappa H_y)p_{t+1} + \beta(I + A')E\left[\left(\frac{\hat{z}_{t+1}}{\hat{z}_t}\right)\varphi_{t+1}|\mathcal{F}_t\right]. && (4)
\end{aligned}
$$

Focus on the Euler equation (3) for λ. Solve it forward and substitute the solution into first-order condition (3) to represent u_t. After that it is easy to compute φ_t by solving the third equation forward.

Solving Euler equation (3) forward leads to

$$
\lambda_t = E\left[\sum_{j=0}^{\infty}\beta^j\left(\frac{\hat{z}_{t+j}}{\hat{z}_t}\right)(\kappa y_{t+j} + c_{t+j} + c^*)\Big|\mathcal{F}_t\right] \tag{5}
$$

where $\frac{\hat{z}_{t+j}}{\hat{z}_t}$ is used to model the conditional distribution of the private sector relative to our benchmark specification over a j-period forecast horizon. Recalling that $\lambda_t = (p_{t+1} - p_t)$, we recognize (5) as a forward-looking version of a New Keynesian Phillips curve. The current inflation rate is a geometric sum of a linear combination of output and the cost-push shock.

Many researchers take "Euler equation" (3) as a starting point sometimes combined with a "consumption Euler equation" of a kind that we will discuss later.

Initially, we will take $\hat{z}$ as given. It thereby represents an exogenous specification of beliefs. Later we will describe a sense in which the government *chooses* $\hat{z}$. In that case, the evolution of the co-state φ will play an important role in shaping the worst-case distributions that the government uses to construct robust decision rules.

2.3 Continuous-time version

We study consequences of shrinking the discrete time increment. In the limit we will obtain the continuous-time robust control problem analyzed in Hansen et al. (2006) for a single-agent decision problem. Index the time increment by $\epsilon = \frac{1}{2^j}$ for some positive integer j.

[4]Calculating first-order conditions for the uncontrollable process x is a device for obtaining laws of motion for the multiplier vector ϕ_t on x.

Replace β by $\exp(-\delta\epsilon)$ for $\delta > 0$. Consider the following counterpart to (3):

$$\lambda_t = \epsilon(\kappa y_t + c_t + c^*) + \exp(-\delta\epsilon)E\left[\lambda_{t+\epsilon}|\mathcal{F}_t\right]$$

and the following counterpart to (1) Suppose that

$$c_{t+\epsilon} - c_t \approx \epsilon\nu_c c_t + \sigma_c(w_{t+\epsilon} - w_t),$$

where $\nu_c < 0$ and w is a scalar Brownian motion. Also suppose that

$$\hat{z}_{t+\epsilon} - \hat{z}_t \approx \hat{z}_t\hat{h}_t(w_{t+\epsilon} - w_t).$$

Represent the motion of λ as

$$\lambda_{t+\epsilon} - \lambda_t \approx \epsilon\mu_{\lambda,t} + \sigma_{\lambda,t}(w_{t+\epsilon} - w_t),$$

where

$$\mu_{\lambda,t} = \lim_{\epsilon\downarrow 0}\frac{E(\lambda_{t+\epsilon} - \lambda_t|\mathcal{F}_t)}{\epsilon}$$

and E denotes a mathematical expectation.

Now use Ito's Lemma to compute

$$\hat{\mu}_{\lambda,t} = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}E\left(\frac{\hat{z}_{t+\epsilon}}{\hat{z}_t}\lambda_{t+\epsilon} - \lambda_t|\mathcal{F}_t\right) = \mu_{\lambda,t} + \sigma_{\lambda,t}\hat{h}_t. \tag{6}$$

Next, approximate

$$\lambda_t = \epsilon(\kappa y_t + c_t + c^*) + \exp(-\delta\epsilon)E\left[\frac{\hat{z}_{t+\epsilon}}{\hat{z}_t}\lambda_{t+\epsilon}|\mathcal{F}_t\right]$$

as

$$0 = (\kappa y_t + c_t + c^*) - \delta\lambda_t + \hat{\mu}_{\lambda,t}.$$

The approximation becomes arbitrarily good as ϵ declines to zero. Thus, when the firm has beliefs represented by the martingale $\hat{z}$, we can describe its optimal choices of (u, p) by

$$\begin{aligned} u_t &= \lambda_t \\ \hat{\mu}_{\lambda,t} &= \delta\lambda_t - (\kappa y_t + c_t + c^*). \end{aligned}$$

If we integrate the second equation forward, we obtain the following counterpart to (5):

$$\lambda_t = E\left[\int_0^\infty \exp(-\delta s)\left(\frac{\hat{z}_{t+s}}{\hat{z}_t}\right)(\kappa y_{t+s} + c_{t+s} + c^*)ds\,\Big|\mathcal{F}_t\right]. \tag{7}$$

3 Problem of government under belief heterogeneity

The government chooses processes y and λ to maximize

$$-\frac{1}{2}E\left(\sum_{t=0}^{\infty}\tilde{z}_t\beta^t\left[\lambda_t^2 + \zeta(y_t - y^*)^2\right]|\mathcal{F}_0\right) \tag{8}$$

subject to the following constraints:

$$\begin{aligned}
\lambda_t &= (\kappa y_t + c_t + c^*) + \beta E\left[\frac{\hat{z}_{t+1}}{\hat{z}_t}\lambda_{t+1}|\mathcal{F}_t\right] \\
c_{t+1} - c_t &= \nu_c c_t + \sigma_c(w_{t+1} - w_t)
\end{aligned} \tag{9}$$

and $\tilde{z}$ and $\hat{z}$ are positive martingales with mathematical expectations one conditioned on $\mathcal{F}_0$. The presence of the martingale $\tilde{z}_t$ in the government's objective (8) and the martingale $\hat{z}_t$ in firm's Euler equation (9) indicates that the government and the firm can have different beliefs. We focus on the following two situations:

1. $\hat{z} = \tilde{z}$. Here the government presumes that the firm knows a correct model $\hat{z}$ that differs from the approximating model except when $\hat{z} \equiv 1$. The Stackelberg leader does not know the $\tilde{z}$ model and seeks to be robust. Part of the equilibrium computation involves imposing $\hat{z} = \tilde{z}$.

2. $\hat{z} = 1$ but $\tilde{z}$ is not identically one. Here the firm trusts the approximating model but the government believes another model, or at least acts as if it believes another model.

For both cases, we eventually want to study the situation in which the government in effect *chooses* (worst-case) beliefs $\tilde{z}$ in order to design a robust Ramsey plan.

4 Government distrusts and does not know the private sector beliefs

We first suppose that the private sector has (unique) beliefs represented by the martingale $\hat{z}$. But the government does not know these beliefs. The government explores the consequence of alternative probability specifications imposing that $\tilde{z} = \hat{z}$. In effect, the government believes that the private sector knows the correct model, which is different from the common approximating model, but that the government itself does *not* know the correct model.

4.1 No robustness

Before studying a problem in which the government seeks robustness, we begin by temporarily assuming that the government completely trusts $\tilde{z}$ and has no concerns about robustness. We focus on case (1) in which the government believes that $\hat{z} = \tilde{z}$. To compute the government's Ramsey plan, form the Lagrangian

$$\begin{aligned} - \quad & \frac{1}{2} E \left[\sum_{t=0}^{\infty} \tilde{z}_t \beta^t \left[\lambda_t^2 + \zeta (y_t - y^*)^2 \right] | \mathcal{F}_0 \right] \\ & + E \left[\sum_{t=0}^{\infty} \tilde{z}_{t+1} \beta^t \left\{ \psi_{t+1} \left[\lambda_t - (\kappa y_t + c_t + c^*) - \beta \lambda_{t+1} \right] \right\} | \mathcal{F}_0 \right] \\ & + E \left[\sum_{t=0}^{\infty} \tilde{z}_{t+1} \beta^{t+1} \left\{ \phi_{t+1} \left[(1 + \nu_c) c_t + \sigma_c (w_{t+1} - w_t) - c_{t+1} \right] \right\} | \mathcal{F}_0 \right]. \end{aligned} \tag{10}$$

Because the constraint is cast in terms of a mathematical expectation conditioned on time t information, we restrict the Lagrange multiplier ψ_{t+1} to depend on date t information. The first-order conditions with respect to λ_t, y_t, and c_t, respectively, are:

$$\begin{aligned} \psi_{t+1} - \psi_t - \lambda_t &= 0 \\ -\zeta (y_t - y^*) - \kappa \psi_{t+1} &= 0 \\ \beta (1 + \nu_c) E \left[\left(\frac{\tilde{z}_{t+1}}{\tilde{z}_t} \right) \phi_{t+1} | \mathcal{F}_t \right] - \phi_t - \psi_{t+1} &= 0. \end{aligned} \tag{11}$$

Combine these with

$$\beta E\left[\left(\frac{\tilde{z}_{t+1}}{\tilde{z}_t}\right)\lambda_{t+1}|\mathcal{F}_t\right] = \lambda_t - (\kappa y^* + c_t + c^*) + \frac{\kappa^2}{\zeta}\psi_{t+1}$$
$$E\left[\left(\frac{\tilde{z}_{t+1}}{\tilde{z}_t}\right)c_{t+1}|\mathcal{F}_t\right] = (1+\nu_c)c_t + E\left[\left(\frac{\tilde{z}_{t+1}}{\tilde{z}_t}\right)\sigma_c(w_{t+1}-w_t)|\mathcal{F}_t\right].$$

So far we have presumed that the $\tilde{z}$ is known by the government. We now turn to the situation in which the government does not know $\tilde{z}$, though the government believes that the firm knows $\tilde{z}$.

4.2 Measuring and managing ambiguity

To manage model ambiguity under this structure, the government chooses $\tilde{z}$ to minimize and y and λ to maximize

$$-\frac{1}{2}E\left[\sum_{t=0}^{\infty}\tilde{z}_t\beta^t\left(\lambda_t^2+\zeta(y_t-y^*)^2\right)|\mathcal{F}_0\right] + \theta E\left[\sum_{t=0}^{\infty}\tilde{z}_{t+1}\beta^{t+1}\left(\log\tilde{z}_{t+1}-\log\tilde{z}_t\right)|\mathcal{F}_0\right] \tag{12}$$

subject to the same constraints faced in the earlier problem and $E\left(\frac{\tilde{z}_{t+1}}{\tilde{z}_t}|\mathcal{F}_t\right) = 1$. To analyze this problem, we add the term

$$\theta E\left[\sum_{t=0}^{\infty}\tilde{z}_{t+1}\beta^{t+1}\left(\log\tilde{z}_{t+1}-\log\tilde{z}_t\right)|\mathcal{F}_0\right] \tag{13}$$

to the Lagrangian (10) and minimize over $\tilde{z}$ while maximizing over plans for λ and y. But this problem is formulated more easily and transparently in continuous time.

The term

$$E\left[\sum_{t=0}^{\infty}\beta^{t+1}\tilde{z}_{t+1}\left(\log\tilde{z}_{t+1}-\log\tilde{z}_t\right)|\mathcal{F}_0\right] = (1-\beta)E\left[\sum_{t=0}^{\infty}\beta^{t+1}\tilde{z}_{t+1}\left(\log\tilde{z}_{t+1}\right)|\mathcal{F}_0\right]$$

measures discounted relative entropy between the $\tilde{z}$ probability model and the approximating model. The component terms: $E\left[\tilde{z}_{t+1}\left(\log\tilde{z}_{t+1}\right)|\mathcal{F}_t\right]$ measures relative entropy for the for assigning probabilities to date $t+1$ events conditioned on date zero information. Similarly, $E\left[\left(\frac{\tilde{z}_{t+1}}{\tilde{z}_t}\right)\left(\log z_{t+1}-\log z_t\right)|\mathcal{F}_t\right]$ measures conditional relative entropy for assigning probabilities to date $t+1$ events conditioned on date t information. Notice that relative entropy measures an expected log-likelihood ratio where the expectation is computed using the altered probability distribution. The parameter θ penalizes the alternative probability models when the governments explores the consequences of alternative probability specifications.

We acquire some simplifications by considering a continuous-time counterpart. Let the sample interval ϵ shrink. For instance, index the interval by $\epsilon = \frac{1}{2^j}$ for some positive integer j. When we alter the probability model, the drift of the Brownian motion changes in a way that we now describe. Use a multiplicative representation of the martingale $\tilde{z}$:

$$d\tilde{z}_t = \tilde{z}_t \tilde{h}_t dw_t. \tag{14}$$

Under the alternative model implied by the martingale z, the drift of dw_t turns out to be $\tilde{h}_t dt$. From Ito's lemma

$$d \log \tilde{z}_t = -\frac{1}{2}\left(\tilde{h}_t\right)^2 dt + \tilde{h}_t dw_t. \tag{15}$$

By an application of Ito's formula,

$$\lim_{\epsilon \downarrow 0} \frac{E\left[\left(\frac{\tilde{z}_{t+\epsilon}}{\tilde{z}_t}\right)(\log \tilde{z}_{t+\epsilon} - \log \tilde{z}_t)|\mathcal{F}_t\right]}{\epsilon} = -\frac{1}{2}\left(\tilde{h}_t\right)^2 + (\tilde{h}_t)^2 = \frac{1}{2}(\tilde{h}_t)^2, \tag{16}$$

which is the local measure of relative entropy used by Hansen et al. (2006). The resulting discounted relative entropy measure in continuous time is:

$$\frac{1}{2}E\left[\int_0^\infty \exp(-\delta t)\tilde{z}_t(\tilde{h}_t)^2 dt|\mathcal{F}_0\right] = \delta E\left[\int_0^\infty \exp(-\delta t)\tilde{z}_t \log \tilde{z}_t dt|\mathcal{F}_0\right]$$

Woodford (2010) solves a different robust Stackelberg problem in which the government embraces the approximating model but does know not the beliefs of the private sector. He uses:

$$E\left[\sum_{t=0}^\infty \beta^{t+1}\left(\frac{\tilde{z}_{t+1}}{\tilde{z}_t}\right)(\log \tilde{z}_{t+1} - \log \tilde{z}_t)|\mathcal{F}_0\right]$$

multiplied by θ to penalize the search for alternative private sector beliefs. Notice that this differs from our discrete-time measure. Whereas at date zero we weight $(\log \tilde{z}_{t+1} - \log \tilde{z}_t)$ by $\tilde{z}_t$, he weights it only by the ratio $\frac{\tilde{z}_{t+1}}{\tilde{z}_t}$. Thus the continuous-time counterpart to his measure is:

$$\frac{1}{2}E\left[\int_0^\infty \exp(-\delta t)(\tilde{h}_t)^2 dt|\mathcal{F}_0\right]$$

There is a different way to arrive at this same continuous time expression. In light of equation (15):

$$\frac{1}{2}E\left[\int_0^\infty \exp(-\delta t)(\tilde{h}_t)^2 dt|\mathcal{F}_0\right] = -\delta E\left[\int_0^\infty \exp(-\delta t)\log \tilde{z}_t dt|\mathcal{F}_0\right]$$

where the right-hand side measure relative entropy by changing the roles of the $\tilde{z}$ model and the approximating model. Thus the continuous-time limit of Woodford (2010)'s discrepancy is a measure of relative entropy, although the discrete-time specification mixes the role of approximating model and alternative model when weighting the one-period conditional measure of entropy. Using $-E(\log \tilde{z}_t|\mathcal{F}_0)$ as a discrepancy measure over an interval of time t in discrete evidently does not give tractable solutions, but the continuous-time counterpart may be more promising.

4.3 Robustness in continuous time

The preceding problem takes $\tilde{z}$ as given and known by the government. We now turn to the situation in which the government does not know $\tilde{z}$, though the government believes that the firm knows $\tilde{z}$. Suppose that

$$dc_t = \nu_c c_t dt + \sigma_c dw_t$$

where $\nu_c < 0$ and w is a scalar Brownian motion with drift. When we alter the probability model, the drift of the Brownian motion changes in a way that we now describe. Pursuing an argument like the one culminating in (6) leads to

$$\tilde{\mu}_{c,t} = \lim_{\epsilon \downarrow 0} \frac{\tilde{E}(c_{t+\epsilon} - c_t|\mathcal{F}_t)}{\epsilon} = \nu_c c_t + \sigma_c \tilde{h}_t$$

where the mathematical expectation $\tilde{E}$ is computed using the martingale $\tilde{z}$.

When the firm adheres to the $\tilde{z}$ model:

$$\begin{aligned} u_t &= \lambda_t \\ \tilde{\mu}_{\lambda,t} = \lim_{\epsilon \downarrow 0} \frac{\tilde{E}(\lambda_{t+\epsilon} - \lambda_t|\mathcal{F}_t)}{\epsilon} &= \delta\lambda_t - (\kappa y_t + c_t + c^*). \end{aligned}$$

In continuous time the first-order conditions with respect to λ, y, c corresponding to the discrete time conditions (11) for the government's problem are

$$\begin{aligned} d\psi_t &= \lambda_t dt && (17) \\ y_t &= -\frac{\kappa}{\zeta}\psi_t + y^* && (18) \\ \tilde{\mu}_{\phi,t} &= \lim_{\epsilon\downarrow 0}\frac{\tilde{E}(\phi_{t+\epsilon} - \phi_t|\mathcal{F}_t)}{\epsilon} = (\delta - \nu_c)\phi_t + \psi_t && (19) \end{aligned}$$

We complete the government's robust control problem by supposing that the government does not know the actual model and wishes to adjust decisions for that ignorance. It does that by minimizing the objective (12) with respect to the martingale $\tilde{z}$. Because this martingale is characterized by the process $\tilde{h}$ in equation (14), we can think of $\tilde{h}_t$ as a date t control for a minimizing agent. In light of (16), the date t contribution to relative entropy is $\frac{1}{2}(\tilde{h}_t)^2$ under the distorted model. Consequently, the first-order conditions for minimization are:

$$\theta\tilde{h}_t + \sigma_c\phi_t = 0,$$

or

$$\tilde{h}_t = -\frac{1}{\theta}\sigma_c\phi_t. \tag{20}$$

Thus, the following system of equations characterizes the government's choices when the government believes that the private sector adheres to a model $\tilde{z}$ that the government does not know:

$$\begin{aligned} \tilde{\mu}_{c,t} &= \nu_c c_t - \tfrac{(\sigma_c)^2}{\theta}\phi_t \\ \tilde{\mu}_{\lambda,t} &= \delta\lambda_t + \tfrac{\kappa^2}{\zeta}\psi_t - \kappa y^* - c_t - c^* \\ \tilde{\mu}_{\phi,t} &= (\delta - \nu_c)\phi_t + \psi_t \\ \tfrac{d\psi_t}{dt} &= \lambda_t \end{aligned}$$

The last equation resembles a state equation (it is a co-state equation associated with a co-state). An equivalent to represent this system is

$$\begin{aligned} dc_t &= \nu_c c_t dt - \frac{\sigma_c^2}{\theta}\phi_t dt + \sigma_c d\tilde{w}_t \\ d\lambda_t &= \delta\lambda_t dt + \frac{\kappa^2}{\zeta}\psi_t dt - \kappa y^* dt - c_t dt - c^* dt + \sigma_{\lambda,t} d\tilde{w}_t \\ d\phi_t &= (\delta - \nu_c)\phi_t dt + \psi_t dt + \sigma_{\phi,t} d\tilde{w}_t \\ d\psi_t &= \lambda_t dt \end{aligned}$$

$$d\tilde{w}_t \quad = \quad \frac{1}{\theta}\sigma_c\phi_t dt + dw_t. \tag{21}$$

Two observations about (21) are pertinent. First, $\sigma_{\lambda,t}$ and $\sigma_{\phi,t}$ are not prespecified. They are shock exposures of Lagrange multipliers that are governed by forward-looking expectational differential equations that emerge from the firm's first-order conditions. We can simplify the task of computing these by solving a deterministic counterpart of system (21), subject to a stability constraint and initial conditions on c and ψ. We can think of w as an exogenous input with an unknown probability distribution; it is a Brownian motion only under the approximating model. The process $\tilde{w}$ depends on $\tilde{h}_t = -\frac{1}{\theta}\sigma_c\phi_t$. Since $\tilde{h}$ is associated with a change of measure, what is important for calculation is that $\tilde{w}$ is a standard Brownian motion under this alternative probability measure.

We can compute the government's decision rules by solving the deterministic version of this system that emerges upon setting $\tilde{w} \equiv 0$. We want to find the stabilizing solution of the deterministic system

$$\frac{d}{dt}\begin{bmatrix} c_t \\ \lambda_t \\ \phi_t \\ \psi_t \end{bmatrix} = H \begin{bmatrix} \lambda_t \\ \phi_t \\ c_t \\ \psi_t \end{bmatrix} + G \tag{22}$$

where $H - \frac{\delta}{2}I$ is a Hamiltonian matrix, ψ_t and c_t and state variables (with initial conditions) and ϕ_t and λ_t are "jump variables" (with initializations to be determined). The stabilizing solution has representation

$$\begin{bmatrix} \phi_t \\ \psi_t \end{bmatrix} \quad = \quad L \begin{bmatrix} c_t \\ \lambda_t \end{bmatrix} + K$$

$$\frac{d}{dt}\begin{bmatrix} c_t \\ \psi_t \end{bmatrix} \quad = \quad N \begin{bmatrix} c_t \\ \psi_t \end{bmatrix} + M \tag{23}$$

where L is a symmetric matrix and N has eigenvalues with real parts that are less than $\frac{\delta}{2}$. With these objects in hand, equations (18) and (20) can be used to get the government's decision rules for y_t and $\tilde{h}_t$ as functions of the state $\begin{bmatrix} c_t \\ \psi_t \end{bmatrix}$.

Recall that the instantaneous shock exposure for the "multiplier on the multiplier", ψ_t, is zero and that the shock exposure for the cost shock, c_t is specified exogenously. To determine the remaining shock exposures, we solve:

$$\begin{bmatrix} \sigma_\phi \\ 0 \end{bmatrix} = L \begin{bmatrix} \sigma_c \\ \sigma_\lambda \end{bmatrix}$$

Given the solution for L, this equation determines σ_ϕ and σ_λ as a function of σ_c.

4.4 Worst-case model

How do we interpret the equilibrium outcomes obtained in the previous subsection? Many match those of Walsh (2004), Leitemo and Soderstrom (2008) as well as a main outcome emphasized by Dennis (2008). If we take the computed equilibrium, plug it back into the firm's optimization, this gives what Walsh (2004) predicts will be the choice for $p_{t+1} - p_t$ as a function of the aggregate state vector. Walsh (2004) argues that the private agents share the government's concern about robustness, so when the government chooses beliefs in a robust fashion, agents act on these same beliefs. We think that interpretation is incorrect and prefer another one. We are not quite sure we understand what Walsh means by the private sector 'sharing the government's concern about robustness', because in selecting a worst-case model the private firm should look at its *own* objective function and constraints, not the government's. That reasoning implies that the government's worst-case model would differ from the government's. Even if the government and the private agents were to share the same value of θ, they would typically compute different worst-case models. We do not mean to pick unfairly on Walsh (2004). In fact, we regard it as a strength that his interpretation is more transparent and criticizable than those provided by most other papers in this line of work.

We have a rather different interpretation of Walsh's robust Stackelberg equilibrium. Like Woodford (2010), we suppose that government does not know the beliefs of the private agents, but that private agents are committed to those beliefs, whatever they are. Private agents know the correct probability model but the government does not. The government cannot correctly infer private agents' model from observing their decisions. Even though we solved the Stackelberg equilibrium while imposing $\hat{z} = \tilde{z}$, we did this as part of a device to impose robustness. The resulting $\tilde{z}$ is intended to be the government's robust or cautious inference about the private agents' beliefs. However, the firm's decision rule as a function of the aggregate states, obtained by solving the right-hand side (7) with the minimizing $\tilde{z}$ used for $\hat{z}$, will not produce the observed value of $p_{t+1} - p_t$. This discrepancy will not unduly concern the government because it understands that the fact that the government can observe that deviation is not enough to reveal the process $\hat{z}$ actually believed by the firm. Thus, the minimization on the part of the government is a device to design a robust government policy, but it is not a prediction of the beliefs of private agents.

To elaborate, there is too little structure on the perturbation $\tilde{z}$ defining firm's model for the government to infer it from observed outcomes on $p_{t+1} - p_t, c_t, y_t$. All the government knows is that the perturbation gives the firm a model that is absolutely continuous over finite intervals and has constrained discounted entropy with respect to the approximating model. This leaves the immense set of models so unstructured that it would be a daunting if not impossible task for the government to infer the private sector's model from histories of outcomes for y, c, and λ and its knowledge of (5) or (7). Therefore, our government does not attempt to reverse engineer $\tilde{z}$ from observed outcomes.

To be more concrete, consider for instance the discrete time specification and suppose that after observing inflation the government solving an Euler equation forward to infer a discounted expected sum of output and a cost shock. The government could compare this to the outcome of the analogous calculation using the approximating model. Comparing outcomes from these calculations would reveal a distorted expectation. There would be many ways to rationalize this distortion. One among many possibilities is that the distortion is concentrated on only the next period transition but not on the transitions to other future time periods. But many other possibilities are also consistent with the same observed inflation. The computed worst-case model is only one among many distortions consistent with observed data.

4.5 Representation of worst-case model

Our solution for the condition mean distortion h_t for dw_t depends on ϕ_t and hence implicitly on the endogenous state variable ψ_t. We now construct a worst-case model specification of the cost shock dynamics with an interesting property that we describe in the following:

Claim 4.1. *Suppose that both the private firm and the government agree on this particular model of cost shock. Under these common beliefs about the cost shock process, compute a rational expectations equilibrium in the usual way. Then along the equilibrium path, the inflation and output gap will be the same as in our model with a government that seeks to be robust.*[5]

[5] This inflation could be different that what will be observed as the outcome of the firm decision making

Construction: First suppose that the cost shock evolves as:

$$d\begin{bmatrix} c_t \\ \Psi_t \end{bmatrix} = N\begin{bmatrix} c_t \\ \Psi_t \end{bmatrix} dt + Mdt + \begin{bmatrix} \sigma_c \\ 0 \end{bmatrix} d\tilde{w}_t$$

where Ψ is viewed as an exogenous forcing process. Initialize $\Psi_0 = \psi_0$. Let both private agents and the government take this as the correct model for the evolution of the cost shock process. In computing the rational expectations equilibrium referred to in the proposition, there will now be two exogenous state variables, c_t and Ψ_t, and an endogenous state variable, ψ_t. Along the equilibrium path $\Psi_t = \psi_t$, but this will emerge as an endogenous outcome given our choice of initialization.

Notice that under this rational expectations formulation, we can give a precise description of private agents' actions as function of the states because of the way we have taken a precise stand on the private sector beliefs. In contrast the observed inflation rate

4.6 Dynamic programming formulation

It is computationally convenient to use the following dynamic programming formulation that is applicable in this setup in which the government and firm both use the $\tilde{z}$ model that the firm believes and the government distrusts. This dynamic programming problem allows us to compute the government's decision rule and worst-case model by solving a matrix Riccati equation, then swapping states and co-states in a fashion described, for example, by Hansen and Sargent (2008, ch. 16).

The dynamic programming problem under the $\tilde{z}$ model is:

$$\min_{\tilde{h}} \max_{\lambda, y} \frac{1}{2} E\left\{ \int_0^\infty \exp(-\delta t)\left[-\lambda_t^2 - \zeta(y_t - y^*)^2 + \theta \tilde{h}_t^2\right] dt \middle| \mathcal{F}_0 \right\}$$

subject to

$$\begin{aligned} d\lambda_t &= \delta\lambda dt - (\kappa y_t + c_t + c^*)dt + \sigma_{\lambda,t} d\tilde{w}_t \\ dc_t &= \nu_c c_t dt + \sigma_c \tilde{h}_t dt + \sigma_c d\tilde{w}_t \\ (d\tilde{w}_t &= -h_t dt + dw_t) \end{aligned}$$

under the Robust Stackelberg equilibrium.

where we can exploit the fact that the $\tilde{w}$ is a standard Brownian motion and solve the model ignoring the equation in parentheses. We can also employ certainty equivalence, and thereby solve by setting $\tilde{w}$ to zero and so solve the certainty counterpart to the robust Ramsey plan. We can do this without knowing $\sigma_{\lambda,t}$. However, when we turn to model 2, things are not so simple.

4.7 Why not infer $\hat{z}$?

There is too little structure on the perturbation $\tilde{z}$ defining firm's model for the government to infer it from observed outcomes on λ, c, y. All the government knows is that the perturbation gives the firm a model that is absolutely continuous over finite intervals and has constrained discounted entropy with respect to the approximating model. This leaves the immense set of models so unstructured that it would be a daunting if not impossible task for the government to infer the private sector's model from histories of outcomes for y, c, and λ and its knowledge of (5) or (7). Therefore, our government does not attempt to reverse engineer $\tilde{z}$ from observed outcomes.

Consider for instance the discrete time specification and suppose that the government by observing inflation infers a discounted expected sum of output and a cost shock by solving an Euler equation forward. The government could compare this to the same calculation using the approximating model. This reveals a distorted expectation. One among many possibilities is that the distortion is concentrated on only the next period transition but not on the transitions to other future time periods. But many other possibilities are also consistent with the same observed inflation. The computed worst-case model is one among many distortions consistent with observed data.

5 Government distrusts but knows firms trust approximating model

We change assumptions about the beliefs that the government imputes to the competitive firm. The government now believes that the firm adheres to the approximating model, which we express by setting $\hat{z} \equiv 1$. But because the government distrusts the approximating model, $\tilde{z} \neq 1$, we find it convenient to use governments' $\tilde{z}$ beliefs when computing an equilibrium. To represent the firms' beliefs under the $\tilde{z}$ model, note that the process $\{\frac{1}{\tilde{z}_t} : t \geq 0\}$ is a martingale with unit expectation (under the $\tilde{z}$ probablity distribution) given date zero information. Since under the approximating model:

$$\mu_{\lambda,t} = \delta\lambda_t + \frac{\kappa^2}{\zeta}\psi_t - \kappa y^* - c_t - c^*,$$

in solving the model from the perspective of the government we use

$$\lim_{\epsilon\downarrow} \frac{\tilde{E}\left[\left(\frac{\tilde{z}_t}{\tilde{z}_{t+\epsilon}}\right)\lambda_{t+\epsilon} - \lambda_t|\mathcal{F}_t\right]}{\epsilon} = \delta\lambda_t + \frac{\kappa^2}{\zeta}\psi_t - \kappa y^* - c_t - c^* - \sigma_{\lambda,t}\tilde{h}_t$$

Notice that the coefficient $\sigma_{\lambda,t}$ has a minus sign. This corrects for the belief heterogeneity between the firms and the government. Specifically, it imposes the private sector commitment to the approximating model and is needed because the equation we solve the the expectations associated with $\tilde{z}$.

The first-order conditions for the government's minimization with respect to $\tilde{h}$ are:

$$\theta\tilde{h}_t + \sigma_c\phi_t - \sigma_{\lambda,t}\psi_t = 0.$$

Hence

$$\tilde{h}_t = -\frac{\sigma_c}{\theta}\phi_t + \frac{\sigma_{\lambda,t}}{\theta}\psi_t. \tag{24}$$

Notice the appearance of $\sigma_{\lambda,t}$ here and its absence from the corresponding equation (20) in our earlier model. Thus, the robust Ramsey plan satisfies the following system of differential equations:

$$\begin{aligned}
\tilde{\mu}_{c,t} &= \nu_c c_t - \tfrac{(\sigma_c)^2}{\theta}\phi_t + \tfrac{\sigma_c\sigma_{\lambda,t}}{\theta}\psi_t \\
\tilde{\mu}_{\lambda,t} &= \delta\lambda_t + \tfrac{\kappa^2}{\zeta}\psi_t - \kappa y^* - c_t - c^* + \tfrac{\sigma_c\sigma_{\lambda,t}}{\theta}\phi_t - \tfrac{(\sigma_{\lambda,t})^2}{\theta}\psi_t \\
\tilde{\mu}_{\phi,t} &= (\delta - \nu_c)\phi_t + \psi_t \\
\tfrac{d\psi_t}{dt} &= \lambda_t.
\end{aligned}$$

As earlier, another way to represent this equation includes the exposures of variables to shocks:

$$\begin{aligned}
dc_t &= \nu_c c_t dt - \frac{\sigma_c^2}{\theta}\phi_t dt - \frac{\sigma_c\sigma_{\lambda,t}}{\theta}\psi_t dt + \sigma_c d\tilde{w}_t \\
d\lambda_t &= \delta\lambda_t dt + \frac{\kappa^2}{\zeta}\psi_t dt - \kappa y^* dt - c_t dt - c^* dt + \frac{\sigma_c\sigma_{\lambda,t}}{\theta}\phi_t dt - \frac{(\sigma_{\lambda,t})^2}{\theta}\psi_t + \sigma_{\lambda,t} d\tilde{w}_t \\
d\phi_t &= (\delta - \nu_c)\phi_t dt + \psi_t dt + \sigma_\phi d\tilde{w}_t \\
d\psi_t &= \lambda_t dt
\end{aligned}$$

$$d\tilde{w}_t = \frac{\sigma_c}{\theta}\phi_t dt - \frac{\sigma_{\lambda,t}}{\theta}\psi_t dt + dw_t \tag{25}$$

where $\tilde{w}$ is again a Brownian motion under the change of measure.

Comparing (25) with our earlier system (21) reveals that $\sigma_{\lambda,t}$ now appears in the systematic part of the right hand side of system (25), while it does not in the systematic part of the right side of (21). By systematic we mean the parts other than the terms in $d\tilde{w}_t$. This makes it more challenging to solve system (25) than it was to solve system (21). In a related discrete time problem, Hansen and Sargent (2008) proposed and implemented the following iterative algorithm. Guess σ_λ constant. Solve the differential equation system under the worst case conditional means but without shocks

$$\begin{aligned}
\frac{dc_t}{dt} &= \nu_c c_t - \frac{\sigma_c^2}{\theta}\phi_t + \frac{\sigma_c\sigma_{\lambda,t}}{\theta}\psi_t \\
\frac{d\lambda_t}{dt} &= \delta\lambda_t + \frac{\kappa^2}{\zeta}\psi_t - \kappa y^* - c_t - c^* + \frac{\sigma_c\sigma_{\lambda,t}}{\theta}\phi_t - \frac{(\sigma_{\lambda,t})^2}{\theta}\psi_t \\
\frac{d\phi_t}{dt} &= (\delta - \nu_c)\phi_t + \psi_t \\
\frac{d\psi_t}{dt} &= \lambda_t
\end{aligned} \tag{26}$$

This differential equation system has initial conditions for c_t and ψ_t. Again, we can use invariant subspace methods to find the stabilizing solution given by (23) and (23). The candidate stochastic counterpart has representation

$$d\begin{bmatrix} c_t \\ \psi_t \end{bmatrix} = N(\sigma_\lambda)\begin{bmatrix} c_t \\ \psi_t \end{bmatrix} dt + M(\sigma_\lambda)dt + \begin{bmatrix} \sigma_c \\ 0 \end{bmatrix} d\tilde{w}_t$$

Since

$$\begin{bmatrix} \phi_t \\ \psi_t \end{bmatrix} = L(\sigma_\lambda)\begin{bmatrix} c_t \\ \lambda_t \end{bmatrix} + K(\sigma_\lambda)$$

where $L(\sigma_\lambda)$ is a symmetric matrix. The implied shock exposure coefficients σ_ϕ^* and σ_λ^* for ϕ_t and λ_t satisfy:

$$\begin{bmatrix} \sigma_\phi^* \\ 0 \end{bmatrix} = L(\sigma_\lambda)\begin{bmatrix} \sigma_c \\ \sigma_\lambda^* \end{bmatrix}.$$

We are particularly interested in σ_λ^* which is typically is different from our previous guess σ_λ. We search for a fixed-point for this problem in which the σ_λ we feed in agrees with the outcome of this calculation: σ_λ^*. Thus, we are heavily exploiting the restriction that ψ_t

is locally predictable in our iterative algorithm for finding a σ_λ. Instead of solving for all objects simultaneously, we iterate to convergence on a time-invariant σ_λ.

Hansen and Sargent (2008, ch. 16) suggested this approach in a discrete-time formulation, but their analysis has an important difference in the sign of σ_λ. We use this parameter to offset the government's distortion in beliefs, which induces a minus sign that is missing in Hansen and Sargent (2008, ch. 16). Woodford (2010) also adopted a similar approach. He solved his model by first conditioning on the shock exposure for a counterpart to λ_t.[6]

5.1 Representation of worst-case model

In contrast to the analysis in section 4.4 where we constructed a corresponding rational expectations Stackelberg equilibrium, in this setting we construct a corresponding heterogeneous belief economy.

Our solution for the condition mean distortion h_t for dw_t depends on ϕ_t and hence implicitly on the endogenous state variable ψ_t. We now construct a worst-case model specification of the cost shock dynamics with an interesting property that we describe in the following:

Claim 5.1. *Suppose that both the private sector firms commit to the approximating model and that the governments to an alternative particular model of cost shock. Under these heterogeneous beliefs about the cost shock process, compute a Stackelberg equilibrium. Then along the equilibrium path, the inflation and output gap will be the same as in our model with a government that seeks to be robust.*

Construction: First suppose that the cost shock evolves as:

$$d\begin{bmatrix} c_t \\ \Psi_t \end{bmatrix} = N\begin{bmatrix} c_t \\ \Psi_t \end{bmatrix} dt + M dt + \begin{bmatrix} \sigma_c \\ 0 \end{bmatrix} d\tilde{w}_t$$

where Ψ is viewed as an exogenous forcing process. Initialize $\Psi_0 = \psi_0$. Let the government take this as the correct model for the evolution of the cost shock process while the firms take the approximating model as correct. In computing the rational expectations equilibrium referred to in the proposition, there will now be two exogenous state variables, c_t and Ψ_t, and an endogenous state variable, ψ_t. Along the equilibrium path $\Psi_t = \psi_t$, but this will emerge as an endogenous outcome given our choice of initialization.

[6]As a second robust Stackelberg equilbrium, Dennis (2008) adopts that the same sign convention as used in the Hansen and Sargent (2008, ch. 16). We are uncertain how to interpret this equilibrium.

5.2 Dynamic programming formulation

For computational convenience, it is again convenient to use a dynamic programming formulation. The following approach works applicable in this setup in which the government distrusts the approximating model while the firm adheres to it. As before, a key step is to solve a matrix Riccati equation, then swap states and co-states appropriately. However, now we have to iterate over σ_λ as well.

The dynamic programming problem under the $\tilde{z}$ model is:

$$\min_{\tilde{h}} \max_{\lambda, y} \frac{1}{2} E \left\{ \int_0^\infty \exp(-\delta t) \left[-\lambda_t^2 - \zeta (y_t - y^*)^2 + \theta \tilde{h}_t^2 \right] dt \middle| \mathcal{F}_0 \right\}$$

subject to

$$\begin{aligned} d\lambda_t &= \delta \lambda dt - (\kappa y_t + c_t + c^*) dt - \sigma_{\lambda,t} \tilde{h}_t dt + \sigma_{\lambda,t} d\tilde{w}_t \\ dc_t &= \nu_c c_t dt + \sigma_c \tilde{h}_t dt + \sigma_c d\tilde{w}_t \\ (d\tilde{w}_t &= -h_t dt + dw_t) \end{aligned}$$

We will use this formulation of a dynamic programming problem together with an iterative algorithm over a time invariant $\sigma_{\lambda,t} = \sigma_\lambda$. At each step of the algorithm, we employ certainty equivalence by setting $\tilde{w}$ to zero and thereby solving the certainty counterpart to the robust Ramsey plan.

6 An additional Euler equation

Often new Keynesian models include a consumption Euler equation. In discrete time this equation is given by:

$$y_t = E(y_{t+1} | \mathcal{F}_t) - \frac{1}{\rho} (i_{t+1} - p_{t+1} - p_t)$$

where i_{t+1} is the one period nominal interest rate set at date t. In continuous time this specification becomes:

$$i_t = \mu_{p,t} - \rho \mu_{y,t}.$$

For a robust counterpart we would include an additional z process to model the beliefs of consumers/investors. For such a model, we are led to consider the interplay among three collections of conditional expectations, those of the firm, those of the consumers, and those of the government. If we constrain firms and consumers to have the same beliefs, we could obtain counterparts to our two types of models. Following Leitemo and Soderstrom (2008) we expect the consequences to be very modest unless we directly enter nominal interest rates into the government objective.

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A Computational Appendix

This appendix contains details of the computational algorithm and the impulse responses of various endogenous objects with and without concerns for robustness. The endogenous objects are

A. State Variables -

- c_t - Cost Push Shock
- ψ_t - The multiplier on multiplier in the Government's Problem
- λ_t - The firms multiplier

B. Controls

- y_t - Output
- h_t - Belief distortion

Finally for each model we plot the impulse responses under the worst case model and the approximating model separately

B Algorithm

This section details the algorithm for implementing the Dynamic programing formulation described in section 5.2 for Model 2 . The key difference is to note that $\sigma_{\lambda,t}$ is endogenous and has to be determined in equilibrium. This is achieved by iterating on the following steps

1. **Guess the shock exposures** - Start with some guess for σ_λ
2. **Formulate the quadratic form for the Government's problem**

   This step involves restating the Government's problem as follows

$$\min_{h} \max_{y,\lambda} -\frac{1}{2} E \int_0^\infty e^{-\delta t} \left\{ \begin{bmatrix} c_t \\ \lambda_t \end{bmatrix}' \bar{Q} \begin{bmatrix} c_t \\ \lambda_t \end{bmatrix} + \begin{bmatrix} y_t \\ h_t \end{bmatrix}' \bar{R} \begin{bmatrix} y_t \\ h_t \end{bmatrix} \right\}$$

   s.t

$$D \begin{bmatrix} c_t \\ \lambda_t \end{bmatrix} = A \begin{bmatrix} c_t \\ \lambda_t \end{bmatrix} + B(\sigma_\lambda) \begin{bmatrix} y_t \\ h_t \end{bmatrix} + C w_t$$

In this case the matrices involved are as follows

$$\bar{Q} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \bar{R} = \begin{bmatrix} \zeta & 0 \\ 0 & -\theta \end{bmatrix}$$

$$A = \begin{bmatrix} \nu_c & 0 \\ -1 & \delta \end{bmatrix} \quad B(\sigma_\lambda) = \begin{bmatrix} 0 & \sigma_c \\ -\kappa & -\sigma_\lambda \end{bmatrix} \quad C = \begin{bmatrix} \sigma_c \\ \sigma_\lambda \end{bmatrix}$$

3. **Obtain the solution to Robust LQR problem**

This step involves solving for the value function and the decision rules for the Robust LQR problem

Let $V(c_t, \lambda_t)$ be the maximum present discounted value for the Government defined as follows

$$V(c_t, \lambda_t) = \min_h \max_{y,\lambda} -\frac{1}{2} E \int_0^\infty e^{-\delta t} \left\{ \begin{bmatrix} c_t \\ \lambda_t \end{bmatrix}' \bar{Q} \begin{bmatrix} c_t \\ \lambda_t \end{bmatrix} + \begin{bmatrix} y_t \\ h_t \end{bmatrix}' \bar{R} \begin{bmatrix} y_t \\ h_t \end{bmatrix} \right\}$$

The LQ structure implies that

$$V(c_t, \lambda_t) = -\frac{1}{2} \begin{bmatrix} c_t \\ \lambda_t \end{bmatrix}' P \begin{bmatrix} c_t \\ \lambda_t \end{bmatrix} - \rho$$

and P solves the algebraic matrix Riccati equation.

$$\bar{Q} + A'P + PA - \delta P - PB\bar{R}^{-1}B'P = 0$$

Adjust Riccati equation for discounting (Magill, *JET*, 1977). Define

$$A_a = A - \frac{\delta}{2}$$

The Riccati equation can be reformulated as

$$\bar{Q} + A_a'P + PA_a - PB\bar{R}^{-1}B'P = 0$$

Finally the robust decision rule for output and belief distortion is given by

$$\begin{bmatrix} y_t \\ h_t \end{bmatrix} = -G \begin{bmatrix} c_t \\ \lambda_t \end{bmatrix}$$

where

$$G = \bar{R}^{-1} B' P$$

Define the closed loop for state as $\tilde{A} = A - BG$

To obtain the outcomes under the approximating model, in the calculations that follow just replace $\tilde{A}$ with

$$\breve{A} = A - BG(1, :).$$

This choice sets the mean distortion $h_t \equiv 0$ but incorporates the robust decision rule for the Government.

Note that explicit dependence of B and intrun P on σ_λ is dropped for notational convenience.

4. **Flip roots**

   This step involves making the co-state variable ψ_t ('multiplier on multiplier') as a state variable and the λ_t as a jump variable. This keeps the solution in a stabilizing subspace

   The envelope theorem implies

$$\psi_t = P_{21} Z_t + P_{22} \lambda_t$$

   We flip the roots to obtain

$$\lambda_t = -P_{22}^{-1} P_{21} c_t + P_{22}^{-1} \psi_t$$

$$\begin{aligned} D\psi &= \begin{bmatrix} P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Dc \\ D\lambda \end{bmatrix} \\ &= \begin{bmatrix} P_{21} & P_{22} \end{bmatrix} \tilde{A} \begin{bmatrix} c \\ \lambda \end{bmatrix} + \begin{bmatrix} P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \sigma_c \\ \sigma_\lambda \end{bmatrix} w \end{aligned}$$

But since ψ has no contemporaneous shock exposure it should be that

$$\begin{bmatrix} P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} \sigma_c \\ \sigma_\lambda \end{bmatrix} = 0$$

and so

$$\sigma_\lambda = -P_{22}^{-1} P_{21} \sigma_c.$$

5. **Update for σ_λ**

   The preceding step points towards a new guess for σ_λ

   Let $\hat{\sigma}_\lambda = -P_{22}^{-1} P_{21} \sigma_c$. For some relaxation parameter $\gamma \in (0, 1)$

$$\sigma_\lambda^* = \gamma \hat{\sigma}_\lambda + (1 - \gamma) \sigma_\lambda$$

   Now repeat the iteration with σ_λ^*

B.1 Equilibrium Representation

We can represent the equilibrium under the stable state space. Continuing from step 4, we have

$$D\psi_t = F_{\psi c} c_t + F_{\psi\psi} \psi_t$$

where

$$\begin{aligned} F_{\psi c} &= P_{21}(\tilde{A}_{cc} - \tilde{A}_{c\lambda} P_{22}^{-1} P_{21}) + P_{22}(\tilde{A}_{\lambda c} - \tilde{A}_{\lambda\lambda} P_{22}^{-1} P_{21}) \\ F_{\psi\psi} &= P_{21} \tilde{A}_{Z\lambda} P_{22}^{-1} + P_{22} \tilde{A}_{\lambda\lambda} P_{22}^{-1} \end{aligned}$$

Thus,

$$D \begin{bmatrix} c_t \\ \psi_t \end{bmatrix} = F \begin{bmatrix} c_t \\ \psi_t \end{bmatrix}.$$

Assembling our findings, the law of motion is

$$D \begin{bmatrix} c_t \\ \psi_t \end{bmatrix} = F \begin{bmatrix} c_t \\ \psi_t \end{bmatrix} + \tilde{C} w_t$$

where

$$\tilde{C} = \begin{bmatrix} \sigma_c \\ 0 \end{bmatrix}$$

Form the state-space system

$$\begin{aligned} D \begin{bmatrix} c_t \\ \psi_t \end{bmatrix} &= F \begin{bmatrix} c_t \\ \psi_t \end{bmatrix} + \tilde{C} w_t \\ \begin{bmatrix} c_t \\ \psi_t \\ \lambda_t \end{bmatrix} &= M \begin{bmatrix} c_t \\ \psi_t \end{bmatrix} \end{aligned} \tag{27}$$

where

$$M = \begin{bmatrix} I & 0 \\ -P_{22}^{-1} P_{21} & P_{22}^{-1} \end{bmatrix}.$$

We can use this representation to compute the impulse responses as shown below

C Calibration

For the computations we use the following set of calibrations

Parameter	Value	Description
κ	0.05	Inflation - Output trade off
y^*	0	Target output
ζ	0.08	Government's relative penalty for output fluctuation
δ	0.01	Time Discount Rate
ν_c	-0.2	Persistence of the cost-push shock
σ_c	1	Volatility of the cost-push shock
θ	1000(2500)	Concerns for Ambiguity - Model 1(Model 2)

Table 1: Parameter Values

D Model 1

D.1 Worst Case model

D.1.1 State Variables

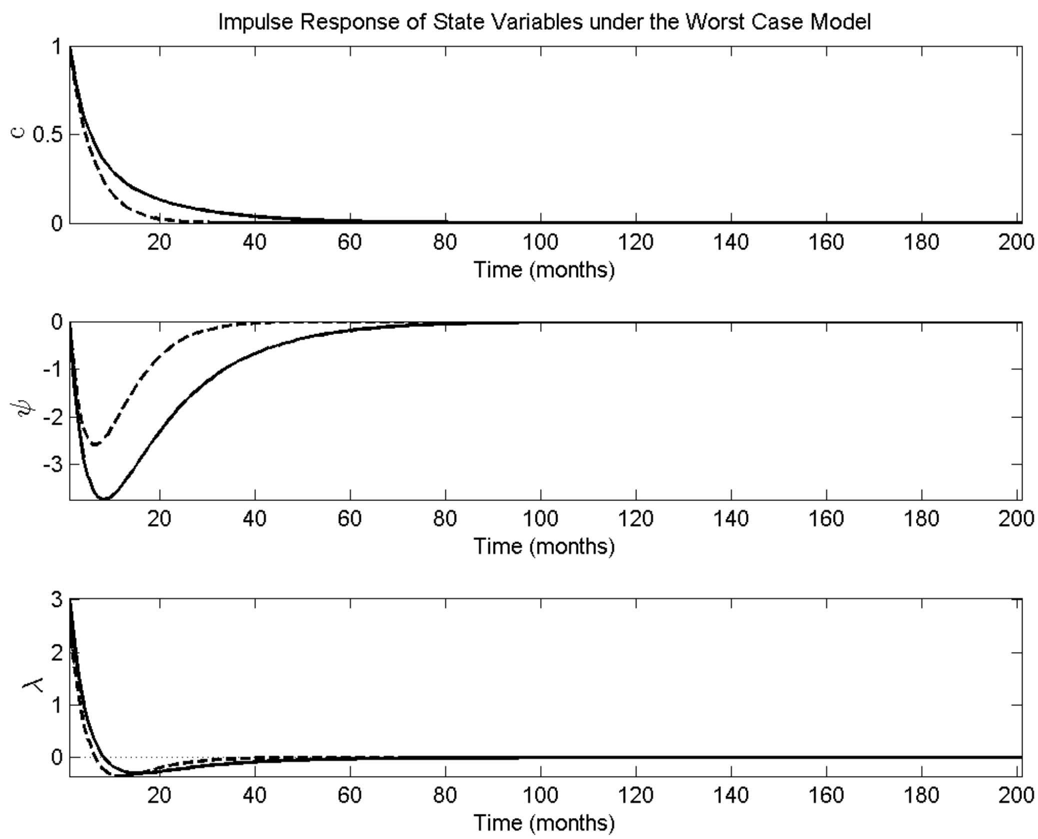

Figure 1: Response of state variables : Cost shock - c_t, multiplier on muliplier - ψ_t and inflation λ_t to w under Model 1 - Worst case. The dashed line is the response without concerns for robustness and the solid line is with concerns for robustness

D.1.2 Control Variables

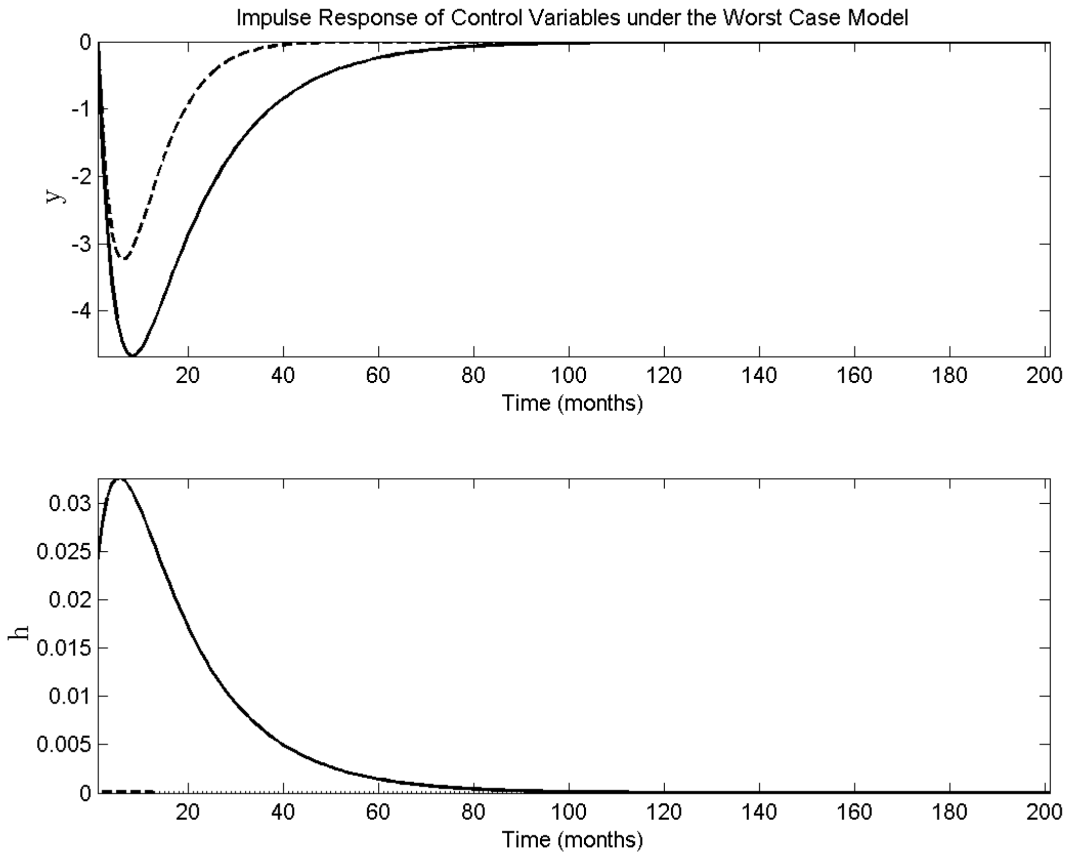

Figure 2: Response of control variables : Output - y_t and belief distortion h_t to w under Model 1 - Worst case. The dashed line is the response without concerns for robustness and the solid line is with concerns for robustness

D.2 Approximating model

D.2.1 State Variables

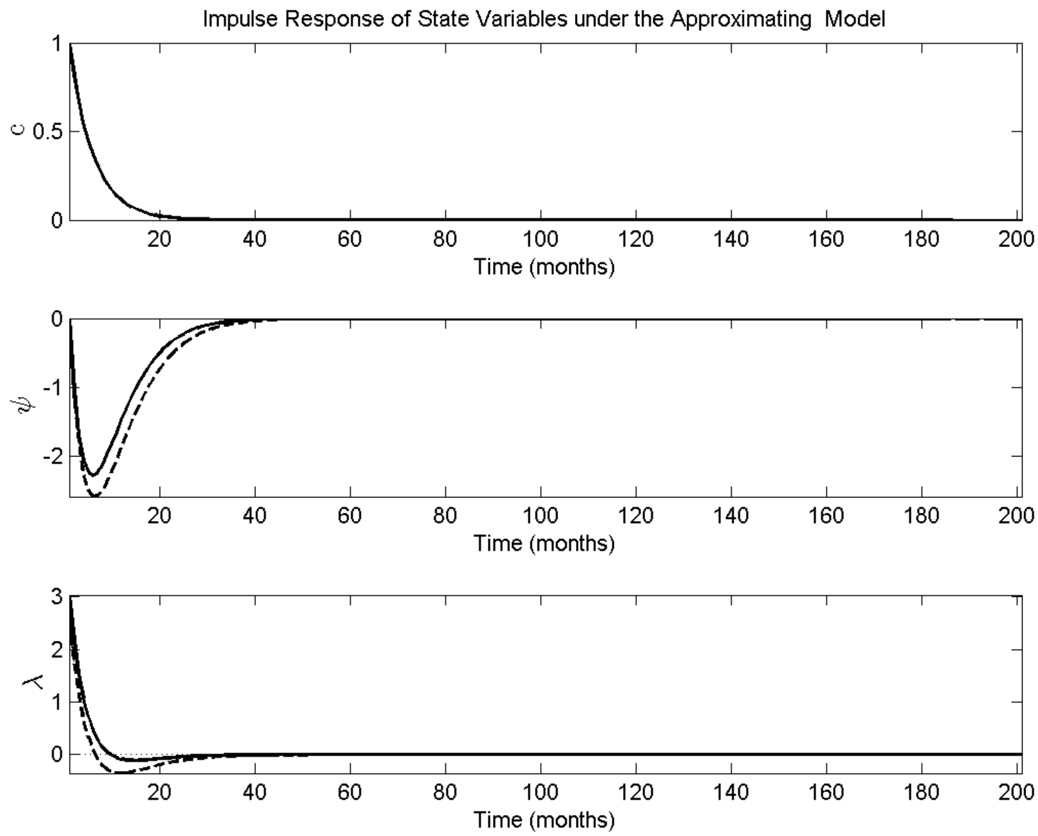

Figure 3: Response of state variables : Cost shock - c_t, multiplier on muliplier - ψ_t and inflation λ_t to w under Model 1 - Approximating Model. The dashed line is the response without concerns for robustness and the solid line is with concerns for robustness

D.2.2 Control Variables

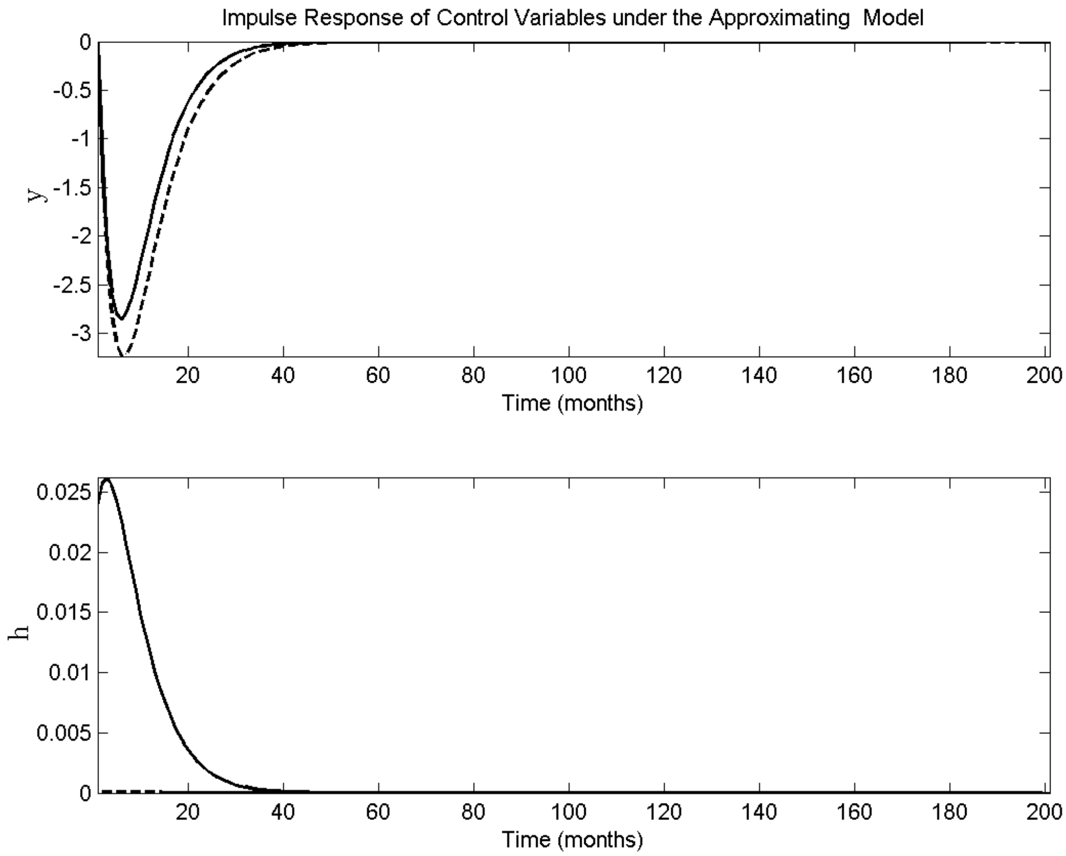

Figure 4: Response of control variables : Output - y_t and belief distortion h_t to w under Model 1 - Approximating Model. The dashed line is the response without concerns for robustness and the solid line is with concerns for robustness

E Model 2

E.1 Worst Case model

E.1.1 State Variables

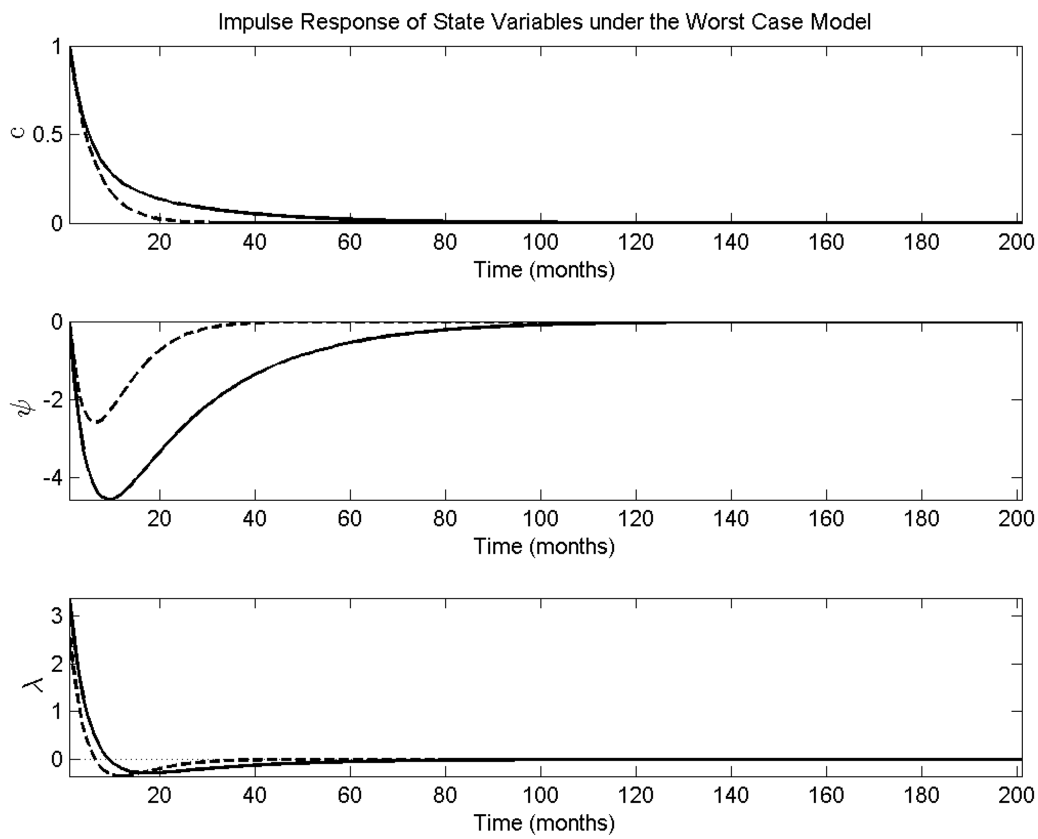

Figure 5: Response of state variables : Cost shock - c_t, multiplier on muliplier - ψ_t and inflation λ_t to w under Model 2 - Worst case. The dashed line is the response without concerns for robustness and the solid line is with concerns for robustness

E.1.2 Control Variables

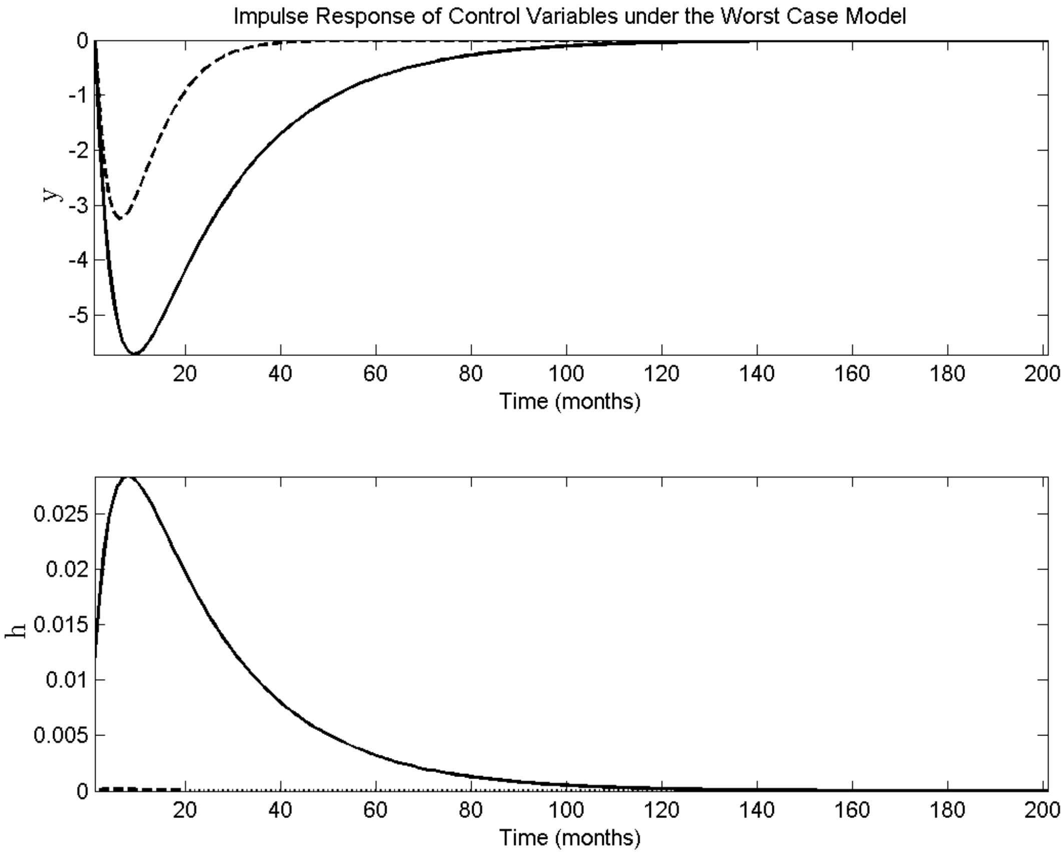

Figure 6: Response of control variables : Output - y_t and belief distortion-h_t to w under Model 2 - Worst case. The dashed line is the response without concerns for robustness and the solid line is with concerns for robustness

E.2 Approximating model

E.2.1 State Variables

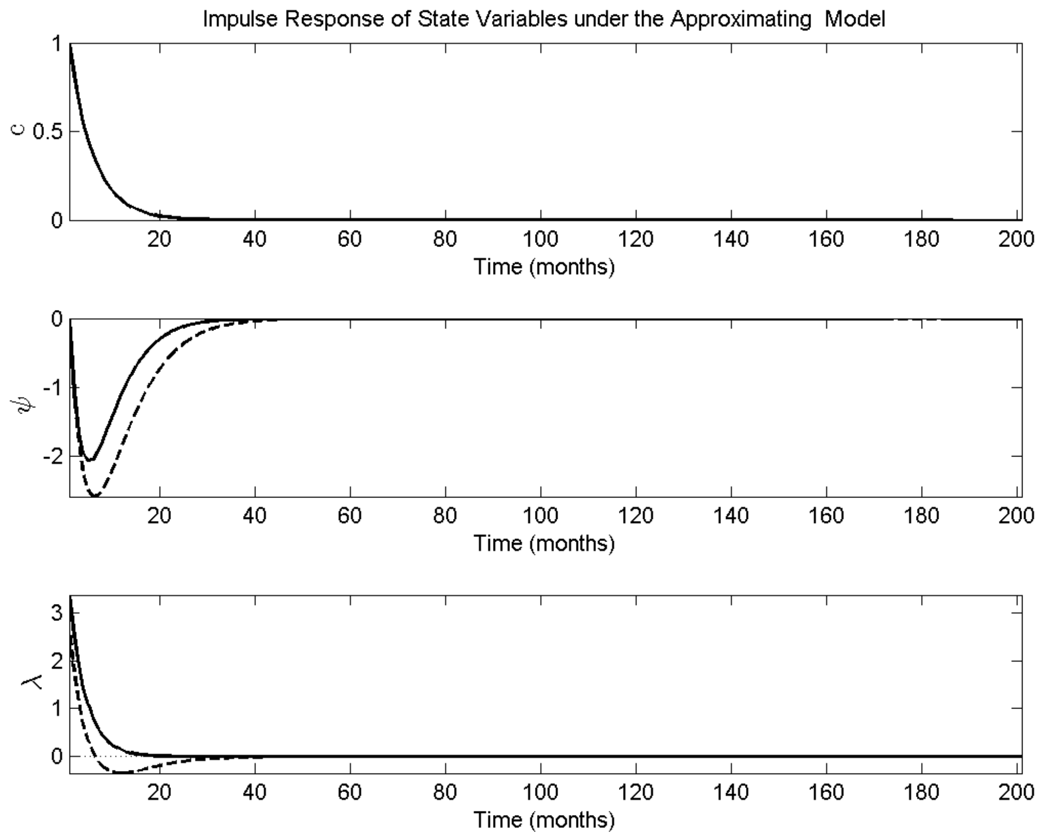

Figure 7: Response of state variables : Cost shock - c_t, multiplier on muliplier - ψ_t and inflation λ_t to w under Model 2 - Approximating Model. The dashed line is the response without concerns for robustness and the solid line is with concerns for robustness

E.2.2 Control Variables

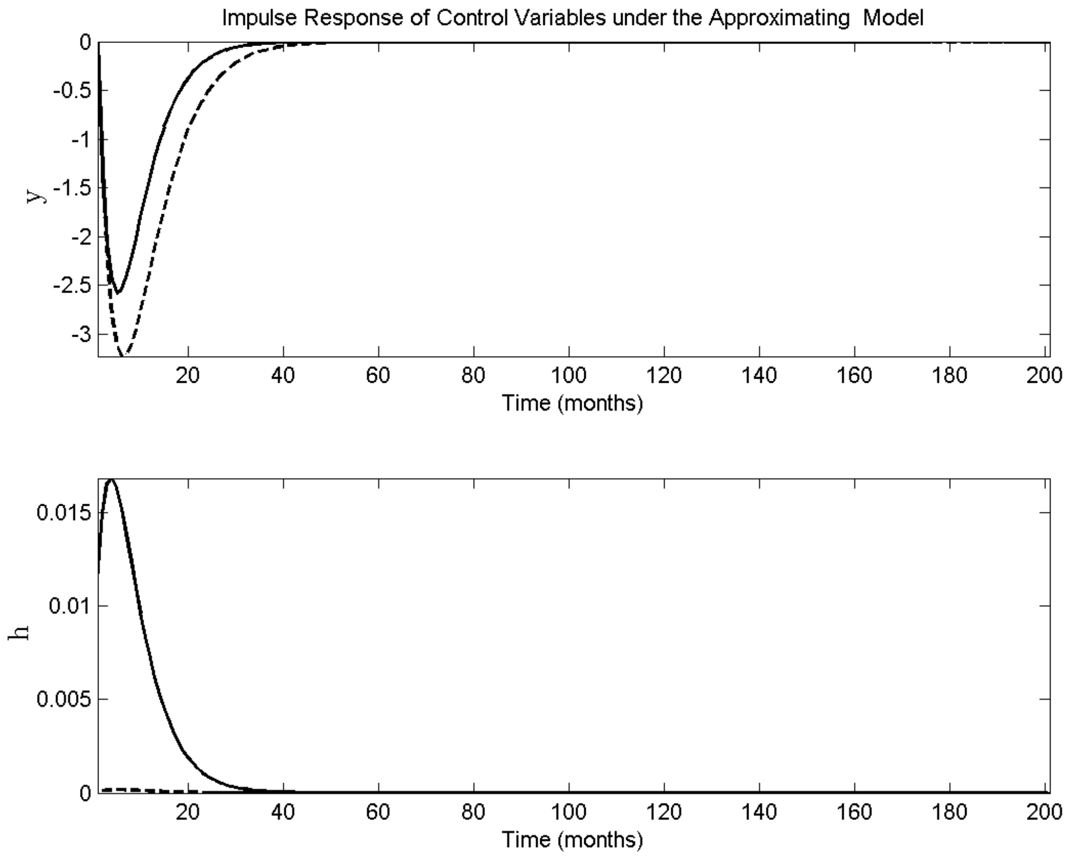

Figure 8: Response of control variables : Output - y_t and belief distortion - h_t to w under Model 2 - Approximating Model. The dashed line is the response without concerns for robustness and the solid line is with concerns for robustness