Ambiguous Business Cycles*

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Abstract

This paper considers business cycle models with agents who are averse not only to risk, but also to ambiguity (Knightian uncertainty). Ambiguity aversion is described by recursive multiple priors preferences that capture agents' lack of confidence in probability assessments. While modeling changes in risk typically calls for higher order approximations, changes in ambiguity in our models work like changes in conditional means. Our models thus allow for uncertainty shocks but can still be solved and estimated using simple 1st order approximations. In an otherwise standard business cycle model, an increase in ambiguity (that is, a loss of confidence in probability assessments), acts like an 'unrealized' news shock: it generates a large recession accompanied by ex-post positive excess returns.

1 Introduction

Recent events have generated renewed interest in the effects of changing uncertainty on macroeconomic aggregates. The standard framework of quantitative macroeconomics is based on expected utility preferences and rational expectations. Changes in uncertainty are typically modeled as *expected and realized changes in risk.* Indeed, expected utility agents think about the uncertain future in terms of probabilities. An increase in uncertainty is described by the expected increase in a measure of risk (for example, the conditional volatility of a shock or the probability of a disaster). Moreover, rational expectations implies that agents' beliefs coincide with those of the econometrician (or model builder). An expected increase in risk must on average be followed by a realized increase in the volatility of shocks or the likelihood of disasters.

This paper studies business cycle models with agents who are averse to ambiguity (Knightian uncertainty). Ambiguity averse agents do not think in terms of probabilities

*Preliminary and incomplete.

– they lack the confidence to assign probabilities to all relevant events. An increase in uncertainty may then correspond to a loss of confidence that makes it more difficult to assign probabilities. Formally, we describe preferences using multiple priors utility (Gilboa and Schmeidler (1989)). Agents act *as if* they evaluate plans using a worst case probability drawn from a set of multiple beliefs. A loss of confidence is captured by an increase in the set of beliefs. It could be triggered, for example, by worrisome news about the future. Agents respond to a loss of confidence as their worst case probability changes.

The paper proposes a simple and tractable way to incorporate ambiguity and shocks to confidence into a business cycle model. Agents' set of beliefs is parametrized by an interval of means for exogenous shocks, such as innovations to productivity. A loss of confidence is captured by an increase in the width of such an interval; in particular it makes the "worst case" mean even worse. Intuitively, a shock to confidence thus works like a news shock: an agent who loses confidence responds *as if* he had received bad news about the future. The difference between a loss of confidence and bad news is that the latter is followed, on average, by the realization of a bad outcome. This is not the case for a confidence shock.

We study ambiguity and confidence shocks in economies that are essentially linear. The key property is that the worst case means supporting agents' equilibrium choices can be written as a linear function of the state variables. It implies that equilibria can be accurately characterized using first order approximations. In particular, we can study agents' responses to changes in uncertainty, as well as time variation in uncertainty premia on assets, without resorting to higher order approximations. This is in sharp contrast to the case of changes in risk, where higher order solutions are critical. We illustrate the tractability of our method by estimating a medium scale DSGE model with ambiguity about productivity shocks.

The effects of a lack of confidence are intuitive. On average, less confident agents engage in precautionary savings and, other things equal, accumulate more steady state capital. A sudden loss of confidence about productivity generates a wealth effect – due to more uncertain wage and capital income in the future, and also a substitution effect since the return on capital has become more uncertain. The net effect on macroeconomic aggregates depends on the details of the economy. In our estimated medium scale DSGE model, a loss of confidence generates a recession in which consumption, investment and hours decline together. In addition, a loss of confidence generates increased demand for safe assets, and opens up a spread between the returns on ambiguous assets (such as capital) and safe assets. Business cycles driven by changes in confidence thus give rise to countercyclical spreads or premia on uncertain assets.

Our paper is related to several strands of literature. The decision theoretic literature on ambiguity aversion is motivated by the Ellsberg Paradox. Ellsberg's experiments suggest that decision makers' actions depend on their confidence in probability assessments – they treat lotteries with known odds differently from bets with unknown odds. The multiple priors model describe such behavior as a rational response to a lack of information about the odds. To model intertemporal decision making by agents in a business cycle model, we use a recursive version of the multiple priors model that was proposed by Epstein and Wang (1994) and has recently been applied in finance (see Epstein and Schneider (2010) for a discussion and a comparison to other models of ambiguity aversion). Axiomatic foundations for recursive multiple priors were provided by Epstein and Schneider (2003).

Hansen et al. (1999) and Cagetti et al. (2002) study business cycles models with robust control preferences. Under the robust control approach, preferences are smooth – utility contains a smooth penalty function for deviations of beliefs from some reference belief. Smoothness rules out first order effects of uncertainty. Models of changes in uncertainty with robust control thus rely on higher order approximations, as do models with expected utility. In contrast, multiple priors utility is not smooth when belief sets differ in means. As a result, there are first order effects of uncertainty – this is exactly what our approach exploits to generate linear dynamics in response to uncertainty shocks.

The mechanics of our model are related to the literature on news shocks (for example Beaudry and Portier (2006), Christiano et al. (2008), Schmitt-Grohe and Uribe (2008), Jaimovich and Rebelo (2009), Blanchard et al. (2009), Christiano et al. (2010a) and Barsky and Sims (2011)). In particular, Christiano et al. (2008) have considered the response to temporary unrealized good news about productivity to study stock market booms. When we apply our approach to news about productivity, a loss confidence works like a (possibly persistent) unrealized decline in productivity. Recent work on changes in uncertainty in business cycle models has focused on changes in realized risk – looking either at stochastic volatility of aggregate shocks (see for example Fernández-Villaverde and Rubio-Ramirez (2007), Justiniano and Primiceri (2008), Fernández-Villaverde et al. (2010) and the review in Fernández-Villaverde and Rubio-Ramírez (2010)) or at changes in idiosyncratic volatility in models with heterogeneous firms (Bloom et al. (2009), Bachmann et al. (2010)). We view our work as complementary to these approaches. In particular, confidence shocks can generate responses to uncertainty – triggered by news, for example – that is not connected to later realized changes in risk.

The paper proceeds as follows. Section 2 describes a stylized business cycle model with ambiguity. Section 3 presents a general framework for adapting business cycle models to incorporate ambiguity aversion. Section 4 discusses the applicability of linear methods. Section 5 describes the estimation of a DSGE model for the US.

2 Ambiguous business cycles: a simple example

To illustrate the role of ambiguity in business cycles, we consider a stylized business cycle model. Our main criterion for this model is simplicity. We abstract from internal propagation of shocks through endogenous state variables such as capital or sticky prices or wages. For uncertainty about productivity to have an effect on labor hours and output, we assume that labor has to be chosen before productivity is known. This introduces an intertemporal decision that depends on both risk and ambiguity. In fact, with the special preferences and technology we choose, the effects of both ambiguity and risk can be read off a loglinear closed form solution, which facilitates comparison.

A representative agent has felicity over consumption and labor hours

$$U(C,N) = \frac{1}{1-\gamma}C^{1-\gamma} - \beta N$$

where γ is the coefficient of relative risk aversion (CRRA) or equivalently the inverse of the intertemporal elasticity of consumption (IES). Agents discount the future with the discount factor β. Setting the marginal disutility of labor equal to β simplifies some algebra below by eliminating constant terms.

Output Y_t is made from labor N_t according to the linear technology.

$$Y_t = Z_t N_{t-1},$$

where $\log Z_t$ is random. The fruits of labor effort made at date $t-1$ thus only become available at date t. One interpretation is that goods have to be stored for some time before they can be consumed. It may be helpful to think of the period length as very short, such as a week.

For simplicity, we assume that log productivity $z_t = \log Z_t$ is serially independent and normally distributed. The productivity process takes the form

$$z_{t+1} = \mu_t - \frac{1}{2}\sigma_u^2 + u_{t+1} \tag{2.1}$$

Here u is an iid sequence of shocks, normally distributed with mean zero and variance σ_u^2. The sequence μ is deterministic and unknown to agents – its properties are discussed further below.

Agents perceive the unknown component μ_t to be ambiguous. We parametrize their one-step-ahead set of beliefs at date t by a set of means $\mu_t \in [-a_t, a_t]$. Here a_t captures agents' lack of confidence in his probability assessment of productivity z_{t+1}. We allow confidence itself to change over time to reflect, for example, news agents receive. We assume an AR(1) process for a_t:

$$a_{t+1} = (1 - \rho_a)\,\bar{a} + \rho_a a_t + \varepsilon_{t+1}^a \tag{2.2}$$

with $\bar{a} > 0$ and $0 < \rho_a < 1$. The lack of confidence a_t thus reverts to a long run mean $\bar{a}$. Periods of low $a_t < \bar{a}$ represent unusually high confidence in future productivity, whereas $a_t > \bar{a}$ describes periods of unusual lack of confidence. We further assume that ε_t^a is independent of u_t. This is for simplicity only – more generally, it could may be interesting to allow for confidence to be correlated with the level or the conditional variance of z_t.

Consider now the Bellman equation of the social planner problem

$$V(Y, a) = \max_N \left\{ U(Y, N) + \beta \min_{\mu \in [-a, a]} E^\mu \left[V\left(e^{\tilde{z}} N, \tilde{a}\right) \right] \right\}$$

where tildes indicate random variables. In particular, the conditional distribution of $\tilde{z}$ is given by (2.1) with μ_t equal to the superscript μ on the expectation operator. The transition law of the exogenous state variable a is given by the last line in (2.2).

It is natural to conjecture that the value function is increasing in output. The "worst case" mean is then always $\mu = -a$. Combining the first order condition for labor with the envelope condition, we obtain

$$\beta = E^{-a}\left[\beta \left(\tilde{Z} N\right)^{-\gamma} \tilde{Z}\right] \tag{2.3}$$

The constant marginal disutility of labor is equal to the marginal product of labor, weighted by future marginal utility because labor is chosen one period in advance.

2.1 The effect of uncertainty on hours

With our special preferences and technology, optimal hours are independent of current productivity (or output). Taking logarithms and using normality of the shocks, we can write the decision rule for hours as

$$n = -(1/\gamma - 1)\left(a + \frac{1}{2}\gamma\sigma_u^2\right) \tag{2.4}$$

The first term describes the effect of uncertainty on aggregate hours. Here uncertainty works the same way whether it is ambiguity, as measured by a, or risk, as measured by the product of the quantity of risk σ_u^2 and risk aversion γ.

As usual, an increase in uncertainty has wealth and substitution effects. Consider first an increase in risk. On the one hand, higher risk lowers the certainty equivalent of future production, which, in the absence of ambiguity, is given by $N \exp(-\frac{1}{2}\gamma\sigma_u^2)$. Other things equal, the resulting wealth effect leads the planner to reduce consumption of leisure and increase hiring. However, higher risk also lowers the risk adjusted return on labor. Other things equal, the resulting substitution effect leads the planner to reduce hiring. The net effect depends on the curvature in felicity from consumption, determined by γ. With a strong enough substitution effect, an increase in risk lowers hiring.

Consider now an increase in ambiguity. When a increases, the planner acts *as if* expected future productivity has declined. Mechanically,. an increase in ambiguity thus entails wealth and substitution effects familiar from the analysis of news shocks. The interpretation of these effects, however, is the same as in the risk case. On the one hand, higher ambiguity lowers the certainty equivalent of future production, which, in the absence of risk, is given by $N \exp(-a_t)$. On the other hand, higher ambiguity lowers the uncertainty-adjusted return on labor. Again, with a strong enough substitution effect an increase in uncertainty lowers hiring.

Given separable felicity and the iid dynamics of z_t, inspection of the Bellman equation shows that the value function depends on current output only through the utility of consumption – the other terms depend only on the state variable a_t, not on current productivity or past hours. It follows that the value function is increasing in output, verifying our conjecture above. Below, we argue that the "guess-and-verify" approach to finding the worst case belief that we have used here to solve the planner problem is applicable much more widely.

The complete dynamics of the model are then given by the productivity equation (2.1) as well as

$$
\begin{aligned}
y_t &= z_t + n_t, \\
n_t &= -\left(1/\gamma - 1\right)\left(a_t + \frac{1}{2}\gamma\sigma_u^2\right), \\
a_t &= (1-\rho_a)\,\bar{a} + \rho_a a_{t-1} + \varepsilon_t^a,
\end{aligned}
$$

The economy is driven by productivity and ambiguity shocks. Productivity shocks temporarily change output but have no effect on hours. In contrast, ambiguity shocks have persistent effects on both hours and output.

With a strong enough substitution effect ($1/\gamma > 1$), a loss of confidence (an increase in a) generates a recession. During that recession, productivity is not unusually low. Hours are nevertheless below steady state: since the marginal product of labor is more uncertain, the planner finds it optimal not to make people work. Conversely, an unusual increase in confidence – a drop of a_t below its long run mean – generates a boom in which employment and output are unusually high, but productivity is not.

2.2 Decentralization

Suppose that agents have access to a set of contingent claims. Write $q_t(\tilde{z},\tilde{a})$ for the date t price of a claim that pays one unit of consumption at date $t+1$ if $(z_{t+1},a_{t+1})=(\tilde{z},\tilde{a})$ is realized and denote the spot wage by w_t. The agent's date t budget constraint is

$$C_t+\int q_t(\tilde{z},\tilde{a})\,\theta_t(\tilde{z},\tilde{a})\,d(\tilde{z},\tilde{a})=w_tN_t+\theta_{t-1}(z_t,a_t),$$

where $\theta_t(\tilde{z},\tilde{a})$ is the amount of claims purchased at t that pays off one unit of the consumption good if $(\tilde{z},\tilde{a})$ is realized at $t+1$. Since aggregate labor is determined one period in advance, this set of contingent claims completes the market – claims on $(\tilde{z},\tilde{a})$ can be used to form any portfolio contingent on the aggregate state (Y,a).

Assume that there are time invariant functions for prices $q(\tilde{z},\tilde{a};Y,a)$ and $w(Y,a)$ as well as aggregate labor $N(Y,a)$ that depend only on the aggregate state (Y,a). Assume further that the agent knows those price functions. The Bellman equation is

$$\begin{aligned}W(\theta,Y,a)&=\max_{C,N,\theta'(.)}\left\{U(C,N)+\beta\min_{\mu\in[-a,a]}E^{\mu}\left[W\left(\theta'(\tilde{z},\tilde{a}),e^{\tilde{z}}N(Y,a),\tilde{a}\right)\right]\right\}\\ w(Y,a)N+\theta&=C+\int q(\tilde{z},\tilde{a};Y,a)\,\theta'(\tilde{z},\tilde{a})\,d(\tilde{z},\tilde{a})\end{aligned}$$

Conjecture again that utility depends negatively on the state variable Y. The worst case mean is then once more $\mu=-a$ and the maximization problem becomes standard.

In particular, prices are related to the agent's marginal rates of substitution through Euler equations. Letting $f^{\mu}(\tilde{z},\tilde{a}|a)$ denote the conditional density of the exogenous variables (z,a) implied by (2.1) and (2.2) with $\mu_t=\mu$, we have

$$w=\chi C(\theta,Y,a)^{\gamma} \tag{2.5}$$

$$q(\tilde{z},\tilde{a};Y,a)=\beta f^{-a}(\tilde{z},\tilde{a}|a)\left(\frac{C\left(\theta'(\tilde{z},\tilde{a}),e^{\tilde{z}}N(Y,a),\tilde{a}\right)}{C(\theta,Y,a)}\right)^{-\gamma} \tag{2.6}$$

The wage is equal to the marginal rate of substitution of consumption for hours. State prices are equal to the marginal rate of substitutions of current for future consumption. Importantly, state prices are based on the worst case conditional density f^{-a}. This is how ambiguity aversion contributes to asset premia and how it shapes firms' decisions in the face of uncertainty.

For simplicity, we consider two-period lived firms that hire workers only at date t and sell output only at date $t+1$. To pay the wage bill at date t, they issue contingent claims which they subsequently pay back out of revenue at date $t+1$. The profit maximization problem is

$$\max_{N,\theta(.)} \int q(\tilde{z},\tilde{a};Y,a)\,(e^{\tilde{z}}N - \theta(\tilde{z},\tilde{a}))d(\tilde{z},\tilde{a})$$
$$\text{s.t. } wN = \int q(\tilde{z},\tilde{a};Y,a)\,\theta(\tilde{z},\tilde{a})\,d(\tilde{z},\tilde{a})$$

As usual, the financial policy of the firm is indeterminate. Substituting the constraint into the objective, the first order condition with respect to labor equates the wage to the marginal product of labor

$$w = \int q(\tilde{z},\tilde{a};Y,a)\,e^{\tilde{z}}d(\tilde{z},\tilde{a}) \tag{2.7}$$

Since labor is chosen one period in advance, the marginal product of labor involves state prices, which in turn reflect uncertainty perceived by agents. Substituting for prices from (2.5)-(2.6), we find that the planner's first order condition for labor (2.3) must hold in any equilibrium.

From the first order conditions, wages and state prices can be solved out in closed form. Let $Q^f(Y,a) = \int q(\tilde{z},\tilde{a};Y,a)\,d(\tilde{z},\tilde{a})$ denote the price of a riskless bond. We can then write

$$\begin{aligned} w(Y,a) &= \beta Y^{\gamma}, \\ Q^f(Y,a) &= \beta Y^{\gamma}\exp\left(a+\gamma\sigma_u^2\right), \\ q(\tilde{z},\tilde{a};Y,a) &= Q^f(Y,a)\,f^0(\tilde{z},\tilde{a}|a)\exp\left(-\frac{1}{2}\sigma_u^2\left(a/\sigma_u^2+\gamma\right)^2 - \left(a/\sigma_u^2+\gamma\right)\left(\tilde{z}-\frac{1}{2}\sigma_u^2\right)\right), \end{aligned} \tag{2.8}$$

where f^0 is the density o the exogenous variables $(\tilde{z},\tilde{a})$ if $\mu_t = 0$.

With utility linear in hours, labor supply is perfectly elastic at a wage tied to current output. Since output does not react to uncertainty shocks on impact, neither does the wage. Uncertainty shocks are transmitted to the labor market because asset prices affect labor demand. The bond price increases with both ambiguity and risk. Intuitively, either type of uncertainty encourages precautionary savings and thereby lowers the riskless interest rate

$$r^f(Y,a) = -\log Q^f(Y,a) = -\log\beta - \gamma\log Y - a - \gamma\sigma_u^2$$

The price of a claim on a particular state $(\tilde{z},\tilde{a})$ is equal to the riskless bond price multiplied by an "uncertainty neutral" density. We have written the latter as the density for $\mu_t = 0$ times an exponential term that collects uncertainty premia.

If agents do not care about either type of uncertainty ($a = \gamma = 0$), then uncertainty premia are zero and the exponential term is one. More generally, the relative price of a "bad" state (that is, lower productivity $\tilde{z}$) is higher when confidence is lower (or a is higher). Intuitively, when confidence is lower, then agents value the insurance provided by claims on bad states more highly. This change in relative prices also affects firms' hiring decision. Indeed, since firms can pay out more in good (high $\tilde{z}$) states, a loss of confidence that makes claims on good states less valuable increases firms' funding cost. Conversely, an increase in confidence makes claims on good states more valuable; lower funding costs then induce more hiring.

The functional form of the state price density is that of an affine pricing model with "time-varying market prices of risk" (that is, time varying coefficients multipling the shocks). This type of pricing model is widely used in empirical finance. Here time variation in confidence drives the coefficient $a/\sigma_u^2 + \gamma$ on the shock $\tilde{z}$ and thus permits a structural interpretation of the functional form. A convenient feature of affine models is that conditional expected returns on many interesting assets are linear functions of the state variables. Consider, for example, a claim to consumption next period. Its price and excess return are

$$\begin{aligned} Q^c(Y,a) &= \beta E^{-a}\left[\left(e^{\tilde{z}}N(Y,a)\right)^{1-\gamma}Y^{\gamma}\right] = Q^f(Y,a)\,N(Y,a)\exp\left(-a-\gamma\sigma_u^2\right) \\ r^e(\tilde{z},Y,a) &= \log\left(e^{\tilde{z}}N(Y,a)\right) - \log Q^c(Y,a) - \log Q^f(Y,a) \\ &= \tilde{z} + a + \gamma\sigma_u^2 \end{aligned}$$

Long run average excess returns have an ambiguity and a risk component. Moreover, the conditional expected excess return depends positively on a. In other words, a loss of confidence not only generates a recession, but also increases the *conditional* premium on the consumption claim.

2.3 Bounding ambiguity by measured volatility

Consider now the connection between the true dynamics of log productivity z in (2.1) and the agents' set of beliefs. In our model, productivity consists of two components, the iid shock u that agents view as risky, and the deterministic sequence μ that agents view as ambiguous. In line with agents' lack of knowledge about μ, we do not impose a particular sequence as "the truth". Instead, we restrict only the long run average and variability of μ, and thereby also of productivity z. We then develop a bound on the process a_t that ensures that the belief set is "small enough" relative to the variability in the data observed by agents. For quantitative modeling, the bound imposes discipline on how much the process a_t can vary relative to the volatility in the data measured by the modeler.

Consider first the long run behavior of μ. Let I denote the indicator function and let $\Phi\left(., m, s^2\right)$ denote the cdf of a univariate normal distribution with mean m and variance s^2. We assume that the empirical distribution of μ converges to that of an iid normal stochastic process with mean zero and variance σ_μ^2. Formally, for any integers $k, \tau_1, ..., \tau_k$ and real numbers $\bar{\mu}_1, ..., \bar{\mu}_k$,

$$\lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} I\left(\left\{\mu_{t+\tau_j} \leq \bar{\mu}_j; \quad j = 1, .., k\right\}\right) = \prod_{j=1}^{k} \Phi\left(\bar{\mu}_j; 0, \sigma_\mu^2\right).$$

For example, if we were to observe μ and record the frequency of the event $\{\mu_t \leq \bar{\mu}\}$ then that frequency would converge to $\Phi\left(\bar{\mu}, 0; \sigma_\mu^2\right)$. For a two-dimensional example, consider the frequency of the event $\{\mu_t \leq \bar{\mu}_1, \mu_{t+\tau} \leq \bar{\mu}_2\}$ – it is assumed to converge to $\Phi\left(\bar{\mu}_1, 0; \sigma_\mu^2\right) \Phi\left(\bar{\mu}_2, 0; \sigma_\mu^2\right)$. Similarly, recording frequencies of any joint event that restricts elements of μ spaced in time as described by the τ_js always delivers in the long run the cdf of and iid multivariate normal.distribution. At the same time, almost every draw from an iid normal process with mean zero and variance σ_μ^2 would deliver a sequence μ_t that satisfies the condition.

Our assumption on the long run empirical distribution of μ also has implications for long run empirical distribution of log producitivity z. Indeed, given a true sequence μ that satisfies the above condition, the law of large numbers says that, for almost every realization of the shocks u, the empirical mean $\frac{1}{T}\sum_t z_t$ converges to zero, the empirical variance $\frac{1}{T}\sum_t z_t^2$ converges to $\sigma_z^2 = \sigma_\mu^2 + \sigma_u^2$, and the empirical autocovariances at all leads and lags converge to zero. In other words, to an observer who sees a large sample, the data look iid with mean zero and variance σ_z^2 regardless of the true sequence μ. If the observer tries to fit a stationary statistical model, he recovers this iid process.

Ambiguity averse agents look at the data differently. Even though they know the limiting properties of μ and hence of z, when they make decisions at date t, they are concerned that they do not know the current conditional mean μ_t needed to forecast z_{t+1}. They understand that statistical tools cannot help them learn μ_t in real time. They deal with their lack of knowledge at date t by behaving *as if* they minimize over a set of forecasting rules (that is, a set of one-step-ahead conditional probabilities) indexed by the interval $[-a_t, a_t]$. It makes sense to assume that this interval should be smaller the less variable the data are (lower σ_z^2) and, in particular, the less variability in the data is attributed to ambiguity as opposed to risk (lower σ_μ^2).

We thus develop a bound on the process a_t, denoted $a^{\max}$, that is increasing in σ_μ^2. The basic idea is that even the boundary forecasts indexed by $\pm a^{\max}$ should be "good enough" in the long run. To define "good enough", we calculate the frequency with which one of the boundary forecasting rules is the best forecasting rule in the interval $[-a^{\max}, a^{\max}]$. The forecasting rule with mean $a^{\max}$ is the best rule at date t if its mean $a^{\max} - \frac{1}{2}\sigma_u^2$ is closest to the true conditional mean $\mu_t - \frac{1}{2}\sigma_u^2$, that is, if $\mu_t \geq a^{\max}$. Similarly, the rule $-a^{\max}$ is the best rule if $\mu_t \leq -a^{\max}$. We now require that the frequency with which μ_t falls outside the interval $[-a^{\max}, a^{\max}]$, thus making the boundary forecasts the best forecasts, converges in the long run to a number $\alpha \in (0.1)$. Given our assumption on the long run behavior of μ above, the bound is defined by

$$\Phi\left(a^{\max}; 0, \sigma_\mu^2\right) = \alpha/2$$

The number α determines the tightness of the bound. For example, $\alpha = 5\%$ implies $a^{\max} \approx 2\sigma_\mu$.

The bound $a^{\max}$ restricts the variability in the worst case mean relative to measured volatility in the data. Suppose the variance of the productivity is measured to be σ_z^2. Denote by $\rho = \sigma_\mu^2/\sigma_z^2$ the share of the variability in the data that agents attribute to ambiguity. Then with $\alpha = 5\%$ we require $a_t \leq 2\sqrt{\rho}\sigma_z$. The bound is tighter if less of the variability in the data is due to ambiguity. In the extreme case of $\rho = 0$, the process a_t must be identically equal to zero – agents treat all variability in z as risk. In practice, the bound dictates parameter restrictions on the law of motion for a_t. In a discrete time model, we cannot impose exactly that $a_t \in [0, 2\sqrt{\rho}\sigma_z]$. However, small enough volatility of ε_t^a in (2.2) ensures that those conditions are virtually always satisfied – this is the approach we follow in our quantitative work below.

It is interesting to compare how risk and ambiguity affect the long run behavior of business cycles variables in our simple model. Consider, for example, the empirical mean and variance of output in a large sample

$$\bar{y} = -\frac{1}{2}\sigma_u^2 - (1/\gamma - 1)\left(\bar{a} + \frac{1}{2}\gamma\sigma_u^2\right)$$
$$\bar{\sigma}_y^2 = \sigma_z^2 + (1/\gamma - 1)^2 \, var\,(a_t)$$

Here we have used the law of large numbers for u together with our assumptions on μ, which imply that the long run moments are the same for every possible sequence μ. The bound puts discipline on the role of ambiguity in explaining business cycles. For example, suppose that we assume $\bar{a} > 3\sqrt{var\,(a_t)}$ and $\bar{a} + 3\sqrt{var\,(a_t)} < 2\sqrt{\rho}\sigma_z$ in order to keep a almost always in the interval $[0, 2\sqrt{\rho}\sigma_z]$. Together these conditions imply that $var\,(a_t) < (\rho/9)\sigma_z^2$, which in turn bounds the share of $\bar{\sigma}_y^2$ that can be contributed by time-varying ambiguity.

3 Ambiguous business cycles: a general framework

Uncertainty is represented by a period state space S. One element $s \in S$ is realized every period, and the history of states up to date t is denote $s^t = (s_0, ..., s_t)$.

3.1 Preferences

Preferences order uncertain streams of consumption $C = (C_t)_{t=0}^{\infty}$, where $C_t : S^t \to \Re^n$ and n is the number of goods. Utility for a consumption process $C = \{C_t\}$ is defined recursively by

$$U_t\left(C; s^t\right) = u\left(C_t\right) + \beta \min_{p \in \mathcal{P}(s^t)} E^p\left[U_{t+1}\left(C; s_t, s_{t+1}\right)\right], \tag{3.1}$$

where $\mathcal{P}\left(s^t\right)$ is a set of probabilities on S.

Utility after history s^t is given by felicity from current consumption plus expected continuation utility evaluated under a "worst case" belief. The worst case belief is drawn from a set $\mathcal{P}_t\left(s^t\right)$ that may depend on the history s^t. The primitives of the model are the felicity u, the discount factor β and the entire process of one-step-ahead belief sets $\mathcal{P}_t\left(s^t\right)$. Expected utility obtains as a special case if all sets $\mathcal{P}_t\left(s^t\right)$ contain only one belief. More generally, a nondegenerate set of beliefs captures the agent's lack of confidence in probability assessments; a larger set describes a less confident agent.

For our applications, we assume Markovian dynamics. In particular, we restrict belief sets to depend only on the last state and write $\mathcal{P}_t\left(s_t\right)$. We also assume that the true law of motion – from the perspective of an observer – is given by a transition probability $p^*\left(s_t\right)$. The assumption here is that agents know that the dynamics are Markov, but they are not confident in what transition probabilities to assign. The true DGP is not relevant for decision making, but it matters for characterizing the equilibrium dynamics.

3.2 Environment & equilibrium

We consider economies with many agents $i \in I$. Agent i's preferences are of the form (3.1) with primitives $(\beta_i, u^i, \{\mathcal{P}_t^i(s_t)\})$. Given preferences, it is helpful to write the rest of the economy in fairly general notation that many typical problems can be mapped into. Consider a recursive competitive equilibrium that is described using a vector X of endogenous state variables. Let A^i denote a vector of actions taken by agent i. Among those actions is the choice of consumption – we write $c^i(A^i)$ for agent $i's$ consumption bundle implied when the action is A^i. Finally, let Y denote a vector of endogenous variables not chosen by the agent – this vector will typically include prices, but also variables such as government transfers that are endogenous, but are neither part of the state space nor actions or prices.

The technology and market structure are summarized by a set of reduced form functions or correspondences. A recursive competitive equilibrium consists of action and value functions A^i and V^i, respectively, for all agents $i \in I$, as well as a function describing the other endogenous variables Y. We also write A for the collection of all actions $(A^i)_{i\in I}$ and A^{-i} for the collection of all actions except that of agent i. All functions are defined on the state (X, s) and satisfy

$$W^i(A, X, s; p) = u^i\left(c^i\left(A^i\right)\right) + \beta_i E^p\left[V^i(x'(X, A, Y(X, s), s, s'), s')\right] \quad ; i \in I \tag{3.2}$$

$$A^i(X, s) = \arg\max_{A^i \in B^i(Y(X,s), A^{-i}, X, s)} \min_{p \in \mathcal{P}^i(s)} W(A, X, s; p) \qquad i \in I \tag{3.3}$$

$$V^i(X, s) = \min_{p \in \mathcal{P}^i(s)} W^i(A(X, s), X, s; p) \tag{3.4}$$

$$0 = G(A(X, s), Y(X, s), X, s) \tag{3.5}$$

The first equation simply defines the agent's objective in state (X, s), while the second and third equation provide the optimal policy and value. Here B^i is the agent's budget set correspondence and the function x' describes the transition of the endogenous state variables. The function G summarizes all other contemporaneous relationships such as market clearing or the government budget constraint – there are enough equations in (3.5) to determine the endogenous variables Y.

3.3 Characterizing optimal actions & equilibrium dynamics

For every state (X, s), there is a measure $p^{0i}(X, s)$ that achieves the minimum for agent i in (3.3). Since the minimization problem is linear in probabilities, we can replace $\mathcal{P}_t^i$ by its convex hull without changing the solution. The minimax theorem then implies that we can exchange the order of minimization and maximization in the problem (3.3). It follows that the optimal action A^i is the same as the optimal action if the agent held the probabilistic belief $p^{0i}(X, s)$ to begin with. In other words, for every equilibrium of our economy, there exists an economy with expected utility agents holding beliefs p^{0i} that has the same equilibrium.

The observational equivalence just described suggests the following guess-and-verify procedure to compute an equilibrium with ambiguity aversion:

1. guess the worst case beliefs p^{0i}
2. solve the model assuming that the agents have expected utility and beliefs p^{0i}
3. compute the value functions V^i
4. verify that the guesses p^{0i} indeed achieves the minimum in (3.4) for every i.

Suppose we have found the optimal action functions A as well as the response of the endogenous variables Y and hence the transition for the states X. We are interested in stochastic properties of the equilibrium dynamics that can be compared to the data. We characterize the dynamics in the standard way by calculating (for example or simulating) moments of the economy under the true distribution of the exogenous shocks p^*. The only unusual feature is that this true distribution need not coincide with the distribution p^{0i} that is used to compute optimal actions.

3.4 Shocks to confidence

We now specialize a process of belief sets $\mathcal{P}_t$ to capture random changes in confidence. For simplicity, assume that there is a single exogenous process z that directly affects the economy, for example productivity or a monetary policy shock. In addition to z, the exogenous state s has a second component a_t that captures time variation in confidence. Suppose the true dynamics of exogenous state s can be represented by an AR(1) process

$$z_{t+1} = \rho_z z_t + \varepsilon_{t+1}^z,$$
$$a_{t+1} = (1 - \rho_a)\,\bar{a} + \rho_a a_t + \varepsilon_{t+1}^a$$

where the shocks ε^z and ε^a are iid and normally distributed shock with mean zero and variances σ_z^2 and σ_a^2, respectively.

The component a_t is a tool to describe the evolution of the belief set $\mathcal{P}_t$. In particular, we assume that the agent knows the evolution of a_t, but that he is not sure whether the conditional mean of z_{t+1} is really $\rho_z z_t$. Instead, he allows for a range of intercepts. The set $\mathcal{P}_t$ can thus be represented by the family of processes

$$\begin{aligned} z_{t+1} &= \rho_z z_t + \mu_t + \varepsilon_{t+1}^z, \\ \mu_t &\in [-a_t, -a_t + 2|a_t|] \\ a_{t+1} &= (1-\rho_a)\,\bar{a} + \rho_a a_t + \varepsilon_{t+1}^a \end{aligned} \tag{3.6}$$

If a_t is higher, then the agent is less confident about the mean.of z_{t+1} – his belief set is larger. The worst case belief p^0 is now described by a worst case intercept a_t^0. Changes in ambiguity parametrized by a_t change the worst case conditional mean $\rho_z z_t + a_t^0$ and therefore have 1st order effects on behavior.

As long as $a_t > 0$, the interval of intercepts contemplated by the agent is centered around zero – it can equivalently be described by the condition $\mu_t \leq |a_t|$. In contrast, if a_t becomes negative, then all intercepts are positive - the agent is thus optimistic relative to the truth. In applications, it is thus useful to parametrize the dynamics such that a does not become negative very often. The advantage of the present parametrization is that the lower bound, which plays a special role in the computation, is guaranteed to have linear dynamics.

4 Computation in essentially linear economies

The computation and interpretation of equilibria is particularly simple if all dynamics is approximately linear. We start from the assumption that the environment (given by B^i, x' u^i and G) is such that, *under expected utility and rational expectations*, a first order solution provides a satisfactory approximation to the equilibrium dynamics. Under rational expectations, the innovations to a_t are news shocks that provide information about z_t one period ahead. In many applications, standard tools will reveal whether a first order approximation is satisfactory.

It is now natural to look for an equilibrium such that the worst case intercepts a_t^{0i} are all linear in the state variables. If such an equilibrium exists, we call the economy *essentially linear.* In an essentially linear economy, the choices A and other endogenous variables Y will all be well approximated by linear functions of the state variables. The equilibrium dynamics is thus described by a linear state space system with all shocks – including uncertainty shocks – driven by the (true) linear laws of motion. In many interesting economies, it is straightforward to establish essential linearity. Consider, for example, a baseline stochastic growth model with a representative agent who perceives ambiguity about productivity. The worst case is that the mean of productivity innovations is always as low as possible, so $a_t^0 = -a_t$.

More generally, to check whether some general economy is essentially linear, we specialize the guess-and-verify procedure described above. Step 2 of the procedure consists of solving the model using the guesses p^{0i} that sets $a_t^{0i} = -a_t$ or $a_t^{0i} = a_t$. In other words, the worst case for agent i is always either the highest or lowest bound of the interval. Since these guesses just implies a linear shift of the shock, this step can also be done by first order methods. In particular, we propose to linearize the model around a "zero risk" steady state that sets the variance of the shocks to a very small number while retaining the effect of ambiguity on decisions. Properties of zero risk steady states are described in more detail below.

To implement step 3 of the procedure, let V^{0i} denote the value function for the problem with expected utility and μ_t at the guessed worst case mean for agent i in (3.6). For example, consider an agent with $a_t^{0i} = -a_t$. For this agent, we verify the guess by checking whether for any $X, A, s = (z, a)$ and a', the function

$$\tilde{V}\left(z'\right) := V^0(x'\left(X, A, Y\left(X, s\right), s, z', a'\right), z', a')$$

is strictly increasing. We do the opposite check for an agent with $a_t^{0i} = a_t$. At this stage, nonlinearity of the value function could be important. We thus compute the value function using higher order approximations and form the function $\tilde{V}$ accordingly.

In what follows, we first describe the dynamics and then the calculation of the zero risk steady state.

4.1 Dynamics

Let x_t denote the endogenous variables of interest. Posit a linear equilibrium law of motion:

$$x_t = Ax_{t-1} + Bs_t,$$

where s_t are the exogenous variables. For notational purposes, split the vector s_t into the technology shock $\widehat{z}_t \equiv \log z_t$, the mean distortion (or news) a_t and the rest of the exogenous variables, s_t^*.

Posit another linear relation for the exogenous variables:

$$
\begin{aligned}
s_t = \begin{bmatrix} s_t^* \\ \widehat{z}_t \\ a_t \end{bmatrix} &= P \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ a_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ \varepsilon_t^z \\ \varepsilon_t^a \end{bmatrix} \\
\Xi_t = \begin{bmatrix} \varepsilon_t \\ \varepsilon_t^z \\ \varepsilon_t^a \end{bmatrix} &\sim N(0, \Sigma)
\end{aligned}
$$

Ξ_t denotes the innovations to the exogenous variables s_t. They are defined to be zero mean.

We follow the method of undetermined coefficients of Christiano (2002) to solving a system of linear equations with rational expectations. Let the linearized equilibrium conditions be restated in general as:

$$
\widetilde{E}_t[\alpha_0 x_{t+1} + \alpha_1 x_t + \alpha_2 x_{t-1} + \beta_0 s_{t+1} + \beta_1 s_t] = 0 \tag{4.1}
$$

where $\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1$ are constants determined by the equilibrium conditions. Importantly for us,

$$
\begin{aligned}
\widetilde{E}_t \begin{bmatrix} s_{t+1}^* \\ \widehat{z}_{t+1} \\ a_{t+1} \end{bmatrix} &= P \begin{bmatrix} s_t^* \\ \widehat{z}_t \\ a_t \end{bmatrix} + \widetilde{E}_t \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1}^z \\ \varepsilon_{t+1}^a \end{bmatrix} \\
\widetilde{E}_t \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon_{t+1}^z \\ \varepsilon_{t+1}^a \end{bmatrix} &= 0
\end{aligned}
$$

To reflect the time t information (news, or mean distortion) about $\widehat{z}_{t+1}$, recall that

$$
\widetilde{E}_t \widehat{z}_{t+1} = \rho_z \widehat{z}_t - a_t
$$

so the matrix P satisfies the restriction:

$$P = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_a \end{bmatrix}$$

$$\widetilde{E}_t \begin{bmatrix} s^*_{t+1} \\ \widehat{z}_{t+1} \\ a_{t+1} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_a \end{bmatrix} \begin{bmatrix} s^*_t \\ \widehat{z}_t \\ a_t \end{bmatrix} + \widetilde{E}_t \begin{bmatrix} \varepsilon_{t+1} \\ \varepsilon^z_{t+1} \\ \varepsilon^a_{t+1} \end{bmatrix}$$

where ρ is a diagonal matrix reflecting the autocorrelation structure of the elements in s^*_t.Notice that without the "news" part, the standard form for P is:

$$P = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & 0 \\ 0 & 0 & \rho_a \end{bmatrix}$$

Substitute the posited policy rule into the linearized equilibrium conditions:

$$0 = \widetilde{E}_t[\alpha_0(Ax_t + Bs_{t+1}) + \alpha_1 (Ax_{t-1} + Bs_t) + \alpha_2 x_{t-1} + \beta_0 (Ps_t + \Xi_{t+1}) + \beta_1 s_t]$$

to get:

$$\begin{aligned} 0 &= \left(\alpha_0 A^2 + \alpha_1 A + \alpha_2\right) x_{t-1} + (\alpha_0 AB + \alpha_0 BP + \alpha_1 B + \beta_0 P + \beta_1) s_t + (\alpha_0 B + \beta_0)\widetilde{E}_t \Xi_{t+1} \\ 0 &= \widetilde{E}_t \Xi_{t+1} \end{aligned}$$

Thus, A is the matrix eigenvalue of matrix polynomial:

$$\alpha(A) = \alpha_0 A^2 + \alpha_1 A + \alpha_2 = 0$$

and B satisfies the system of linear equations:

$$F = (\beta_0 + \alpha_0 B)P + [\beta_1 + (\alpha_0 A + \alpha_1) B] = 0$$

The solution to the model obtained so far is one in which the mean distortion a_t to the process for z_{t+1} is realized.

With this solution in hand, look at the variables when the negative mean distortion is not realized at each period t. In the equilibrium defined above:

$$x_t = Ax_{t-1} + Bs_t \tag{4.2}$$

$$s_t = \begin{bmatrix} s_t^* \\ \widehat{z}_t \\ a_t \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_a \end{bmatrix} \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ a_{t-1} \end{bmatrix} + \Xi_t$$

but now we have:

$$x_t = Ax_{t-1} + Bs_t$$

$$s_t = \begin{bmatrix} s_t^* \\ \widehat{z}_t \\ a_t \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & 0 \\ 0 & 0 & \rho_a \end{bmatrix} \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ a_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ \varepsilon_t^z \\ \varepsilon_t^a \end{bmatrix}$$

$$s_t = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_a \end{bmatrix} \begin{bmatrix} s_{t-1}^* \\ z_{t-1} \\ a_{t-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_{t-1}^* \\ z_{t-1} \\ a_{t-1} \end{bmatrix} + \Xi_t$$

$$s_t = P \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ a_{t-1} \end{bmatrix} + C \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ a_{t-1} \end{bmatrix} + \Xi_t, \; C \equiv \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so:

$$x_t = Ax_{t-1} + Bs_t$$

$$x_t = Ax_{t-1} + BP \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ a_{t-1} \end{bmatrix} + B \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} s_{t-1}^* \\ \widehat{z}_{t-1} \\ a_{t-1} \end{bmatrix} + B\Xi_t$$

or in other words, for every j element in the vector x_t, where the superscript j refers to the jth row of the corresponding matrix, we have:

$$x_t^j = A^j x_{t-1} + B^j P s_{t-1} + B^j \Xi_t + B_z^j a_{t-1} \tag{4.3}$$

where the element B_z^j refers to the coefficient of the matrix B that reflects the response of the element x_t^j to the realized state $\widehat{z}_t$.

Notice that the evolution in (4.3) defines the equilibrium law of motion for our economy. From the perspective of the economy in (4.2) it is interpreted as the response to an "unusual" innovation to ε_t^z whose value is not zero (on average) but rather a_{t-1}. However, conditional on this "innovation" and state of the economy x_t the expectations are still governed by the equation (4.1) where the expectation about z_{t+1} is:

$$\widetilde{E}_t \widehat{z}_{t+1} = \rho_z \widehat{z}_t - a_t$$

4.2 Zero risk steady state

We now describe an approach to find the stochastic steady state of our model using the linearized law of motion of the endogenous variables. Take the perceived law of motion:

$$\widehat{z}_{t+1} = \rho_z \widehat{z}_t + \varepsilon^z_{t+1} - \overline{a}$$

where $\overline{a}$ is the steady state level of a_t. We can summarize our procedure in the following steps:

1. Find the deterministic 'distorted' steady state in which the intercept is actually $-\overline{a}$. The steady state technology level is then

$$z^o = \exp\left(\frac{-\overline{a}}{1-\rho_z}\right)$$

Using z^o, one can compute the deterministic 'distorted' steady state, by analyzing the FOC of these economy. Denote these steady state values of the m variables as a vector x_o

$$x^o_{m\times 1}$$

2. Linearize the model around this deterministic 'distorted' steady state.

For example, in the above notations, look for matrices A, B that describe the evolution of the variables as:

$$x_t - x_o = A(x_{t-1} - x_o) + BP(s_{t-1} - s_o) + B(\Xi_t - \Xi_o) \tag{4.4}$$

$$s_t = \begin{bmatrix} s^*_t \\ \widehat{z}_t \\ a_t \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & \rho_z & -1 \\ 0 & 0 & \rho_a \end{bmatrix} \begin{bmatrix} s^*_{t-1} \\ \widehat{z}_{t-1} \\ a_{t-1} \end{bmatrix} + \Xi_t$$

where Ξ_t are the innovations to the stochastic shock processes. Let the size of this vector be $n \times 1$, where the first element refers to the innovation of the technology shock.

Notice for example that when innovations are equal to their expected values, set to zero, i.e. that $\Xi_o = 0_{n\times 1}$ and $s_{t-1} = s_o$, the law of motion recovers that $x_t = x_o$.

So, in step 2 we need to find the matrices A and B. This is done by standard solution techniques of forward looking rational expectations model.

3. Correct for the fact that, from the perspective of the agent's ex-ante beliefs, the average innovation of the technology shock at time t is not equal to 0. The average innovation is equal to $\overline{a}$. Indeed:

$$\widetilde{E}_{t-1}\widehat{z}_t = \rho_z \widehat{z}_{t-1} - \overline{a}$$

but the realized average $\log z_t$ is

$$\widehat{z}_t = \rho_z \widehat{z}_{t-1}$$

Thus, from the perspective of the time $t-1$ expectation:

$$\begin{aligned} \widehat{z}_t &= \widetilde{E}_{t-1}\widehat{z}_t + \overline{\varepsilon}_t \\ \overline{\varepsilon}_t &= \overline{a} \end{aligned}$$

So, take the law of motion in (4.4) and impose that the first element of Ξ_t is equal to $\overline{a}$, while keeping the rest equal to 0 :

$$\widehat{\Xi} = \begin{bmatrix} \overline{a} \\ 0_{(n-1)\times 1} \end{bmatrix}$$

4. Find the steady state of the variables x_t, given that the law of motion is:

$$x_t - x_o = A(x_{t-1} - x_o) + BP(s_{t-1} - s_o) + B(\widehat{\Xi}_t - \Xi_o)$$

The steady state version for the exogenous variables is:

$$s^{SS} - s_o = P(s^{SS} - s_o) + (\widehat{\Xi} - \Xi_o)$$

where s^{SS} are the steady state values of the exogenous variables under their true DGP. So:

$$s^{SS} = s_o + \begin{bmatrix} \overline{a} \\ 0_{(n-1)\times 1} \end{bmatrix} (I-P)^{-1}$$

Then solve for the rest of the endogenous variables by using:

$$x^{SS} - x_o = A(x^{SS} - x_o) + B(s^{SS} - s_o)$$

where x^{SS} are the steady state values of the variables under ambiguity. Thus, x^{SS} can be found as:

$$x^{SS} = x_o + B(s^{SS} - s_o)\,(I-A)^{-1}$$

5 An estimated model with ambiguity

This section describes the model that we use to describe the US business cycles. The model is based on a standard medium scale DSGE model along the lines of Christiano et al. (2005) and Smets and Wouters (2007). The key difference in our model is that decision makers are ambiguity-averse. We now describe the model structure and the shocks.

5.1 The model

5.1.1 The goods sector

The final output in this economy is produced by a representative final good firm that combines a continuum of intermediate goods $Y_{j,t}$ in the unit interval by using the following linear homogeneous technology:

$$Y_t = \left[\int_0^1 Y_{j,t}^{\frac{1}{\lambda_{f,t}}} dj \right]^{\lambda_{f,t}},$$

where $\lambda_{f,t}$ is the markup of price over marginal cost for intermediate goods firms. The markup shock evolves as:

$$\log(\lambda_{f,t}/\lambda_f) = \rho_{\lambda_f} \log(\lambda_{f,t-1}/\lambda_f) + \lambda_{f,t}^x,$$

where $\lambda_{f,t}^x$ is $i.i.d.N(0, \sigma_{\lambda_f}^2)$. Profit maximization and the zero profit condition leads to to the following demand function for good j:

$$Y_{j,t} = Y_t \left(\frac{P_t}{P_{j,t}} \right)^{\frac{\lambda_{f,t}}{\lambda_{f,t}-1}} \tag{5.1}$$

The price of final goods is:

$$P_t = \left[\int_0^1 P_{j,t}^{\frac{1}{1-\lambda_{f,t}}} dj \right]^{(1-\lambda_{f,t})}.$$

The intermediate good j is produced by a price-setting monopolist using the following production function:

$$Y_{j,t} = \max\{z_t K_{j,t}^{\alpha} \left(\epsilon_t L_{j,t}\right)^{1-\alpha} - \Phi \epsilon_t^*, 0\},$$

where Φ is a fixed cost and $K_{j,t}$ and $L_{j,t}$ denote the services of capital and homogeneous labor employed by firm j. Φ is chosen so that steady state profits are equal to zero. The intermediate goods firms are competive in factor markets, where they confront a rental rate, $P_t r_t^k$, on capital services and a wage rate, W_t, on labor services.

The variable, ϵ_t, is a shock to technology, which has a covariance stationary growth rate. The variable, z_t, is a stationary shock to technology. The fixed costs are modeled as growing with the exogenous variable, ϵ_t^* :

$$\epsilon_t^* = \epsilon_t \Upsilon^{\left(\frac{\alpha}{1-\alpha} t\right)}$$

with $\Upsilon > 1$. If fixed costs were not growing, then they would eventually become irrelevant. We specify that they grow at the same rate as ϵ_t^*, which is the rate at which equilibrium output grows. Note that the growth of ϵ_t^*, i.e. $\mu_{\epsilon,t}^* \equiv \Delta \log(\epsilon_t^*)$, exceeds that of ϵ_t, i.e. $\mu_{\epsilon,t} \equiv \Delta \log(\epsilon_t)$:

$$\mu_{\epsilon,t}^* = \mu_{\epsilon,t} \Upsilon^{\frac{\alpha}{1-\alpha}}.$$

This is because we have another source of growth in this economy, in addition to the upward drift in ϵ_t. In particular, we posit a trend increase in the efficiency of investment. We discuss this process as well as the time series representation for the the transitory technology shock further below. The process for the stochastic growth rate is:

$$\log(\mu_{\epsilon,t}^*) = (1 - \rho_{\mu_\epsilon^*}) \log \mu_\epsilon^* + \rho_{\mu_\epsilon^*} \log \mu_{\epsilon,t-1}^* + \mu_{\epsilon,t}^{x*},$$

where $\mu_{\epsilon,t}^{x*}$ is $i.i.d.N(0, \sigma_{\mu_\epsilon^*}^2)$.

We now describe the intermediate good firms pricing opportunities. Following Calvo (1983), a fraction $1 - \xi_p$, randomly chosen, of these firms are permitted to reoptimize their price every period. The other fraction ξ_p cannot reoptimize. Of these, a (randomly selected) fraction $(1 - \iota_P)$ must set $P_{it} = \bar{\pi} P_{i,t-1}$ and a fraction ι_P set $P_{it} = \pi_{t-1} P_{i,t-1}$, where $\bar{\pi}$ is steady state inflation. The j^{th} firm that has the opportunity to reoptimize its price does so to maximize the expected present discounted value of the future profits:

$$E_t^{p^0} \sum_{s=0}^{\infty} (\beta \xi_p)^s \frac{\lambda_{t+s}}{\lambda_t} \left[P_{j,t+s} Y_{j,t+s} - W_{t+s} L_{j,t+s} - P_{t+s} r_{t+s}^k K_{j,t+s} \right], \tag{5.2}$$

subject to the demand function (5.1), where λ_t is the marginal utility of nominal income for the representative household that owns the firm.

It should be noted that the expectation operator in these equations is, in the notation of the general representation in section 3, the expectation under the worst-case belief p^0. This is because state prices in the economy reflect ambiguity. We will describe the household's problem further below.

We now describe the problem of the perfectly competitive "employment agencies". The households specialized labor inputs are aggregated by these agencies into a homogeneous labor service according to the following function:

$$L_t = \left[\int_0^1 (l_{i,t})^{\frac{1}{\lambda_w}} \, di \right]^{\lambda_w}.$$

These employment agencies rent the homogeneous labor service L_t to the intermediate goods firms at the wage rate W_t. In turn, these agencies pay the wage $W_{i,t}$ to the household supplying labor of type i. Similarly as for the final goods producers, the profit maximization and the zero profit condition leads to to the following demand function for labor input of type i:

$$l_{i,t} = L_t \left(\frac{W_t}{W_{i,t}} \right)^{\frac{\lambda_w}{\lambda_w - 1}}, \tag{5.3}$$

We follow Erceg et al. (2000) and assume that the household is a monopolist in the supply of labor by providing $l_{i,t}$ and it sets its nominal wage rate, $W_{i,t}$. It does so optimally with probability $1 - \xi_w$ and with probability ξ_w is does not reoptimize its wage. In case it does not reoptimize, it sets the wage as:

$$W_{i,t} = \pi_{t-1}^{\iota_w} \bar{\pi}^{1-\iota_w} \mu_{z^*} W_{i,t-1},$$

where μ_{z^*} is the steady state growth rate of the economy. When household i has the chance to reoptimize, it does do by maximizing the expected present discounted value of future net utility gains of working:

$$E_t^{p^0} \sum_{s=0t}^{\infty} \sum_{s=0}^{\infty} (\beta \xi_w)^s \left[\lambda_{t+s} W_{i,t+s} l_{i,t+s} - \frac{\psi_L}{1 + \sigma_L} l_{i,t+s}^{1+\sigma_L} \right]. \tag{5.4}$$

subject to the demand function (5.3).

5.1.2 Households

The model described here is a special case of the general formulation of recursive multiple priors of section 3. In particular, recall the recursive representation in (3.1), where preferences were defined over uncertain streams of consumption $C = (C_t)_{t=0}^{\infty}$, where $C_t : S^t \to \Re^n$ and n is the number of goods. In the model described here, there are 2 goods, the consumption of the final good Y_t and leisure. We then only have to define the per-period felicity function, which for agent i is:

$$u^i(c_t) = \log(C_t - \theta C_{t-1}) - \frac{\psi_L}{1 + \sigma_L} l_{i,t}^{1+\sigma_L}. \tag{5.5}$$

Here, θ is an internal habit parameter, C_t denotes individual consumption of the final good and $l_{i,t}$ denotes a specialized labor service supplied by the household. Also, $\psi_L > 0$ is a parameter.[1]

Utility follows a recursion similar to (3.1):

$$U_t\left(C; s^t\right) = u^i\left(C_t, C_{t-1}\right) + \beta \min_{p \in \mathcal{P}(s^t)} E^p\left[U_{t+1}\left(C; s_t, s_{t+1}\right)\right], \tag{5.6}$$

Let the solution to the minimization problem in (5.6) be denoted by p^0. This minimizing p^0 is the same object that appears in the expectation operator of equations (5.4) and (5.2).

The type of Knightian uncertainty we consider in this model is over the transitory technology level. In section 5.1.1, we described the production side of the economy and showed where technology enters this economy. Ambiguity here is reflected by the set of one-step ahead conditional beliefs $\mathcal{P}\left(s^t\right)$ about the future transitory technology. We follow the description in section 3 to describe the stochastic process. Specifically, we will assume that there is time-varying ambiguity about the future technology. This time-variation is captured by an exogenous component a_t. The true dynamics of the transitory productivity shock z_t can be represented by an AR(1) process:

$$\log z_t = \rho_z \log z_{t-1} + \sigma_z z_t^x. \tag{5.7}$$

The set $\mathcal{P}_t$ of one step conditional beliefs about future technology can then be represented by the family of processes:

$$\log z_{t+1} = \rho_z \log z_t + \sigma_z z_{t+1}^x + \mu_t \tag{5.8}$$

$$\mu_t \in \left[-a_t, -a_t + 2|a_t|\right] \tag{5.9}$$

$$a_{t+1} = \left(1 - \rho_a\right)\overline{a} + \rho_a a_t + \sigma_a a_{t+1}^x \tag{5.10}$$

where the shocks z^x and a^x are standard normal iid shocks. As in Section 3, we assume that the agent knows the evolution of a_t, but that he is not sure whether the conditional mean of $\log z_{t+1}$ is really $\rho_z \log z_t$. Instead, the agent allows for a range of intercepts. If a_t is higher, then the agent is less confident about the mean of $\log z_{t+1}$ – his belief set is larger.

In this model, it is easy to what is the worst-case scenario, i.e. what is the belief about μ_t that solves the minimization problem in (5.6). In section 4 we described a general procedure to find the worst-case belief. In the present model, the environment (given by B, x' u and G in the general formulation of section 3) is such that, *under expected utility and rational*

[1] Note that consumption is not indexed by i because we assume the existence of state contingent securities which implies that in equilibrium consumption and asset holdings are identical across households.

expectations, a first order solution provides a satisfactory approximation to the equilibrium dynamics. In this environment, it is easy to check that the value function, under expected utility, is increasing in z_t. This monotonicity implies that the worst-case scenario belief that solves the minimization problem in (5.6) is given by the lower bound of the set $[-a_t, -a_t + 2|a_t|]$. Intuitively, it is natural that the agents take into account that the worst case is always that the mean of productivity innovations is as low as possible.

The laws of motion in (5.8) and (5.10) are linear. To maintain the interpretation that μ_t is the worst-case scenario solution to the minimization problem, a_t should be positive. Thus, it is useful not to have a_t become negative very often. We are then guided to parameterize the ambiguity process in the following way. We compute the unconditional variance of the process in (5.10) and insist that the mean level of ambiguity is high enough so that even a large negative shock to a_t of m unconditional standard deviations away from the mean will remain in the positive domain. We can write this constraint as:

$$\overline{a} \geq m \frac{\sigma_a}{\sqrt{1-\rho_a^2}} \tag{5.11}$$

The formula in (5.11) provides a constraint on our parameterization. If the constraint is binding, then, for a given m, there is a fixed relationship on our parameterization between $\overline{a}$, σ_a and ρ_a. In that case, we are left with essentially choosing two out of three parameters.

The household accumulates capital subject to the following technology:

$$\bar{K}_{t+1} = (1-\delta)\bar{K}_t + \left[1 - S\left(\zeta_t \frac{I_t}{I_{t-1}}\right)\right] I_t,$$

where ζ_t is a disturbance to the marginal efficiency of investment with mean unity, $\bar{K}_t$ is the beginning of period t physical stock of capital, and I_t is period t investment. The function S reflects adjustment costs in investment. The function S is convex, with steady state values of $S = S' = 0, S'' > 0$. The specific functional form for $S(.)$ that we use is:

$$S\left(\zeta_t \frac{I_t}{I_{t-1}}\right) = \exp\left[\sqrt{\frac{S''}{2}}\left(\zeta_t \frac{I_t}{I_{t-1}} - 1\right)\right] + \exp\left[-\sqrt{\frac{S''}{2}}\left(\zeta_t \frac{I_t}{I_{t-1}} - 1\right)\right] - 2$$

The marginal efficiency of investment follows the process:

$$\log(\zeta_t) = \rho_\zeta \log(\zeta_{t-1}) + \zeta_t^x,$$

where ζ_t^x is $i.i.d.N(0, \sigma_\zeta^2)$.

Households own the physical stock of capital and rent out capital services, K_t, to a competitive capital market at the rate $P_t\widetilde{r}_t^k$, by selecting the capital utilization rate u_t:

$$K_t = u_t\bar{K}_t,$$

Increased utilization requires increased maintenance costs in terms of investment goods per unit of physical capital measured by the function $a\left(u_t\right)$. The function $a(.)$ is increasing and convex, $a\left(1\right) = 0$ and u_t is unity in the nonstochastic steady state. We assume that $a''\left(u\right) = \vartheta r^k$, where r^k is the steady state value of the rental rate of capital. Then, $a''\left(u\right)/a'\left(u\right) = \vartheta$ is a parameter that controls the degree of convexity of utilization costs.

The i^{th} household's budget constraint is:

$$P_tC_t + P_t\frac{I_t}{\mu_{\Upsilon,t}\Upsilon^t} + B_t = B_{t-1}R_{t-1} + P_t\overline{K}_t[\widetilde{r}_t^k u_t - a(u_t)\Upsilon^{-t}] + W_{t,i}l_{t,i} - T_tP_t$$

where B_t are holdings of government bonds, R_t is the gross nominal interest rate and T_t is net lump-sum taxes.

When we specify the budget constraint, we will assume that the cost, in consumption units, of one unit of investment goods, is $\left(\Upsilon^t\mu_{\Upsilon,t}\right)^{-1}$. Since the currency price of consumption goods is P_t, the currency price of a unit of investment goods is therefore, $P_t\left(\Upsilon^t\mu_{\Upsilon,t}\right)^{-1}$. The stationary component of the relative price of investment follows the process:

$$\log(\mu_{\Upsilon,t}/\mu_\Upsilon) = \rho_{\mu_\Upsilon}\log(\mu_{\Upsilon,t-1}/\mu_\Upsilon) + \mu_{\Upsilon,t}^x,$$

where $\mu_{\Upsilon,t}^x$ is $i.i.d.N(0, \sigma_{\mu_\Upsilon}^2)$.

5.1.3 The government

The market clearing condition for this economy is:

$$C_t + \frac{I_t}{\mu_{\Upsilon,t}\Upsilon^t} + G_t = Y_t^G$$

where G_t denotes government expenditures and Y_t^G is our definition of measured GDP, i.e. $Y_t^G \equiv Y_t - a(u_t)\Upsilon^{-t}\overline{K}_t$. We model government expenditures as $G_t = g_t\epsilon_t^*$, where g_t is a stationary stochastic process. This way of modeling G_t helps to ensure that the model has a balanced growth path. The fiscal policy is Ricardian. The government finances G_t by issuing short term bonds B_t and adjusting lump sum taxes T_t.The law of motion for g_t is:

$$\log(g_t/g) = \rho_g\log(g_{t-1}/g) + g_t^x$$

where g_t^x is.$i.i.d.N(0, \sigma_g^2)$.

The nominal interest rate R_t is set by a monetary policy authority according to the following feedback rule:

$$\frac{R_t}{R} = \left(\frac{R_{t-1}}{R}\right)^{\rho_R} \left[\left(\frac{\pi_t}{\pi}\right)^{a_\pi} \left(\frac{Y_t^G}{Y_t^*}\right)^{a_y} \left(\frac{Y_t^G}{\mu_z^* Y_{t-1}^G}\right)^{a_{gy}}\right]^{1-\rho_R} \exp(\epsilon_{R.t}),$$

where $\epsilon_{R.t}$ is a monetary policy shock $i.i.d.N(0, \sigma_{\epsilon_R}^2)$.

5.1.4 Model Solution

The model economy fluctuates along a stochastic growth path. Some variables are stationary: the nominal interest rates, the long-term interest rate, inflation and hours worked. Consumption, real wages and output grow at the rate determined by ϵ_t^*. The capital stock and investment grow faster, due to increasing efficiency in the investment producing sector, at a rate determined by $\epsilon_t^* \Upsilon^t$, with $\Upsilon > 1$. The solution of our model with ambiguity follows the general steps described in section 4. The solution builds on the standard approach to solve for a rational expectations equilibrium with the difference that we need to take into account that the worst-case scenario expectations do not materialize on average ex-post. Therefore, the solution involves the following procedure. First, we solve the model as a rational expectations model in which expectations are correct on average. Here we follow the standard approach of solving these type of models: we rewrite the model in terms of stationary variables by detrending each variable using its specific trend growth rate. Then we find the non-stochastic steady state for this detrended system and construct a log-linear approximation around it. We then solve the resulting linear system of rational expectations equations. With that law of motion in hand, we then correct for the fact that the true dynamics of the productivity process follow the process in (5.7) in which $\mu_t = 0$.

The model has 7 fundamental shocks:

$$\left[z_t^x, \mu_{\epsilon,t}^{x,*}, \epsilon_{R.t}, g_t^x, \mu_{\Upsilon,t}^x, \lambda_{f,t}^x, \zeta_t^x\right]$$

and the ambiguity shock a_t^x.

5.1.5 Estimation and Data

The linearity of the state space representation of the model and the assumed normality of the shocks allow us to estimate the model using standard Bayesian methods as discussed for example in An and Schorfheide (2007) and Smets and Wouters (2007). We estimate the posterior distribution of the structural parameters by combining the likelihood function with prior information. The likelihood is based on the following vector of observable variables:

$$\left[\Delta \log Y_t^G, \Delta \log I_t, \Delta \log C_t, \log L_t, \log \pi_t, \log R_t, \Delta \log P_{I,t}\right]$$

where Δ denotes the first difference operator. The vector of observables refers to data for US on GDP growth rate, investment growth rate, consumption growth rate, real wage growth rate, log of hours per capita, log of gross inflation rate, log of gross short term nominal interest rate and price of investment growth rate. The sample period used in the estimation is 1984Q1-2010Q1. In the state space representation we do not allow for a measurement error on any of the observables.

We now discuss the priors on the structural parameters. The only parameter we calibrate is the share of government expenditures in output which is set to match the observed empirical ratio of 0.22. The rest of the structural parameters are estimated. The priors on the parameters not related to ambiguity and thus already present in the standard medium scale DSGE are broadly in line with those adopted in previous studies (e.g. Justiniano et al. (2011), Christiano et al. (2010b)). The prior for each of the autocorrelation parameter of the shock processes is a Beta distribution with a mean of 0.5 and a standard deviation of 0.15. The prior distribution for the standard deviation of the 7 fundamental shocks is an Inverse Gamma with a mean of 0.01 and a standard deviation of 0.01.

Regarding the choice over the ambiguity parameters, recall the discussion in 5.1.2 and in 2.3. There are two constraints on the process for a_t that come out of those arguments. One is that a_t should not be negative too often which then implies the constraint in (5.11). The second is that the upper bound on the amount of ambiguity is related to how much variability is there in the z_t process. In section 2.3 we argued that this upper bound is equal to $2\sqrt{\rho}\sigma_z$, where $\rho \in [0, 1]$ is the share of the variability in the data that agents attribute to ambiguity. When $\rho = 1$ we obtain the largest upper bound, i.e. $a_t \leq 2\sigma_z$. Then, a similar argument as for the lower bound in (5.11) leads us to consider a process for a_t such that a realization which is higher by m unconditional standard deviations than the mean is still below this upper bound:

$$\overline{a} + m\frac{\sigma_a}{\sqrt{1-\rho_a^2}} \leq 2\sigma_z \tag{5.12}$$

In preliminary estimations of the model, we find that the constraint in (5.11) between the amount of unconditional volatility and mean ambiguity is binding.[2] When that constraint

[2]More precisely, if the three parameters characterizing ambiguity are separately estimated we find that the implied unconditional volatility of the a_t process is so large that it implies very frequent negative realizations to a_t. Additionally there are further issues of identification: if we do not fix other structural parameters, such

holds with equality it means that we can estimate two parameters out of the three: $\overline{a}$, σ_a and ρ_a. Notice also that in that case, the constraint in (5.12) implies that

$$\overline{a} \leq \sigma_z. \tag{5.13}$$

For computational reasons, we choose to work with the following scaling between the mean amount of ambiguity and the standard deviation of the technology shock:

$$\overline{a} = n\sigma_z \tag{5.14}$$

where (5.13) implies that $n \in [0, 1]$. Given this scaling, the binding constraint in (5.11) implies that:

$$\sigma_a = n\sigma_z \frac{\sqrt{1-\rho_a^2}}{m}$$

We fix $m = 3$ so that a negative shock of three unconditional standard deviations is still in the positive domain and a positive shock of three unconditional standard deviations will still be below the upper bound of ambiguity given by $2\sigma_z$. So, the structural parameters to be estimated are n and ρ_a.

Alternatively, instead of the scaling in (5.14), we could estimate directly the parameters ρ_a and σ_a. The mean ambiguity would then be given by the binding constraint in (5.11). We find that the results are very robust to this choice. An advantage of the scaling in (5.14) is that it shows a direct link between how much ambiguity about the mean innovation $\sigma_z z_t^x$ and the standard deviation of that innovation. Another advantage is that the parameter n belongs to the unit interval which is computationally convenient.

In the benchmark model, the prior on the scaling parameter n is a Beta distribution with mean 0.5 and standard deviation equal to 0.25. The prior is loose and it allows a wide range of plausible values. The prior on ρ_a follows the pattern of the other autocorrelation coefficients and is a Beta distribution with a mean of 0.5 and a standard deviation of 0.15.

The prior and posterior distributions are described in Table 2. The posterior estimates of our structural parameters that are unrelated to ambiguity are in line with previous estimations of such medium scale DSGE models (Del Negro et al. (2007), Smets and Wouters (2007), Justiniano et al. (2011), Christiano et al. (2010b)). These parameters imply that there are significant 'frictions' in our model: price and wage stickiness, investment adjustment costs and internal habit formation are all substantial. The estimated policy rule is inertial and responds strongly to inflation but also to output gap and output growth. Given that

as β, we can easily run into identification problems of $\overline{a}$. This is a further reason to impose a relationship between the ambiguity parameters.

these parameters have been extensively analyzed in the literature, we now turn attention to the role of ambiguity in our estimated model.

5.2 Results

We evaluate the importance of ambiguity in our model along two dimensions: the steady state effect of ambiguity and its role in business cycle fluctuations. We will argue that ambiguity plays an important part along both of these dimensions.

5.2.1 Steady state

The posterior mode of the structural parameters of ambiguity implies that the mean level of ambiguity is

$$\overline{a} = n\sigma_z = 0.963 \times 0.0045 = 0.00435$$

which means that the ambiguity averse agent is on average concerned about a mean one-step ahead future technology level that is 0.435% lower than the true technology, normalized to 1. In the long run, the agent expects the technology level to be z^*, which solves:

$$\log z^* = \rho_z \log z^* - \overline{a}.$$

For the estimated $\rho_z = 0.955$, we get that $z^* = 0.903$. Thus, the ambiguity-averse agent expects under his worst-case scenario evaluations the long run mean technology to be approximately 9% lower than the true mean. Based on these estimates and using (5.11) we can directly find that the standard deviation of the innovations to ambiguity is $\sigma_a = 0.000405$.

Our interpretation of the reason why we find a relatively large $\overline{a}$ is the following: the estimation prefers to have a large σ_a because the ambiguity shock provides a channel in the model that delivers dynamics that seem to be favored by the data. Indeed, as detailed in the next section, the ambiguity shocks generate comovement between variables that enter in the observation equation. This is a feature that is strongly in the data and is not easily captured by other shocks. Given the large role that the fit of the data places on the ambiguity shock, the implied estimated σ_a is relatively large. Because of the constraint on the size of the mean ambiguity in (5.11), this results also in a large required steady state ambiguity. Thus, given also the estimated σ_z, the posterior mode for n is relatively large. The picture that comes out of these estimates is that ambiguity is large in the steady state, it is volatile and persistent.

The estimated amount of ambiguity has substantial effects on the steady state of endogenous variables. To describe these effects we perform the following calculations. We fix all the estimated parameters of the model but change the estimated standard deviation of the transitory technology shock, σ_z, from its estimated value of $\overline{\sigma}_z = 0.0045$ to being equal to 0. When $\sigma_z = 0$, then the level of ambiguity $\overline{a}$ is also equal to 0. By reporting the difference between the steady states with $\sigma_z = \overline{\sigma}_z > 0$ and with $\sigma_z = 0$ we calculate the effect on steady states of fluctuations in transitory technology that goes through the estimated amount of ambiguity. In Table 1 we present the net percent difference of some variables of interest between the two cases, i.e. for a variable X we report $100[X_{SS,(\sigma_z=\overline{\sigma}_z)}/X_{SS,(\sigma_z=0)} - 1]$, where $X_{SS,(\sigma_z=\overline{\sigma}_z)}$ and $X_{SS,(\sigma_z=0)}$ are the steady states of variable X under $\sigma_z = \overline{\sigma}_z$ and respectively $\sigma_z = 0$.[3]

Table 1: Steady state percent difference from zero fluctuations

Variable	$Welfare$	$Output$	$Capital$	$Consumption$	$Hours$	$Nom.Rate$
	-13.1	-15	-14	-16.4	-14.8	-42.5

As evident from Table 1, the effect of fluctuations in the transitory technology shock that goes through ambiguity is very substantial. Output, capital, consumption, hours are all significantly smaller when $\sigma_z = \overline{\sigma}_z$. The nominal interest rate is smaller by 42%, which corresponds to the quarterly steady state interest rate being lower by 73 basis points. Importantly, the welfare cost of fluctuations in this economy is also very large, of about 13% of steady state consumption. These effects are much larger than what it is implied by the standard analysis featuring only risk. By standard analysis we mean the strategy of shutting down all the other shocks except the transitory technology and computing a second order approximation of the model in which there is no ambiguity but $\sigma_z = \overline{\sigma}_z$. For such a calculation, we find that the welfare cost of business cycle fluctuations is around 0.01% of steady state consumption. The effects on the steady state values of the other variables reported in Table 1 is negligible. This latter result is consistent with the conclusions of the literature on the effect of business cycle fluctuations and risk.

5.2.2 Business cycle fluctuations

In this section we analyze the role of time-varying ambiguity in generating business cycles. We highlight the role of ambiguity by discussing three main points: a theoretical variance decomposition of variables; a historical variance decomposition based on the smoothed shocks and impulse responses experiments.

[3]Note that for welfare, we report the difference between $Welf_{SS,(\sigma_z=\overline{\sigma}_z)}$ and $Welf_{SS,(\sigma_z=0)}$ in terms of steady state consumption under $\sigma_z = 0$.

Variance decomposition Table 3 reports the theoretical variance decomposition of several variables of interest. For each structural shock we compute the share of the total variation in the corresponding variable that is accounted by that shock at two horizons: one is at the business cycle frequency which incorporates periodic components with cycles between 6 and 32 quarters, as in Stock and Watson (1999). The second is at a long-run horizon which is the theoretical variance decomposition obtained by solving the dynamic Lyapunov equation characterizing the law of motion of the model. These two shares are reported in the first two rows of Table 3. In the third row, we also report for comparison the variance decomposition in an estimated model without ambiguity.

The main message of this exercise is that ambiguity shocks can be a very influential factor in explaining the variance of key economic variables. At business cycle frequency this shock accounts for about 27% of GDP variability and it simultaneously explains a large share of real variables such as consumption (52%), investment (14%), hours (31%) and less for inflation (2%) and the nominal interest rate (7%). The long-run theoretical decomposition implies that the shock is even more important. It explains about 55% of GDP variability and it is a very significant driver of the rest of the variables: consumption (62%), investment (51%), hours (52%), inflation (29%) and the nominal interest rate (38%). Based on these two sets of numbers we can conclude that the ambiguity shock is an important driver of business cycle fluctuations while also having a low-frequency component that magnifies its role in the total variance decomposition. The simultaneous large shares of variation explained by ambiguity suggest that time-variation in the confidence of the agents about transitory technology shocks can be a unified source of macroeconomic variability.

For comparison, we can analyze the estimated model without ambiguity. The business cycle frequency variance decomposition for a model that sets the level of ambiguity to zero, i.e. $n = 0$, is reported in the third row of Table 3. There the largest share of GDP variability is explained by the marginal efficiency of investment shock, confirming the results of many recent studies, such as Christiano et al. (2010b) and Justiniano et al. (2011). Introducing time-varying ambiguity reduces the importance of the other shocks, except for the transitory technology shock z_t, in explaining the decomposition of the level of observed variables. The reduction in effects are especially strong for the marginal efficiency of investment and growth rate shocks. With ambiguity, the shock z_t becomes more important. The reason is that ambiguity enters in the model indirectly through the variance of σ_z. Thus, with ambiguity the estimated variance of z_t affects the likelihood evaluation through two channels: one direct, through the shock z_t, and one indirect through the variance of the shock a_t.

Impulse response We now turn to analyzing the impulse responses for the ambiguity shock in the estimated model. As suggested already in the discussion, an increase in ambiguity generates a recession, in which hours worked, consumption and investment fall. The fact that this shock predicts comovement between this variables is an important feature that helps explain why the estimation prefers in the likelihood maximization such a shock. Figure 3 plots the responses to a one standard deviation increase in ambiguity for the estimated model. On top of the mentioned comovement, the model also predicts a fall in the price of capital, a fall in the real interest rate and a countercyclical excess return.

We briefly explain these results. The main intuition in understanding the effect of this shock is to relate it to its interpretation of a news shock. An increase in ambiguity makes the agent act under a more cautious forecast of the future technology. From an outside observer that analyzes the agent's behavior, it seems that this agent acts under some negative news about future productivity. This negative news interpretation of the increase in ambiguity helps explain the mechanics and economics of the impulse response. As described in detail in Christiano et al. (2008), Christiano et al. (2010a), in a rational expectations model, a negative news about future productivity can produce a significant bust in real economy while simultaneously generating a fall in the price of capital. This result is reflected in our impulse response. In our model, the negative news is on average not materialized, because nothing changed in the true process for technology, as shown in the first panel of Figure 3. However, because of the persistent effect of ambiguity, the economy continues to go through a prolonged recession. The ex-post excess return, defined as the difference between the realized return on capital and the risk-free rate, is positive following the period of the initial increase in ambiguity. The ex-ante excess return is always equal to zero, as we solve a linearized model. The explanation for the countercyclical excess returns is that the negative expectation about future productivity does not materialize, so ex-post, capital pays more. The ex-post excess return is a rational uncertainty premium that ambiguity-averse agents require to invest in the uncertain asset.

Historical shock decomposition We conclude the description of the role of ambiguity shocks in business cycle fluctuations by section by discussing the historical variance decomposition and the smoothed shocks that result from the estimated model. In Figure 1 we plot the smoothed ambiguity shock, as a deviation from its steady state value. The figure first shows that ambiguity is very persistent. After an initial increase around 1991, which also corresponds to an economic downturn, the level of ambiguity was low and declining during the 90's, reaching its lowest values around 2000. It then increases back to levels close to steady state until 2005. Following a few years of relatively small upward deviations from its mean, ambiguity spikes starting in 2008. Ambiguity rapidly increases, so that throughout 2008 it doubles over each quarter. Ambiguity reaches its peak in 2008Q4 when it is 8 times larger than its 2008Q1 value. The figure shows a dotted vertical line at 2008Q3, which corresponds to the Lehman Brothers bankruptcy. Our model interprets the period following 2008Q1 as one in which ambiguity about future productivity has increased dramatically.

Based on these smoothed path of ambiguity shocks we can now calculate what the model implies for the historical evolution of endogenous variables. In Figure 2 we compare the observed data with the hypothetical historical evolution for the growth rate of output, consumption, investment and the level of hours worked when the ambiguity shock is the only shock active in the model economy. The ambiguity shock implies a path for variables that come close to matching the data, especially for output, consumption and hours. The model implied path of investment is less volatile but the correlation with the observed data is still significantly large. It is interesting that the ambiguity shock helps explain some of the business cycle frequency of these variables but also the low-frequency component as present in hours worked.[4]

The ambiguity shock generates the three large recessions observed in this sample. Indeed, if we analyze the smoothed path of the shock in figure 2, the time-varying ambiguity helps explain the recession of the 1991, the large growth of the 1990's (as a period of low ambiguity), and then the recession of 2001. Given that the estimated ambiguity still continues to rise through 2005, the model misses by predicting a more prolonged recession than in the data, where output picks up quickly. The rise in ambiguity in 2008 predicts in the model that output, investment and consumption fall. The model matches the fall in consumption, but fails to generate a large fall in investment. It is important to highlight that the ambiguity shock implies that in the model consumption and investment comove. Indeed, in the historical decomposition, recessions are times when both of these variables fall. This is an important effect because standard shocks that have been recently found to be quantitatively important, such as the marginal efficiency of investment or intertemporal preference shocks imply a weak, and most often a negative comovement between these two components.

Overall, we draw the conclusion that an increase in Knightian uncertainty (ambiguity) generates in our estimated model a recession, in which consumption, investment and price of capital fall, while producing countercyclical ex-post excess returns. Given these dynamics, we believe that time-varying ambiguity can be an important source of observed business cycle dynamics.

[4] Usually the low-frequency movement in hours worked is attributed to exogenous labor supply shocks, corresponding to shocks to ψ_L in our model. See for example Justiniano et al. (2011).

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Figure 1: Smoothed ambiguity shock

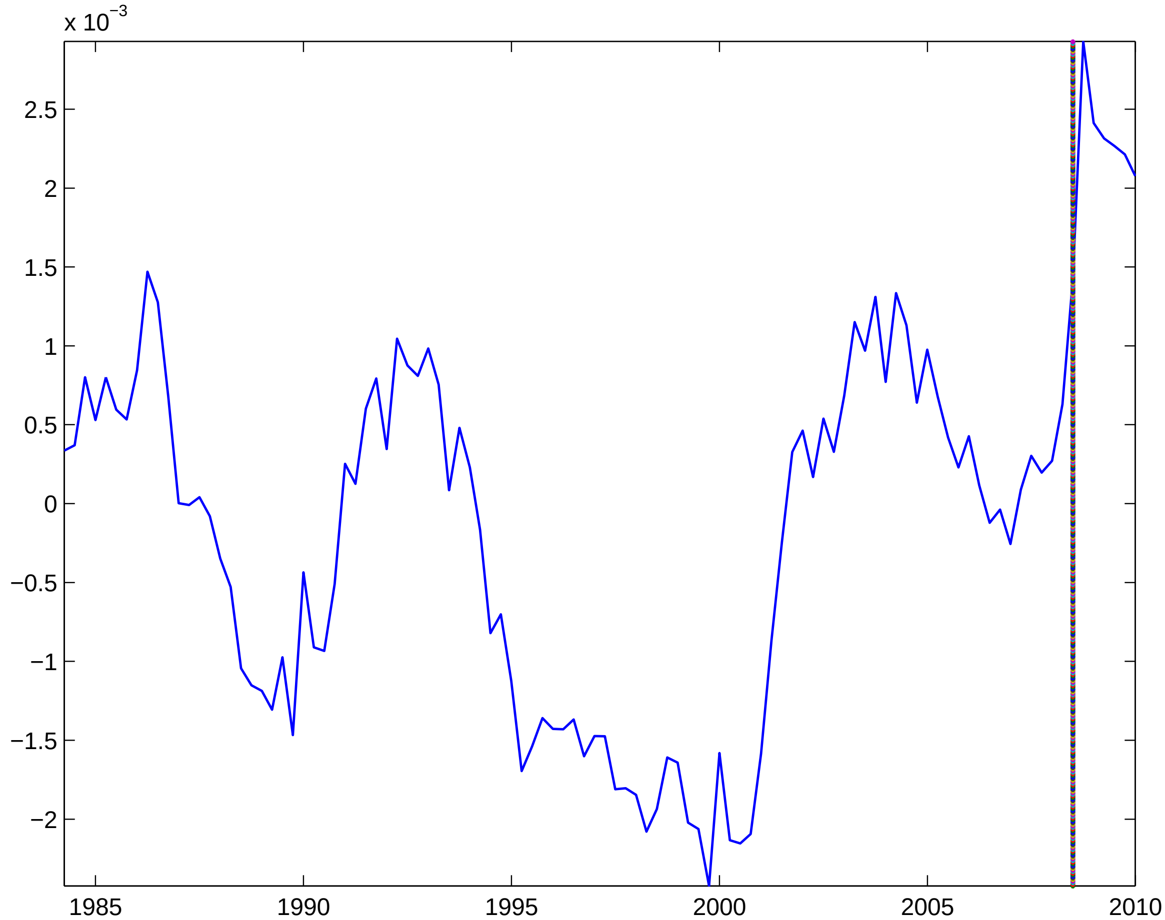

Figure 2: Historical shock decomposition

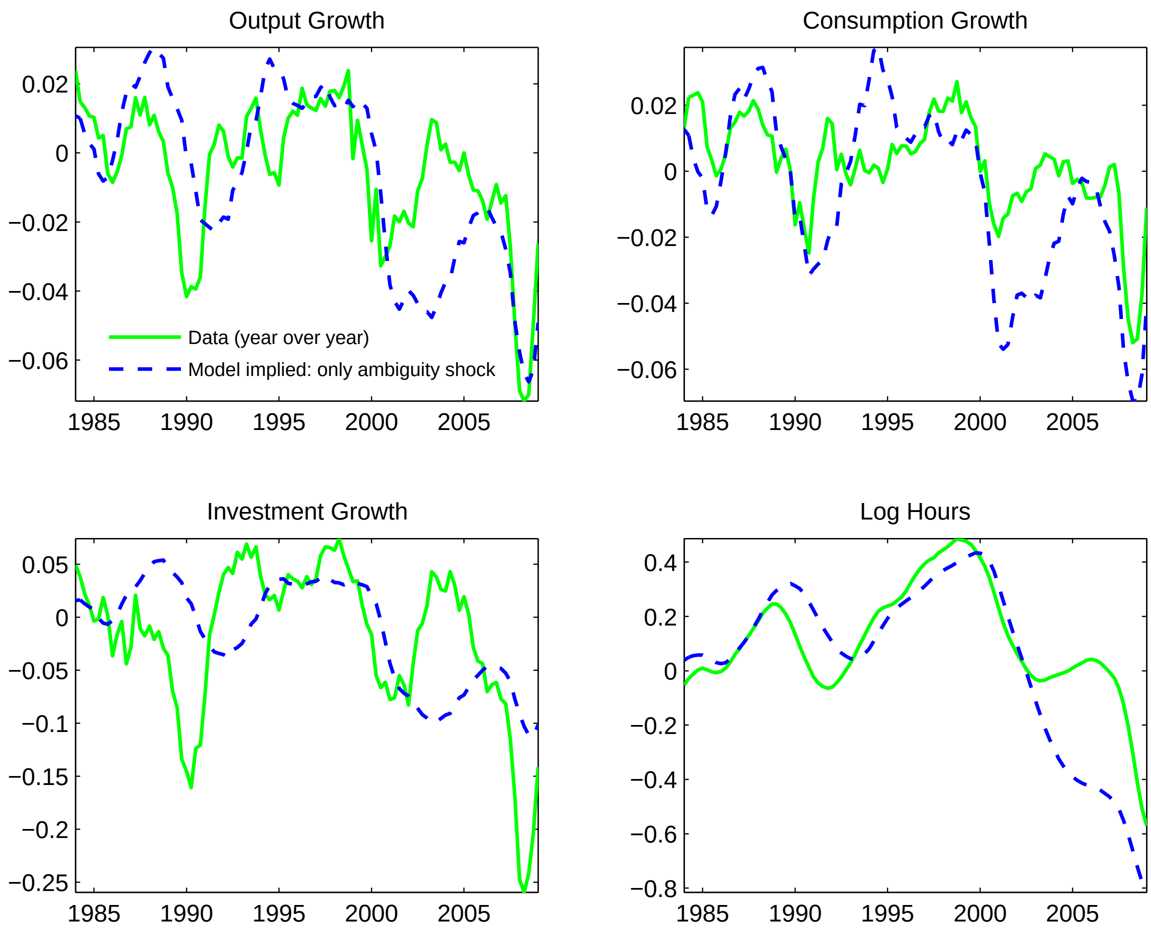

Table 2: Priors and Posteriors for structural parameters

Parameter	Description	Prior			Posterior		
		Type[a]	Mean	St.dev	Mode	[.5 ,	.95][b]
α	Capital share	B	0.4	0.02	0.322	0.291	0.353
δ	Depreciation	B	0.025	0.002	0.0237	0.0206	0.0279
$100(\beta^{-1}-1)$	Discount factor	G	0.3	0.05	0.353	0.2586	0.4728
$100(\mu_\epsilon^*-1)$	Growth rate	N	0.4	0.1	0.5	0.4	0.6
$100(\mu_\Upsilon-1)$	Price of investment growth rate	N	0.4	0.1	0.46	0.43	0.49
$100(\bar{\pi}-1)$	Net inflation	N	0.6	0.2	0.85	0.66	1.17
ξ_p	Calvo prices	B	0.5	0.1	0.743	0.681	0.841
ξ_w	Calvo wages	B	0.5	0.1	0.938	0.912	0.953
S''	Investment adjustment cost	G	10	5	13.92	6.157	27.962
ϑ	Capacity utilization	G	2	1	1.959	0.433	4.279
a_π	Taylor rule inflation	N	1.7	0.3	2.09	1.771	2.473
a_y	Taylor rule output	N	0.15	0.05	0.059	0.013	0.188
a_{gy}	Taylor rule output growth	N	0.15	0.05	0.209	0.116	0.294
ρ_R	Interest rate smoothing	B	0.5	0.15	0.808	0.751	0.842
λ_f-1	SS price markup	N	0.2	0.05	0.22	0.134	0.314
λ_w-1	SS wage markup	N	0.2	0.05	0.135	0.069	0.22
θ	Internal habit	B	0.5	0.1	0.661	0.535	0.729
σ_L	Curvature on disutility of labor	G	2	1	1.886	1.64	2.288
n	Level ambiguity scale parameter	B	0.5	0.25	0.963	0.827	0.999
ρ_z	Transitory technology	B	0.5	0.15	0.955	0.928	0.974
$\rho_{\mu_\epsilon^*}$	Persistent technology	B	0.3	0.15	0.132	0.014	0.509
ρ_ζ	Marginal efficiency of investment	B	0.5	0.15	0.494	0.351	0.722
ρ_{λ_f}	Price mark-up	B	0.5	0.15	0.907	0.62	0.961
ρ_g	Government spending	B	0.5	0.15	0.954	0.923	0.977
ρ_{μ_Υ}	Price of investment	B	0.5	0.15	0.957	0.929	0.983
ρ_a	Level Ambiguity	B	0.5	0.15	0.96	0.936	0.981
σ_z	Transitory technology	IG	0.01	0.01	0.0045	0.0041	0.0058
$\sigma_{\mu_\epsilon^*}$	Persistent technology	IG	0.01	0.01	0.0044	0.0029	0.0064
σ_ζ	Marginal efficiency of investment	IG	0.01	0.01	0.0183	0.016	0.0231
σ_{λ_f}	Price mark-up	IG	0.005	0.01	0.0102	0.007	0.033
σ_g	Government spending	IG	0.01	0.01	0.0195	0.017	0.0236
σ_{μ_Υ}	Price of investment	IG	0.01	0.01	0.003	0.0026	0.0034
σ_{ϵ_R}	Monetary policy shock	IG	0.005	0.01	0.0015	0.0013	0.0017

a B refers to the Beta distribution, N to the Normal distribution, G to the Gamma distribution, and IG to the Inverse-gamma distribution.

b Posterior percentiles obtained from 2 chains of 200,000 draws generated using a Random walk Metropolis algorithm. We discard the initial 50,000 draws and retain one out of every 5 subsequent draws.

Table 3: Theoretical variance decomposition

Shock\Variable	Output	Cons.	Invest.	Hours	Inflation	Int. rate
TFP Ambiguity (a_t)	27.2	52.1	14.4	31.1	2	7.4
	(55.4)	(62.7)	(51.4)	(52.1)	(29.6)	(38.5)
	[-]	[-]	[-]	[-]	[-]	[-]
Transitory technology (z_t)	12.1	13.5	9.6	2.5	23.8	15.9
	(5.5)	(5)	(5.7)	(6.5)	(15.2)	(10.5)
	[4.1]	[7.3]	[3.1]	[3.1]	[17.2]	[15.4]
Persistent technology ($\mu^*_{\epsilon,t}$)	5.9	5.7	5.3	10.4	5.1	1.9
	(8.1)	(7.1)	(8.4)	(9.3)	(7.7)	(6.7)
	[18.8]	[37.4]	[12.8]	[29.8]	[14.2]	[5.9]
Government spending (g_t)	3.4	2.1	0.22	3.3	0.75	1.4
	(0.6)	(0.3)	(0.1)	(0.8)	(0.5)	(0.8)
	[4.8]	[4.1]	[0.3]	[3.9]	[0.3]	[0.7]
Price mark-up ($\lambda_{f,t}$)	13.1	12.5	13.3	14.4	61.6	46.8
	(8.4)	(8.6)	(8.4)	(8.9)	(32.1)	(18.9)
	[16.2]	[27.2]	[14.2]	[15.2]	[65.8]	[58.2]
Monetary policy ($\epsilon_{R,t}$)	3.6	5.7	2.3	4.1	1	13.1
	(1.7)	(1.8)	(1.7)	(1.8)	(1.2)	(6.3)
	[4.1]	[8.8]	[2.1]	[4]	[0.3]	[11.4]
Price of investment ($\mu_{\Upsilon,t}$)	1.7	0.5	2.3	1.6	0.3	0.7
	(5.1)	(4.2)	(6.1)	(5)	(3.3)	(4.4)
	[2.2]	[1.3]	[2.2]	[1.8]	[0.1]	[0.4]
Efficiency of investment (ζ_t)	32.8	7.6	52.5	32.3	5.3	12.6
	(15)	(10.1)	(18)	(15.4)	(10.3)	(13.9)
	[49.6]	[13.8]	[65.1]	[42.1]	[2]	[7.7]

Note: For each variable, the first two rows of numbers refer to the variance decomposition in the estimated model with ambiguity. The first row is the business cycle frequency and the second row is the long-run decomposition. The third row, in squared brackets, refers to the business cycle frequency decomposition in the estimated model without ambiguity.

Figure 3: Impulse response: ambiguity