Information, Liquidity, and Dynamic Limit Order Markets*

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March 15, 2018

Abstract

This paper describes price discovery and liquidity provision in a dynamic limit order market with asymmetric information and non-Markovian learning. In particular, investors condition on information in both the current limit order book and also, unlike in previous research, on the prior trading history when deciding whether to provide or take liquidity. Numerical examples show that the information content of the prior order history can be substantial. In addition, we show that the information content of arriving orders can be non-monotone in both the direction and aggressiveness of arriving orders.

JEL classification: G10, G20, G24, D40

Keywords: Limit order markets, asymmetric information, liquidity, market microstructure

*We thank Sandra Fortini, Thierry Foucault, Paolo Giacomazzi, Burton Hollifield, Phillip Illeditsch, Stefan Lewellen, Marco Ottaviani, Tom Ruchti, and seminar participants at Carnegie Mellon University for helpful comments. We are grateful to Fabio Sist for his significant contribution to the computer code for our model.

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The aggregation of private information and the dynamics of liquidity supply and demand are closely intertwined in financial markets. In dealer markets, informed and uninformed investors trade via market orders and, thus, take liquidity, while dealers provide liquidity and try to extract information from the arriving order flow (as in Kyle (1985) and Glosten and Milgrom (1985)). However, in limit order markets — the dominant form of securities market organization today — the relation between who has information and who is trying to learn it and who supplies and demands liquidity is not well understood theoretically.[1] Recent empirical research highlights the role of informed traders not only as liquidity takers but also as liquidity suppliers. O'Hara (2015) argues that fast informed traders use market and limit orders interchangeably and often prefer limit orders to marketable orders. Fleming, Mizrach, and Nguyen (2017) and Brogaard, Hendershott, and Riordan (2016) find that limit orders play a significant empirical role in price discovery.[2]

Our paper presents the first rational expectations model of a dynamic limit order market with asymmetric information and history-dependent Bayesian learning. In particular, learning is not constrained to be Markovian. The model represents a trading day with market opening and closing effects. Our model lets us investigate the information content of different types of market and limit orders, the dynamics of who provides and demands liquidity, and the non-Markovian information content of the trading history. In addition, we study how changes in the amount of adverse selection — in terms of both asset-value volatility and the arrival probability of informed investors — affect equilibrium trading strategies, liquidity, price discovery, and welfare. We have three main results:

- Increased adverse selection does not always worsen market liquidity as in Kyle (1985). Liquidity can potentially improve if informed traders with better information trade more aggressively by submitting limit-orders at the inside quotes rather than using market orders.

[1] See Jain (2005) for a discussion of the prevalence of limit order markets. See Parlour and Seppi (2008) for a survey of theoretical models of limit order markets. See Rindi (2008) for a model of informed traders as liquidity providers.

[2] Gencay, Mahmoodzadeh, Rojcek, and Tseng (2016) investigate brief episodes of high-intensity/extreme behavior of quotation process in the U.S. equity market (bursts in liquidity provision that happen several hundreds of time a day for actively traded stocks) and find that liquidity suppliers during these bursts significantly impact prices by posting limit orders.

- The relation between limit and market orders and their information content depends on the size of private information shocks relative to the tick size. Indeed, the information content of orders can even be opposite the order direction and aggressiveness.

- The learning dynamics are non-Markovian in that the order history has information in addition to the current state of the limit order book. In particular, the incremental information content of arriving limit and market orders is history-dependent.

Dynamic limit order markets with uninformed investors are studied in a large literature. This includes Foucault (1999), Parlour (1998), Foucault, Kadan, and Kandel (2005), and Goettler, Parlour, and Rajan (2005). There is some previous theoretical research that allows informed traders to supply liquidity. Kumar and Seppi (1994) is a static model in which optimizing informed and uninformed investors use profiles of multiple limit and market orders to trade. Kaniel and Liu (2006) extend the Glosten and Milgrom (1985) dealership market to allow informed traders to post limit orders. Aït-Sahalia and Saglam (2013) also allow informed traders to post limit orders, but they do not allow them to choose between limit and market orders. Moreover, the limit orders posted by their informed traders are always at the best bid and ask prices. Goettler, Parlour, and Rajan (2009) allow informed and uninformed traders to post limit or market orders, but their model is stationary and assumes Markovian learning. Roşu (2016b) studies a steady-state limit order market equilibrium in continuous-time with Markovian learning and additional information-processing restrictions. These last two papers are closest to ours. Our model differs from them in two ways: First, they assume Markovian learning in order to study dynamic trading strategies with order cancellation, whereas we simplify the strategy space (by not allowing dynamic order cancellations and submissions) in order to investigate non-Markovian learning (i.e., our model has a larger state space). Second, we model a non-stationary trading day with opening and closing effects and history-dependent Bayesian learning. Market opens and closes are important daily events in the dynamics of liquidity in financial markets. Bloomfield, O'Hara, and Saar (2005) show in an experimental asset market setting that informed traders sometimes provide more liquidity than uninformed traders. Our model provides equilibrium examples of liquidity provision by informed investors.

A growing literature investigates the relation between information and trading speed (e.g., Biais, Foucault, and Moinas (2015); Foucault, Hombert, and Roşu (2016); and Roşu (2016a)). However, these models assume Kyle or Glosten-Milgrom market structures and, thus, cannot consider the roles of informed and uninformed traders as endogenous liquidity providers and demanders. We argue that understanding price discovery dynamics in limit order markets is an essential precursor to understanding speedbumps and cross-market competition given the real-world prevalence of limit order markets.

1 Model

We consider a limit order market in which a risky asset is traded at five times $t_j \in \{t_1, t_2, t_3, t_4, t_5\}$ over a trading day. The fundamental value of the asset after time t_5 at the end of the day is

$$\tilde{v} = v_0 + \Delta = \begin{cases} \bar{v} = v_0 + \delta & with\ Pr(\bar{v}) = \frac{1}{3} \\ v_0 & with\ Pr(v_0) = \frac{1}{3} \\ \underline{v} = v_0 - \delta & with\ Pr(\underline{v}) = \frac{1}{3} \end{cases} \tag{1}$$

where v_0 is the ex ante expected asset value, and Δ is a symmetrically distributed value shock. The limit order market allows for trading through two types of orders: Limit orders are price-contingent orders that are collected in a limit order book. Market orders are executed immediately at the best available price in the limit order book. The limit order book has a price grid with four prices, $P_i \in \{A_2, A_1, B_1, B_2\}$, two each on the ask and bid sides of the market. The tick size is equal to $\kappa > 0$, and the ask prices are $A_1 = v_0 + \frac{\kappa}{2}$, $A_2 = v_0 + \kappa$, ; and by symmetry the bid prices are $B_1 = v_0 - \frac{\kappa}{2}$, $B_2 = v_0 - \kappa$. Order execution in the limit order book follows time and price priority.

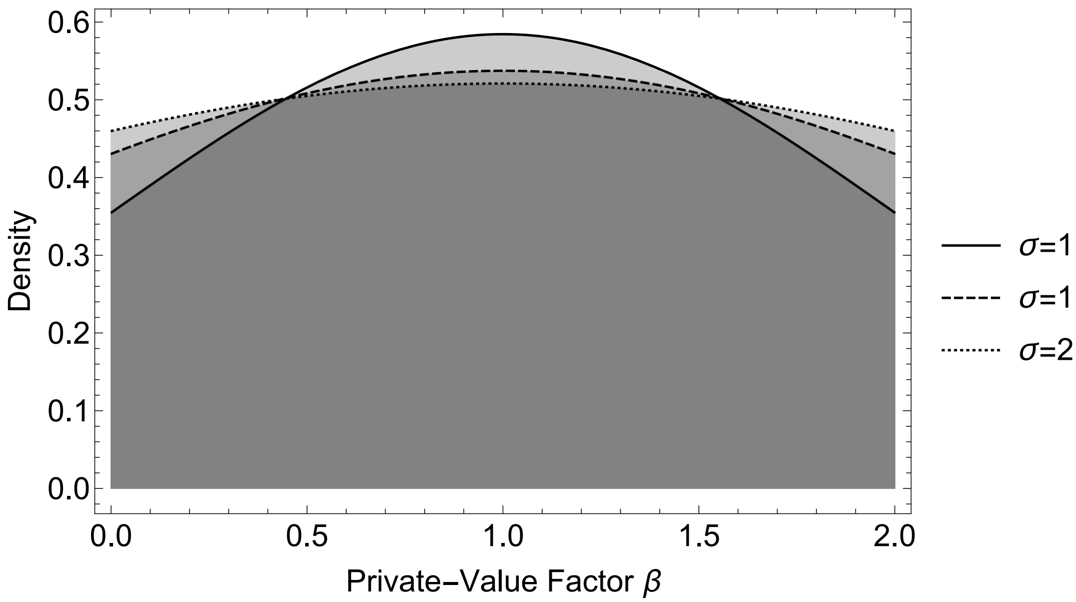
Investors arrive sequentially over time to trade in the market. At each time t_j one investor arrives. Investors are risk-neutral and asymmetrically informed. A trader is informed with probability α and uninformed with probability $1 - \alpha$. Informed investors know the realized value shock Δ perfectly. Uninformed investors do not know Δ, but they use Bayes' Rule and their knowledge of the equilibrium to learn about Δ from the observable market dynamics over time. An investor arriving at time t_j may also have a personal private-value trading motive, which — we assume for tractability — causes them to adjust their valuation of v_0 to $\beta_{t_j} v_0$ where the factor β_{t_j} may be greater than or less than 1. Non-informational private-value motives include preference shocks, hedging needs, and taxation. The absence of a non-informational trading motive would lead to the Milgrom and Stokey (1982) no-trade result. The factor β_{t_j} at time t_j is drawn from a truncated normal distribution, $Tr[\mathcal{N}(\mu, \sigma^2)]$, with support over the interval $[0, 2]$. The mean is $\mu = 1$, which corresponds to a neutral private valuation. Traders with neutral valuations tend to provide liquidity symmetrically on both the buy and sell sides of the market, while traders with extreme private valuations provide one-sided liquidity or actively take liquidity. The parameter σ determines the dispersion of a trader's private-value factor β_{t_j}, as shown in Figure 1, and, thus, the probability of large private gains-from-trade due to extreme investor private valuations.

The sequence of arriving investors is independently and identically distributed in terms of whether they are informed or uninformed and in terms of their individual private-value factors β_{t_j}. In one specification of our model, only uninformed investors have private valuations, while in a second richer specification both informed and uninformed investors have private valuations. A generic informed investor is denoted as I, where we denote the informed investor as $I_{\bar{v}}$ if the value shock is positive ($\Delta = \delta$), as $I_{\underline{v}}$ if the shock is negative ($\Delta = -\delta$), and as I_{v_0} if the shock is zero ($\Delta = 0$). Informed investors arriving at different times during the day all have the identical asset-value information (i.e., there is only one realized Δ). Uninformed investors are denoted as U.

An investor arriving at time t_j can take one of seven possible actions x_{t_j}: One possibility is to submit a buy or sell market order MOA_{i,t_j} or MOB_{i,t_j} to buy or sell immediately at the best available ask or bid respectively in the limit order book at time t_j. A subscript $i = 1$ indicates that the best quote at time t_j is at the inside quote A_1 or B_1, and $i = 2$ means the best quote is at the outside quote. Alternatively, the investor can submit one of four possible limit orders LOA_{i,t_j} and LOB_{i,t_j} on the ask or bid side of the book, respectively. A subscript $i = 1$ denotes an aggressive limit order posted at the inside quote, and $i = 2$ is a less aggressive limit order at the outside quote, A_2 or B_2.[3] Yet another alternative is for an investor to choose to do nothing (NT_{t_j}).

[3]For tractability, it is assumed that investors cannot post buy limit orders at A_1 and sell limit orders at B_1. This is one way in which the investor action space is simplified in our model.

Figure 1: Distribution of Traders' Private-Value Factors - $\boldsymbol{\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]}$.
This figure shows the truncated Normal probability density Function (PDF) of trader private-value factors β_{t_j} with a mean $\mu = 1$ and three different values of dispersion σ.



For tractability, we make a few simplifying assumptions. Limit orders cannot be modified or canceled after submission. Thus, each arriving investor has one and only one opportunity to submit an order. There is also no quantity decision. Orders are to buy or sell one share. Lastly, investors can only submit one order. Taken together, these assumptions let us express the traders' action space as $X_{t_j} = \{MOB_{i,t_j}, LOA_{1,t_j}, LOA_{2,t_j}, NT_{t_j}, LOB_{2,t_j}, LOB_{1,t_j}, MOA_{i,t_j}\}$, where each of the orders denotes an order for one share.

In addition to the arriving informed and uninformed traders, there is a market-making *trading crowd* that submits limit orders to provide liquidity. By assumption, the crowd just posts single limit orders at the outside prices A_2 and B_2. The market opens with an initial book submitted by the crowd at time t_0. After each subsequent order-submission time t_j for arriving informed and uninformed investors, the crowd replenishes the book at the outside prices, if needed, when either side of the book is empty. If there are still limit orders at prices A_2 and B_2 on both sides of the book, then the crowd does not submit any limit orders. For tractability, we assume that public limit orders by the arriving informed and uninformed investors have priority over limit orders from the crowd. The focus of our model is on market dynamics involving information and liquidity given the behavior of optimizing informed and uninformed investors. The crowd is simply a modeling device to insure it is always possible for arriving traders to submit market orders if they so choose.

Market dynamics over the trading day are intentionally non-stationary in our model in order to capture market opening and closing effects. When the market opens at t_1 there are no standing limit orders in the book except from those at prices A_2 or at B_2 from the trading crowd.[4] At the end of the day all unexecuted limit orders are cancelled.

The state of the limit order book at time t_j given orders from arriving investors is

$$L_{t_j} = [q_{t_j}^{A_2}, q_{t_j}^{A_1}, q_{t_j}^{B_1}, q_{t_j}^{B_2}] \tag{2}$$

where $q_{t_j}^{A_i}$ and $q_{t_j}^{B_i}$ indicate the depth at prices A_i and B_i at time t_j. In addition, there are limit orders from the crowd. While the crowd's orders are in the book, we net them out when talking about the informational "state" of the book, since they are perfectly predictable. Let ΔL_{t_j} be the change in the limit order book generated by an arriving informed and uninformed investor's action $x_{t_j} \in X_{t_j}$ at time t_j:[5]

$$\Delta L_{t_j} = [\Delta q_{t_j}^{A_2}, \Delta q_{t_j}^{A_1}, \Delta q_{t_j}^{B_1}, \Delta q_{t_j}^{B_2}] = \begin{cases} [-1,0,0,0] & \text{if } x_{t_j} = MOA_{2,t_j} \\ [0,-1,0,0] & \text{if } x_{t_j} = MOA_{1,t_j} \\ [+1,0,0,0] & \text{if } x_{t_j} = LOA_{2,t_j} \\ [0,+1,0,0] & \text{if } x_{t_j} = LOA_{1,t_j} \\ [0,0,0,0] & \text{if } x_{t_j} = NT \\ [0,0,+1,0] & \text{if } x_{t_j} = LOB_{1,,t_j} \\ [0,0,0,+1] & \text{if } x_{t_j} = LOB_{2,t_j} \\ [0,0,-1,0] & \text{if } x_{t_j} = MOB_{1,t_j} \\ [0,0,0,-1] & \text{if } x_{t_j} = MOB_{2,t_j} \end{cases} \tag{3}$$

where "+1" with a limit order denotes the addition of an order at a particular limit price and "−1"

[4]In practice, daily opening limit order books include uncancelled orders from the previous day and new limit orders from opening auctions. For simplicity, we abstract from these interesting features of markets.

[5]There are nine alternatives in (3) because we allow separately for cases in which the best bid and ask for market sells and buys are at the inside and outside quotes.

denotes execution of an earlier BBO limit order in the book. The resulting dynamics of the limit order book are:

$$L_{t_j} = L_{t_{j-1}} + \Delta L_{t_j} \tag{4}$$

where $j = 1, \ldots, 5$. An important source of information in our model is the observed trading history of orders posted at times $t_1, .., t_j$ in the market. We denote an order-flow history by $\mathscr{L}_{t_j} = \{\Delta L_{t_1}, \ldots, \Delta L_{t_j}\}$. When traders arrive in the market, they observe the history of market activity up through the current standing limit order book at the time they arrive.

Investors trade using optimal order-submission strategies given their information and any private-value motive. If an uninformed investor arrives at time t_j, then his order x_{t_j} is chosen to maximize his expected terminal payoff

$$\begin{aligned} \max_{x \in X_{t_j}} \varphi^U(x \,|\beta_{t_j}, \mathscr{L}_{t_{j-1}}) &= E[(\beta_{t_j} v_0 + \Delta - p(x))\, f(x) | \beta_{t_j}, \mathscr{L}_{t_{j-1}}] \\ &= [\beta_{t_j} v_0 + E[\Delta \,| \mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] - p(x)]\, Pr(\theta^x_{t_j} | \mathscr{L}_{t_{j-1}}) \end{aligned} \tag{5}$$

where $p(x)$ is the price at which order x trades, and $f(x)$ denotes the amount of the submitted order that is actually "filled." If x is a market order, then $f(x) = 1$ (i.e., all of the order is executed), and the execution price $p(x)$ is the best quote on the other side of the book at time t_j. If x is a non-marketable limit order, then the execution price $p(x)$ is its limit price, but the fill amount $f(x)$ is random variable equal to 1 if the limit order is filled and zero if it is not filled. If the investor does not trade — either because no order is submitted or because a limit order is not filled — then $f(x)$ is zero. In the second line of (5), the expression $\theta^x_{t_j}$ denotes the set of future trading states in which order x_{t_j} is executed.[6] This conditioning matters for limit orders because the sequence of subsequent orders in the market, which may or may not result in the execution of a limit order submitted at time t_j, is correlated with the asset value shock Δ. For example, future market buy orders are more likely if the Δ shock is positive (since $I_{\overline{v}}$ investors will want to buy). Uninformed investors rationally take the relation between future orders and Δ into account when forming their expectation $E[\Delta \,| \mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$ of what the asset will be worth in states in which their limit orders

[6] A market orders x_{t_j} is executed immediately at time t_j and so is executed for sure.

are executed. The second line of (5) also makes clear that uninformed investors use the prior order history $\mathscr{L}_{t_{j-1}}$ in two ways: It affects their beliefs about limit order execution probabilities $Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}})$ and their execution-state-contingent asset-value expectations $E[\Delta\,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}]$.

An informed investor who arrives at t_j chooses an order x_{t_j} to maximize her expected payoff

$$\begin{aligned}\max_{x\in X_{t_j}} \varphi^I(x\,|\,v, \beta_{t_j}, \mathscr{L}_{t_{j-1}}) &= E[(\beta_{t_j}\,v_0 + \Delta - p(x))\,f(x)|\,v, \beta_{t_j}, \mathscr{L}_{t_{j-1}}] \\ &= [\beta_{t_j}\,v_0 + \Delta - p(x)]\,Pr(\theta^x_{t_j}|\,v, \mathscr{L}_{t_{j-1}})\end{aligned} \tag{6}$$

The only uncertainty for informed investors is about whether any limit orders they submit will be executed. Their belief about order-execution probabilities $Pr(\theta^x_{t_j}|\,v, \mathscr{L}_{t_{j-1}})$ are conditioned on both the trading history up through the current book and on their knowledge about the ending asset value. Thus, informed traders condition on $\mathscr{L}_{t_{j-1}}$, not to learn about the value shock Δ (which they already know) or about future investor private-value factors β_{t_j} (which are i.i.d. over time), but because they understand that the trading history is an input in the trading behavior of future uninformed investors with whom they might trade in the future. Our analysis considers two model specifications for the informed investors. In one, informed investors have no private-value motive, so that their β factors are equal to 1. In the second specification, their β factors are random and are independently drawn from the same truncated normal distribution $Tr[\mathscr{N}(\mu, \sigma^2)]$ as the uninformed investors.

The optimization problem in (5) defines sets of actions $x_{t_j} \in X_{t_j}$ that are optimal for the uninformed investor at different times t_j given different private-value factors β_{t_j} and order histories $\mathscr{L}_{t_{j-1}}$. These optimal orders can be unique, or there may be multiple orders which make the uninformed investor equally well-off. The *optimal order-submission strategy* for the uninformed investor is a probability function $\gamma^U_j(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}})$ that is zero if the order x is suboptimal and equals a mixing probability over optimal orders. If an optimal order x is unique, then $\gamma_j(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}}) = 1$. Similarly, the optimization problem in (6) can be used to define an optimal order submission strategy $\gamma^I_j(x|\beta, v, \mathscr{L}_{t_{j-1}})$ for informed investors at time t_j given their factor β_{t_j}, their knowledge about the asset value v, and the order history $\mathscr{L}_{t_{j-1}}$.

1.1 Equilibrium

An equilibrium is a set of mutually consistent optimal strategy functions and beliefs for uninformed and informed investors for each time t_j, given each order history $\mathscr{L}_{t_{j-1}}$, private-value factor β_{t_j}, and (for informed traders) private information v. This section explains what "mutually consistent" means and then gives a formal definition of an equilibrium in our model.

A central feature of our model is asymmetric information. The presence of informed traders means that, by observing prices and associated quantities (i.e., past and current states of the book), uninformed traders can infer information about the asset value v and use it in their order-submission strategies. More precisely, uninformed traders rationally learn from the trading history about the probability that v will go up, stay constant, or go down. However, investors cannot learn about the private values (β) or information status (I or U) of future traders since these are both i.i.d over time. Informed traders do not need to learn about v since they know it. However, they do condition their trading behavior on v (since that tells them what types of informed traders will arrive in the future along with the uninformed traders), and they condition on the trading history (since that is informative about the trading behavior of future uninformed traders since the trading history is an input in their order-submission strategy functions).

The underlying *economic state* in our model is the realization of the asset value v and a realized sequence of investors who arrive in the market. The investor who arrives at time t_j is described by two characteristics: their status as being informed or uninformed, I_v or U, and their private-value factor β_{t_j}. The underlying economic state is exogenously chosen over time by Nature. More formally, it follows an exogenous stochastic process described by the model parameters δ, α, μ, and σ. A sequence of arriving investors together with a pair of strategy functions — which we denote here as $\Gamma = \{\gamma_j^U(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}}), \gamma_j^I(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}})\}$ — induce a sequence of trading actions x_{t_j} which results in a sequence of observable changes in the state of the limit order book. Thus, the stochastic process generating paths of trading outcomes (i.e., trading histories in the limit order book) is induced by the economic state process and the strategy functions. Given the trading-outcome path process, there are several things we can compute directly: First, we can compute the unconditional probabilities of different paths $Pr(\mathscr{L}_{t_j})$ and the conditional probabilities $Pr(\Delta L_{t_j}|\mathscr{L}_{t_{j-1}})$ of particular order book changes ΔL_{t_j} given a prior history $\mathscr{L}_{t_{j-1}}$. In particular, we can identify paths of trading outcomes that are *possible* (i.e., have positive probability $Pr(\mathscr{L}_{t_j})$) given the strategy functions $\{\gamma_j^U(x|\beta, \mathscr{L}_{t_{j-1}}), \gamma_j^I(x|\beta, v, \mathscr{L}_{t_{j-1}})\}$ and paths of trading outcomes which are not possible (i.e., for which $Pr(\mathscr{L}_{t_j}) = 0$). Second, the trading-outcome path process also determines the order-execution probabilities $Pr(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}})$ and $Pr(\theta^x_{t_j}|\, \mathscr{L}_{t_{j-1}})$ for informed and uninformed investors for orders submitted at time t_j. Computing each of these probabilities is simply a matter of listing all of the possible underlying economic states, mechanically applying the order-submission rules, identifying the relevant outcomes path-by-path, and then taking expectations across paths.

Let ℓ denote the set of all feasible histories $\{\mathscr{L}_{t_j} : j = 1, \ldots, 4\}$ of physically available orders of lengths up to four trading periods. A four-period long history is the longest history a order-submission strategy can depend on in our model. In this context, *feasible* paths are simply sequences of actions in the action choice set without regard to whether they are *possible* in the sense that they can occur with positive probability given the strategy functions Γ. Let $\ell^{\,in,\Gamma}$ denote the subset of all possible trading paths in ℓ that have positive probability, $Pr(\mathscr{L}_{t_j}) > 0$ given a pair of order strategies Γ. Let $\ell^{\,off,\Gamma}$ denote the complementary set of trading paths that are *feasible* but *not possible* given Γ. This notation will be useful when discussing "off equilibrium" beliefs. In our analysis, strategy functions Γ are defined for all feasible paths in ℓ. In particular, this includes all of the possible paths in $\ell^{\,in,\Gamma}$ given Γ and also the paths in $\ell^{\,off,\Gamma}$. As a result, the probabilities $Pr(\Delta L_{t_j}|\mathscr{L}_{t_{j-1}})$, $Pr(\theta^x_{t_j}|\, v, \mathscr{L}_{t_{j-1}})$ and $Pr(\theta^x_{t_j}|\, \mathscr{L}_{t_{j-1}})$ are always well-defined, because the continuation trading process going forward, even after an unexpected order-arrival event (i.e., a path $\mathscr{L}_{t_{j-1}} \in \ell^{\,off,\Gamma}$), is still well-defined.

The stochastic process for trading-outcome paths and its relation to the underlying economic state also determine the uninformed-investor expectations $E[v\,|\mathscr{L}_{t_j}, \theta^x_{t_j}]$ of the terminal asset value given the previous order history ($\mathscr{L}_{t_j}$) and conditional on future limit-order execution ($\theta^x_{t_j}$). These expectations are determined as follows:

- Step 1: The conditional probabilities $\pi^v_{t_j} = Pr(v|\mathscr{L}_{t_j})$ of a particular final asset value $v = \bar{v}, v_0$ or $\underline{v}$ given a possible trading history $\mathscr{L}_{t_j} \in \ell^{\,in,\Gamma}$ up through time t_j is given by Bayes' Rule.

At time t_1, this probability is

$$\begin{aligned}
\pi_{t_1}^v &= \frac{Pr(v, \mathscr{L}_{t_1})}{Pr(\mathscr{L}_{t_1})} = \frac{Pr(\mathscr{L}_{t_1}|v)Pr(v)}{Pr(\mathscr{L}_{t_1})} = \frac{Pr(\Delta L_{t_1}|v)Pr(v)}{Pr(\Delta L_{t_1})} \\
&= \frac{Pr(\Delta L_{t_1}|v, I)Pr(I) + Pr(\Delta L_{t_1}|U)Pr(U)}{Pr(\Delta L_{t_1})} Pr(v) \\
&= \frac{E^\beta[\gamma_1^I(x_{t_1}|\beta_{t_1}^I, v)|v]\alpha + E^\beta[\gamma_1^U(x_{t_1}|\beta_{t_1}^U)](1-\alpha)}{Pr(\Delta L_{t_1})} \pi_{t_0}^v
\end{aligned} \tag{7}$$

where the prior is the unconditional probability $\pi_{t_0}^v = Pr(v)$, x_{t_1} is the trading action at time t_1 that leads to the order book change ΔL_{t_1}, and $\beta_{t_1}^I$ and $\beta_{t_1}^U$ are independently distributed private-value β realizations for informed and uninformed investors at time t_1.[7] At time $t_j > t_1$, this probability is given recursively by[8]

$$\begin{aligned}
\pi_{t_j}^v &= \frac{Pr(v, \mathscr{L}_{t_j})}{Pr(\mathscr{L}_{t_j})} = \frac{Pr(v, \Delta L_{t_j}, \mathscr{L}_{t_{j-1}})}{Pr(\Delta L_{t_j}, \mathscr{L}_{t_{j-1}})} \\
&= \frac{\begin{pmatrix} Pr(\Delta L_{t_j}|v, \mathscr{L}_{t_{j-1}}, I)Pr(I|\mathscr{L}_{t_{j-1}})Pr(v|\mathscr{L}_{t_{j-1}}) \\ + Pr(\Delta L_{t_j}|v, \mathscr{L}_{t_{j-1}}, U)Pr(U|\mathscr{L}_{t_{j-1}})Pr(v|\mathscr{L}_{t_{j-1}}) \end{pmatrix}}{Pr(\Delta L_{t_j}|\mathscr{L}_{t_{j-1}})} \\
&= \frac{E^\beta[\gamma_j^I(x_{t_j}|\beta_{t_j}^I, v, \mathscr{L}_{t_{j-1}})|v, \mathscr{L}_{t_{j-1}}]\ \alpha + E^\beta[\gamma_j^U(x_{t_j}|\beta_{t_j}^U, \mathscr{L}_{t_{j-1}})|\mathscr{L}_{t_{j-1}}]\ (1-\alpha)}{Pr(\Delta L_{t_j}|\mathscr{L}_{t_{j-1}})} \pi_{t_{j-1}}^v
\end{aligned} \tag{8}$$

These probabilities are then used to compute the uninformed-investor expected asset value conditional on the order history path

$$E[\tilde{v}|\mathscr{L}_{t_{j-1}}] = \pi_{t_{j-1}}^{\bar{v}}\, \bar{v} + \pi_{t_{j-1}}^{v_0}\, v_0 + \pi_{t_{j-1}}^{\underline{v}} \underline{v} \tag{9}$$

- Step 2: The conditional probabilities $\pi_{t_j}^v$ given a "feasible but not possible in equilibrium" order history $\mathscr{L}_{t_j} \in \ell^{\,\mathit{off},\Gamma}$ in which a limit order book change ΔL_{t_j} that is inconsistent with

[7] A trader's information status (I or U) is independent of the asset value v, so $P(I|v) = Pr(I)$ and $Pr(U|v) = Pr(U)$. Furthermore, uninformed traders have no private information about v, so the probability $Pr(\Delta L_{t_1}|U)$ with which they take a trading action ΔL_{t_1} does not depend on v.

[8] A trader's information status is again independent of v, and it is also independent of the past trading history $\mathscr{L}_{t_1}$. While the probability with which an uninformed trader takes a trading action ΔL_{t_1} may depend on the past order history $\mathscr{L}_{t_j}$, it does not depend directly on v which uninformed traders do not know.

the strategies Γ at time t_j are set as follows:

1. If the priors are fully revealing in that $\pi^v_{t_{j-1}} = 1$ for some v, then $\pi^v_{t_j} = \pi^v_{t_{j-1}}$ for all v.
2. If the priors are not fully revealing at time t_j, then $\pi^v_{t_j} = 0$ for any v for which $\pi^v_{t_{j-1}} = 0$ and the probabilities $\pi^v_{t_j}$ for the remaining v's can be any non-negative numbers such that $\pi^{\bar{v}}_{t_j} + \pi^{v_0}_{t_j} + \pi^{\underline{v}}_{t_j} = 1$.
3. Thereafter, until any next unexpected trading event, the subsequent probabilities $\pi^v_{t_{j'}}$ for $j' > j$ are updated according to (8).

- Step 3: The execution-contingent conditional probabilities $\hat{\pi}^v_{t_j} = Pr(v|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j})$ of a final asset value v conditional on a prior path $\mathscr{L}_{t_{j-1}}$ and on execution of a limit order x submitted at time t_j is

$$\begin{aligned} \hat{\pi}^v_{t_j} &= \frac{Pr(\mathscr{L}_{t_{j-1}}) Pr(v|\mathscr{L}_{t_{j-1}}) \Pr(\theta^x_{t_{j-1}}|v, \mathscr{L}_{t_{j-1}})}{Pr(\theta^{x_{t_j}}, \mathscr{L}_{t_{j-1}})} \\ &= \frac{\Pr(\theta^x_{t_j}|v, \mathscr{L}_{t_{j-1}})}{Pr(\theta^x_{t_j}|\mathscr{L}_{t_{j-1}})} \pi^v_{t_{j-1}} \end{aligned} \tag{10}$$

This holds when adjusting for a future execution contingency both when the probabilities $\pi^v_{t_{j-1}}$ given the prior history $\mathscr{L}_{t_{j-1}}$ are for possible paths in $\ell^{\,in,\Gamma}$ (from (7) and (8) in Step 1) and also for feasible but not possible paths in $\ell^{\,off,\Gamma}$ (from Step 2). These execution-contingent probabilities $\hat{\pi}^v_{t_j}$ are used to compute the execution-contingent conditional expected value

$$E[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] = \hat{\pi}^{\bar{v}}_{t_j}\, \bar{v} + \hat{\pi}^{v_0}_{t_j}\, v_0 + \hat{\pi}^{\underline{v}}_{t_j}\, \underline{v} \tag{11}$$

used by uninformed traders to compute expected payoffs for limit orders. In particular, the probabilities in (11) are the execution-contingent probabilities $\hat{\pi}^v_{t_j}$ from (10) rather than the probabilities $\pi^v_{t_j}$ from (8) that just condition on the prior trading history but not on the future states in which the limit order is executed.

Given these updating dynamics, we can now define an equilibrium.

Definition. A *Perfect Bayesian Nash Equilibrium* of the trading game in our model is a collection $\{\gamma_j^{U,*}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}}), \gamma_j^{I,*}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}}), Pr^*(\theta_{t_j}^x|\, v, \mathscr{L}_{t_{j-1}}), Pr^*(\theta_{t_j}^x|\, \mathscr{L}_{t_{j-1}}), E^*[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta_{t_j}^x]\}$ of order-submission strategies, execution-probability functions, and execution-contingent conditional expected asset-value functions such that:

- The equilibrium execution probabilities $Pr^*(\theta_{t_j}^x|\, v, \mathscr{L}_{t_{j-1}})$ and $Pr^*(\theta_{t_j}^x|\, \mathscr{L}_{t_{j-1}})$ are consistent with the equilibrium order-submission strategies $\{\gamma_{j+1}^{U,*}(x|\beta_{t_{j+1}}, \mathscr{L}_{t_j}), \ldots, \gamma_5^{U,*}(x|\beta_{t_5}, \mathscr{L}_{t_4})\}$ and $\{\gamma_{j+1}^{I,*}(x|\beta_{t_{j+1}}, v, \mathscr{L}_{t_j}), \ldots, \gamma_5^{I,*}(x|\beta_{t_5}, v, \mathscr{L}_{t_4})\}$ after time t_j.

- The execution-contingent conditional expected asset values $E^*[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta_{t_j}^x]\}$ agree with Bayesian updating equations (7), (8), (10), and (11) in Steps 1 and 3 when the order x is consistent with the equilibrium strategies $\gamma_j^{U,*}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}})$ and $\gamma_j^{I,*}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}})$ at date t_j and, when x is an off-equilibrium action inconsistent with the equilibrium strategies, with the off-equilibrium updating in Step 2.

- The positive-probability supports of the equilibrium strategy functions $\gamma_j^{U,*}(x|\beta_{t_j}, \mathscr{L}_{t_{j-1}})$ and $\gamma_j^{I,*}(x|\beta_{t_j}, v, \mathscr{L}_{t_{j-1}})$ (i.e., the orders with positive probability in equilibrium) are subsets of the sets of optimal orders for uninformed and informed investors computed from their optimization problems (5) and (6) and the equilibrium execution probabilities and outcome-contingent conditional asset-value expectation functions $Pr^*(\theta_{t_j}^x|\, v, \mathscr{L}_{t_{j-1}})$, $Pr^*(\theta_{t_j}^x|\, \mathscr{L}_{t_{j-1}})$, and $E^*[\tilde{v}|\mathscr{L}_{t_{j-1}}, \theta_{t_j}^x]$.

The Appendix explains the algorithm used to compute the equilibria in our model. To help with intuition, the next section walks through the order-submission and Bayesian updating mechanics for a particular path in the extensive form of the model.

Our equilibrium concept differs from the Markov Perfect Bayesian Equilibrium used in Goettler et al. (2009). Beliefs and strategies in our model are path-dependent; traders use Bayes Rule given the full prior order history when they arrive in the market. In contrast, Goettler et al. (2009) restricts Bayesian updating to the current state of the limit order book but do not allow for conditioning on the previous order history. Roşu (2016b) also assumes a Markov Perfect Bayesian Equilibrium. The quantitative importance of the order history is an issue that is considered when we discuss our results in Section 2.

1.2 Illustration of order-submission mechanics and Bayesian updating

This section uses an excerpt of the extensive form of the trading game in our model to illustrate order-submission and trading dynamics and the associated Bayesian updating process. The particular trading history path in Figure 2 is from the equilibrium for the model specification in which informed and uninformed investors both have random private-value motives. The parameter values are $\kappa = 0.10$, $\sigma = 1.5$, $\alpha = 0.8$, and $\delta = 0.16$, which is a market with a relatively high informed-investor arrival probability and large value shocks. In this example, Nature has chosen an economic state in which there is good news ($\overline{v}$) about the asset, and the realized sequence of arriving traders over time is $\{I, U, U, I, I\}$. Given the purpose of this discussion, Figure 2 just shows the "public" portion of the total book L_{t_j} due to orders from arriving informed and uninformed investors without the additional orders from the crowd. Trading starts at t_1 with an incoming empty public book, $[0, 0, 0, 0]$ (shown here) plus the additional limit orders from the trading crowd (i.e., 1 at the outside prices A_2 and B_2). For simplicity, our discussion here only reports a few nodes of the trading game with their associated equilibrium strategies. For example, we do not include NT at the end of t_1, since, as we show later in the paper, NT is not an equilibrium action at t_1 for these parameters.

The path in Figure 2 also illustrates Bayesian updating in the model. After the investor at t_1 has been observed submitting a limit order LOA_{2,t_1}, the uninformed trader who arrives in this example at time t_2 — who just knows the submitted order at time t_1 but not the identity or information of the trader at time t_1 — updates his equilibrium conditional valuation to be $E[\tilde{v}|LOA_{2,t_1}] = 1.056$ and his execution-contingent expectation given his limit order LOA_{1,t_2} at time t_2 to be $E[\tilde{v}|LOA_{2,t_1}, \theta^{LOA_{1,t_2}}] = 1.089$.[9] In subsequent periods, later investors observe additional realized orders and then further update their beliefs.

Investors in our equilibrium choose from a discrete number of possible orders given their respective information and any private-value trading motives. Along the equilibrium path considered here, the optimal strategies do not involve any randomization across different orders. Optimal orders are unique given the inputs. Figure 2 shows below each order type at each time the probabilities with which the different orders are submitted by the trader who arrived. For example, if an informed

[9] The numerical values of the expected values and the order-submission probabilities discussed here are all taken from our equilibrium calculations.

trader $I_{\overline{v}}$ arrives at t_1, she chooses a limit order LOA_{2,t_1} to sell at A_2 with probability 0.118. Each of these unique optimal orders is associated with a different range of β types (for both informed and uninformed investors) and value signals (for informed investors). Figure 3 illustrates where the order-submission probabilities come from by superimposing the upper envelope of the expected payoffs for the different optimal orders at time t_1 on the truncated Normal β distribution. It shows how different β ranges correspond to a discrete set of optimal orders delimited by the β thresholds. At each trading time, as the trading game progresses along this path, traders submit orders (or do not trade) following their equilibrium order-submission strategies. The equilibrium execution probabilities of their orders depend on the order-submission decisions of future traders, which, in turn, depend on their trading strategies and the input information (i.e., their β realizations, any private knowledge about v, and the order history path at the times they arrive). At time t_1, the initial trader has rational-expectation beliefs that the execution probability of her LOA_{2,t_1} order posted at t_1 is 0.644. This equilibrium execution probability depends on all of the possible future trading paths from the submission time t_1 up through time t_5. For example, one possibility is that the LOA_{2,t_1} order will be hit by an investor arriving at time t_2 who submits a market order. Another possibility (which is what happens along this particular path) is that the next period (at t_2) an uninformed trader could arrive and post a limit order LOA_{1,t_2} to sell at A_1, thereby undercutting the LOA_{2,t_1} order — so that the public portion of the book at the end of t_2 is $[1, 1, 0, 0]$). In this scenario, the initial LOA_{2,t_1} order from t_1 will only be executed provided that the LOA_{1,t_2} order submitted at t_2 is executed first. For example, the probability of a market order MOA_{1,t_3} hitting the limit order at A_1 at t_3 is 0.365, and then the probability of another market order hitting the initial limit sell at A_2 is 0.423 at t_4 or 0.505 at t_5.[10] Therefore, there is a chance that the LOA_{2,t_1} order from t_1 will still be executed after it is undercut by the order LOA_{1,t_2} at t_2.

[10]Due to space constraints, we do not include the t_4 node in Figure 2.

Figure 2: Excerpt of the Extensive Form of the Trading Game. This figure shows one of the possible trading paths of the trading game with parameters $\alpha = 0.8$, $\delta = 0.16$, $\mu = 1$, $\sigma = 1.5$, $\kappa = 0.10$, and 5 time periods. Before trading starts at time t_1, the incoming book from time t_0 consists of an empty public book $[0, 0, 0, 0]$ at all price levels (A_i and B_i with $i = 1, 2$) plus the limit orders from the crowd (at A_2 and B_2). Nature selects a realized final value $v = \{\bar{v}, v_0, \underline{v}\}$ with probabilities $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$. At each trading period nature also selects an informed trader (I) with probability α and an uninformed trader (U) with probability $(1 - \alpha)$. Arriving traders choose the optimal order at each period which may potentially include limit orders LOA_t (LOB_t) or market orders at the best ask, $MOA_{i,t}$, or at the best bid, $MOB_{i,t}$. Below each optimal trading strategy we report in italics its equilibrium order-submission probability. Boldfaced equilibrium strategies and associated states of the book (within double vertical bar) indicate the states of the book that we consider at each node of the chosen trading path.

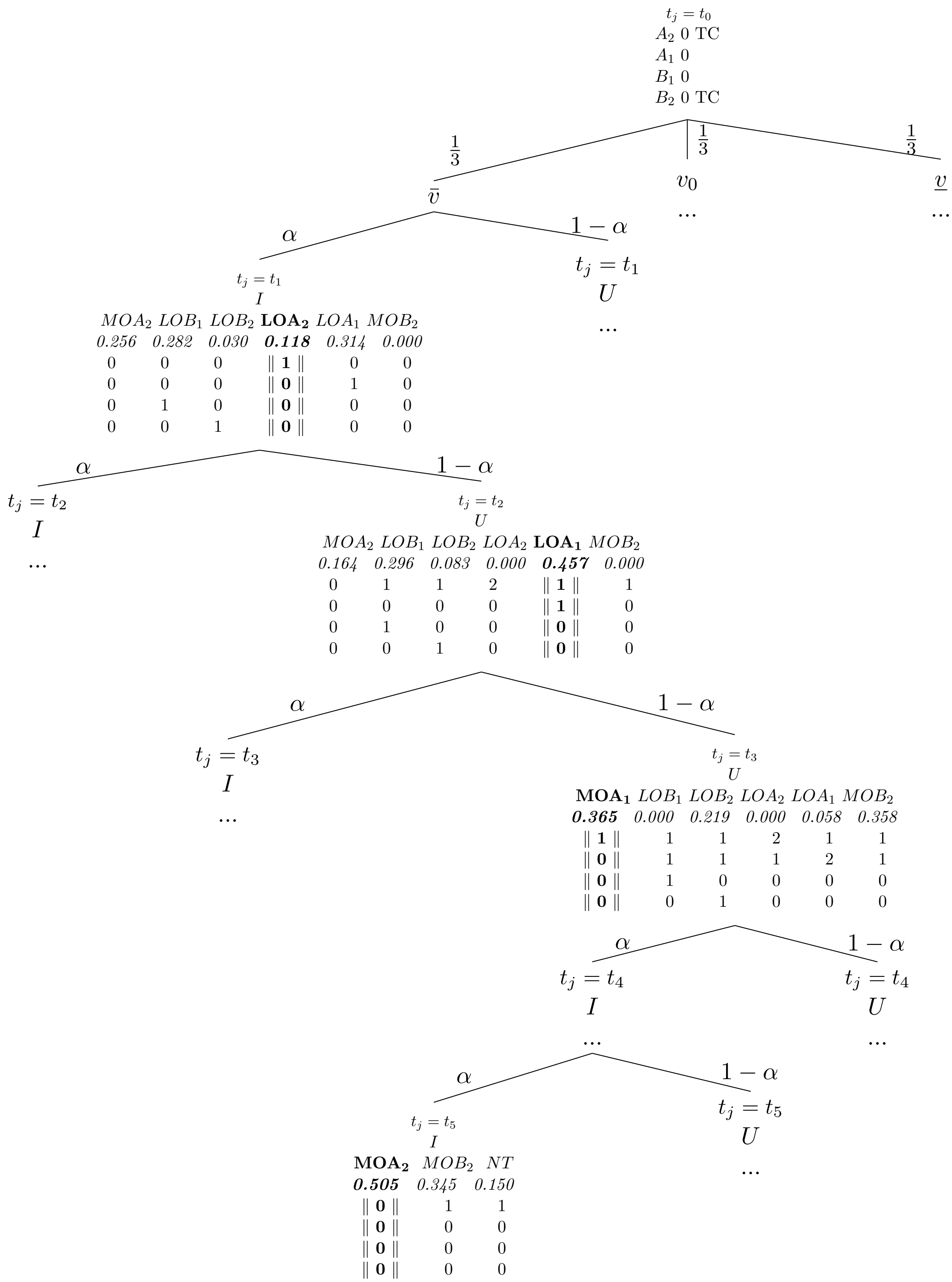
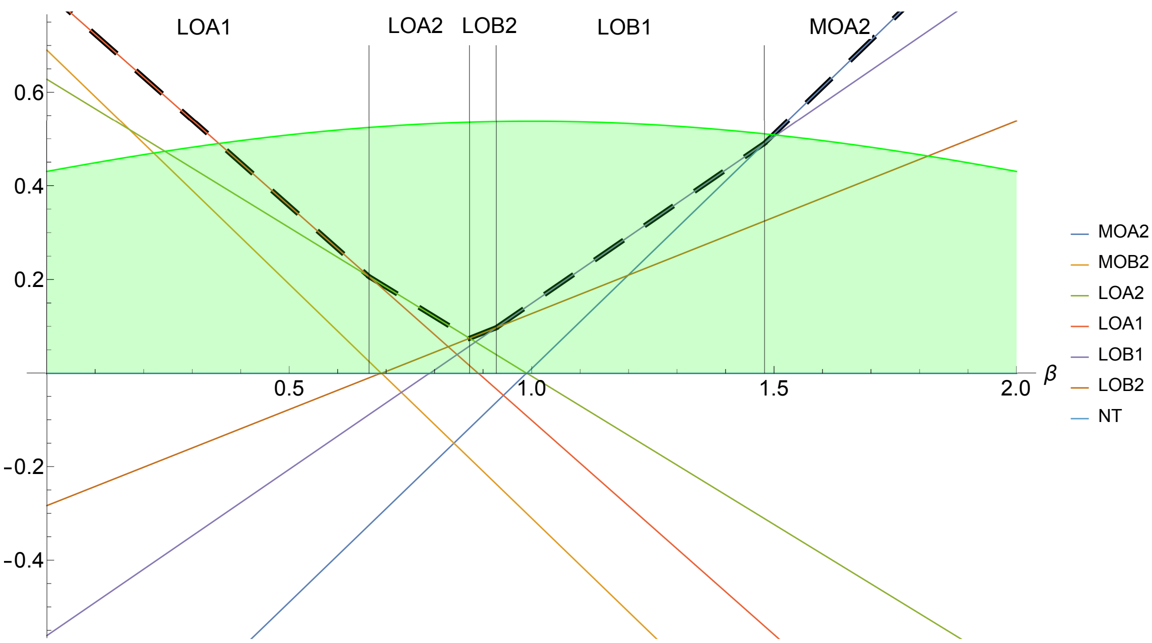


Figure 3: β Distribution and Upper Envelope for Informed Investor $I_{\bar{v}}$ at time t_1. This figure shows the private-value factor $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ distribution superimposed on the plot of the expected payoffs the informed investor $I_{\bar{v}}$ with good news at time t_1 for each equilibrium order type MOA_2, MOB_2, LOA_2, LOA_1, LOB_1, LOB_2, NT, (solid colored lines) when the total book (including crowd limit orders) opens L_{t_0} = [1 0 0 1]. The dashed line shows the investor's upper envelope for the optimal orders. The vertical black lines show the β-thresholds at which two adjacent optimal strategies yield the same expected payoffs. For example LOA_1 is the optimal strategy for values of β between 0 and the first vertical black line; LOA_2 is instead the optimal strategy for the values of beta between the first and the second vertical lines. The parameters are $\alpha = 0.8$, $\delta = 0.16$, $\mu = 1$, $\sigma = 1.5$, and $\kappa = 0.10$.



2 Results

Our analysis investigates how liquidity supply and demand decisions of informed and uninformed traders and the learning process of uninformed traders affect market liquidity, price discovery, and investor welfare. This section presents numerical results for our model. We first consider a model specification in which only uninformed investors have a random private-value trading motive. In a second specification, we generalize the analysis and show the robustness of our findings and extend them. The tick size κ is fixed at 0.10, and the private-value dispersion σ is 1.5 throughout. We investigate comparative statics for the amount of adverse selection. We also show that our model has significant non-Markovian learning that would be missed in constrained Markovian equilibria.

Our analysis focuses on two time windows. The first is when the market opens at time t_1.

The second is over the middle of the trading day from times t_2 through t_4. We look at these two windows because our model is non-stationary over the trading day. Much like actual trading days, our model has start-up effects at the beginning of the day and terminal horizon effects at the market close. When the market opens at time t_1, there are time-dependent incentives to provide rather than to take liquidity: The incoming book is thin (with limit orders only from the crowd), and there is the maximum time for future investors to arrive to hit limit orders from t_1. There are also time-dependent disincentives to post limit orders. Information asymmetries are maximal at time t_1, since there has been no learning from the trading process. Over the day, information is revealed (lessening adverse selection costs), but the book can also become fuller (i.e., there is competition in liquidity provision from earlier limit orders), and the remaining time for market orders to arrive and execute limit orders becomes shorter. Comparing these two time windows shows how market dynamics change over the day. The market close at t_5 is also important, but trading then is straightforward. At the end of the day, investors only submit market orders (or do not trade), because the execution probability for new limit orders submitted at t_5 is zero given our assumption that unfilled limit orders are canceled once the market closes.

We use our model to investigate three questions: First, who provides and takes liquidity, and how does the amount of adverse selection affect investor decisions to take and provide liquidity? Second, how does market liquidity vary with different amounts of adverse selection? Third, how does the information content of different types of orders depend on an order's direction, aggressiveness, and on the prior order history?

The amount of adverse selection can change in two ways: The expected number of informed traders can change, and the magnitude of asset value shocks can change. We consider four different combinations of parameters with high and low informed-investor arrival probabilities ($\alpha = 0.8$ and 0.2) and high and low value-shock volatilities ($\delta = 0.16$ and 0.02). We call markets with $\delta = 0.02$ *low-volatility* markets and markets with $\delta = 0.16$ *high-volatility* markets, because the arriving information is small relative to the tick size $\kappa = 0.10$ in the former parameterization and large relative to the tick size in the later. In high-volatility markets, the final asset value v given good or bad news is beyond the outside quotes A_2 or B_2, and so even market orders at the outside prices are profitable for informed traders. However, in low-volatility markets, v is always within the inside quotes A_1 and B_1, and so market orders are never profitable for informed investors.

2.1 Uninformed traders with random private-value motives

In our first model specification, only uninformed traders have random private values. Informed traders have fixed neutral private-value factors $\beta = 1$. Thus, as in Kyle (1985), there is a clear differentiation between investors who speculate on private information and those who trade for purely non-informational reasons. Our model differs from Kyle (1985) in that informed and uninformed investors can trade using both limit and market orders rather than being restricted to market orders.

2.1.1 Trading strategies

We begin by investigating who supplies and takes liquidity and how these decisions change with the amount of adverse selection. Table 1 reports results about trading early in the day at time t_1 using a 2×2 format. Each of the four cells corresponds to a different combination of parameters. Comparing cells horizontally shows the effect of a change in the value-shock size δ while holding the arrival probability α for informed traders fixed. Comparing cells vertically shows the effect of a change in the informed-investor arrival probability while holding the value-shock size fixed. In each cell corresponding to a set of parameter values, there are four columns reporting conditional results for informed investors with good news, neutral news, and bad news about the asset ($I_{\overline{v}}$, I_{v_0}, $I_{\underline{v}}$) and for an uninformed investor (U) and a fifth column with the unconditional market results ($Uncond$). The table reports the order-submission probabilities and several market-quality metrics. Specifically, we report expected bid-ask spreads conditioning on the three informed-investor types $E[Spread\,|I_v]$ and on the uninformed trader $E[Spread\,|U]$, the unconditional expected market spread $E[Spread]$, and expected depths at the inside prices (A_1 and B_1) and total depths ($A_1 + A_2$ and $B_1 + B_2$) on each side of the market.

Table 1: Trading Strategies, Liquidity, and Welfare at Time t_1 in an Equilibrium with Informed Traders with $\beta = 1$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and two different value-shock volatilities δ (0.16 and 0.02). The private-value factor parameters are $\mu = 1$ and $\sigma = 1.5$, and the tick size is $\kappa = 0.10$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market at time t_1 as well as the welfare expectation of market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 0.16$					$\delta = 0.02$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	LOA_2	0	0.500	0.650	0.143	0.335	0	0.500	1.000	0.052	0.410
	LOA_1	0	0	0.350	0	0.093	0	0	0	0.079	0.016
	LOB_1	0.350	0	0	0	0.093	0	0	0	0.079	0.016
	LOB_2	0.650	0.500	0	0.143	0.335	1.000	0.500	0	0.052	0.410
	MOA_2	0	0	0	0.357	0.071	0	0	0	0.369	0.074
	MOA_1	0	0	0	0	0	0	0	0	0	0
	MOB_1	0	0	0	0	0	0	0	0	0	0
	MOB_2	0	0	0	0.357	0.071	0	0	0	0.369	0.074
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.8$											
	E[Spread $\vert\cdot$]	0.265	0.300	0.265	0.300	0.281	0.300	0.300	0.300	0.284	0.297
	E[Depth A_2+A_1 $\vert\cdot$]	1.000	1.500	2.000	1.143	1.429	1.000	1.500	2.000	1.131	1.426
	E[Depth A_1 $\vert\cdot$]	0	0	0.350	0	0.093	0	0	0	0.079	0.016
	E[Depth B_1 $\vert\cdot$]	0.350	0	0	0	0.093	0	0	0	0.079	0.016
	E[Depth B_1+B_2 $\vert\cdot$]	2.000	1.500	1.000	1.143	1.429	2.000	1.500	1.000	1.131	1.426
	E[Welfare LO $\vert\cdot$]	0.034	0.053	0.034	0.018		0.029	0.069	0.029	0.015	
	E[Welfare MO $\vert\cdot$]	0	0	0	0.337		0	0	0	0.339	
	E[Welfare $\vert\cdot$]	0.034	0.053	0.034	0.355		0.029	0.069	0.029	0.354	
	LOA_2	0	0.500	0.110	0.063	0.091	0	0.500	1.000	0.063	0.150
	LOA_1	0	0	0.890	0.374	0.358	0	0	0	0.397	0.318
	LOB_1	0.890	0	0	0.374	0.358	0	0	0	0.397	0.318
	LOB_2	0.110	0.500	0	0.063	0.091	1.000	0.500	0	0.063	0.150
	MOA_2	0	0	0	0.064	0.051	0	0	0	0.040	0.032
	MOA_1	0	0	0	0	0	0	0	0	0	0
	MOB_1	0	0	0	0	0	0	0	0	0	0
	MOB_2	0	0	0	0.064	0.051	0	0	0	0.040	0.032
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.2$											
	E[Spread $\vert\cdot$]	0.211	0.300	0.211	0.225	0.228	0.300	0.300	0.300	0.221	0.236
	E[Depth A_2+A_1 $\vert\cdot$]	1.000	1.500	2.000	1.436	1.449	1.000	1.500	2.000	1.460	1.468
	E[Depth A_1 $\vert\cdot$]	0	0	0.890	0.374	0.358	0	0	0	0.397	0.318
	E[Depth B_1 $\vert\cdot$]	0.890	0	0	0.374	0.358	0	0	0	0.397	0.318
	E[Depth B_1+B_2 $\vert\cdot$]	2.000	1.500	1.000	1.436	1.449	2.000	1.500	1.000	1.460	1.468
	E[Welfare LO $\vert\cdot$]	0.273	0.146	0.273	0.316		0.081	0.150	0.081	0.360	
	E[Welfare MO $\vert\cdot$]	0	0	0	0.099		0	0	0	0.064	
	E[Welfare $\vert\cdot$]	0.273	0.146	0.273	0.415		0.081	0.150	0.081	0.424	

In addition, we report the probability-weighted contributions to the different investors' gains-from-trade coming from limit orders, market orders, and their total expected gains-from-trade. Table B1 in the Numerical Appendix provides additional results about conditional and unconditional future execution probabilities for the different orders ($P^{EX}(x_{t_1})$) and also the uninformed investor's updated expected asset value $E[v|x_{t_1}]$ given different types of buy orders x_{t_1} at time t_1. Expectations given sell orders are symmetric on the other side of $E[v] = 1$.

Table 2 shows average results for times t_2 through t_4 during the day using a similar 2×2 format. The averages are across time and trading histories. Comparing results for time t_1 with the trading averages for t_2 through t_4 shows intraday changes in properties of the trading process. There is no table for time t_5, because only market orders are used at the market close.

Result 1 Changes in adverse selection due to the value-shock size δ affect trading strategies differently than changes in the informed-investor arrival probability α.

One aspect of this result is about how different forms of adverse selection affect investors' liquidity taking decisions. This can been seen theoretically from first principles. Suppose the informed-investor arrival probability α is close to zero. If the value-shock volatility δ is close to zero, then directionally informed investors $I_{\bar{v}}$ and $I_{\underline{v}}$ with good or bad news never use market orders, since the final asset value v is always between the inside bid and ask prices. However, if δ is increased sufficiently, then at some point investors with good and bad news will start to use market orders given the guaranteed execution of market orders. Thus, the set of orders used by directionally informed investors can change in these small α scenarios when δ changes. In contrast, consider instead a market in which δ is close to zero. Now informed investors with good or bad news never use market orders regardless of how large or small the informed-investor arrival probability α is. Thus, the set of orders used by directionally informed investors never changes to include market orders in these small δ scenarios when α changes.

Table 2: Averages for Trading Strategies, Liquidity, and Welfare across Times t_2 through t_4 for Informed Traders with $\beta = 1$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and for two different asset-value volatilities δ (0.16 and 0.02). The private-value factor parameters are $\mu = 1$ and $\sigma = 1.5$, and the tick size is $\kappa = 0.10$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market at time t_1 as well as the welfare expectation of market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 0.16$					$\delta = 0.02$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
$\alpha = 0.8$	LOA_2	0	0.191	0.051	0.157	0.096	0.399	0.255	0.108	0.026	0.209
	LOA_1	0	0.258	0.257	0.023	0.142	0.192	0.239	0.288	0.064	0.205
	LOB_1	0.257	0.258	0	0.023	0.142	0.288	0.239	0.192	0.064	0.205
	LOB_2	0.051	0.191	0	0.157	0.096	0.108	0.255	0.399	0.026	0.209
	MOA_2	0.493	0	0	0.286	0.189	0	0	0	0.347	0.069
	MOA_1	0.001	0	0	0.031	0.006	0	0	0	0.058	0.012
	MOB_1	0	0	0.001	0.031	0.006	0	0	0	0.058	0.012
	MOB_2	0	0	0.493	0.286	0.189	0	0	0	0.347	0.069
	NT	0.198	0.061	0.198	0.007	0.124	0.013	0.010	0.013	0.011	0.012
	E[Spread $\vert\cdot$]	0.217	0.212	0.217	0.251	0.223	0.227	0.228	0.227	0.278	0.237
	E[Depth A_2+A_1 $\vert\cdot$]	1.047	2.276	2.480	1.755	1.899	2.165	2.300	2.433	1.608	2.161
	E[Depth A_1 $\vert\cdot$]	0	0.438	0.829	0.243	0.387	0.226	0.362	0.506	0.131	0.318
	E[Depth B_1 $\vert\cdot$]	0.829	0.438	0	0.243	0.387	0.506	0.362	0.226	0.131	0.318
	E[Depth B_1+B_2 $\vert\cdot$]	2.480	2.276	1.047	1.755	1.899	2.433	2.300	2.165	1.608	2.161
	E[Welfare LO $\vert\cdot$]	0.010	0.020	0.010	0.106		0.014	0.013	0.014	0.005	
	E[Welfare MO $\vert\cdot$]	0.009	0	0.009	0.298		0	0	0	0.354	
	E[Welfare $\vert\cdot$]	0.019	0.020	0.019	0.405		0.014	0.013	0.014	0.359	
$\alpha = 0.2$	LOA_2	0	0.358	0.508	0.102	0.139	0.375	0.389	0.443	0.093	0.155
	LOA_1	0	0.122	0.258	0.056	0.070	0.044	0.096	0.116	0.066	0.070
	LOB_1	0.258	0.122	0	0.056	0.070	0.116	0.096	0.044	0.066	0.070
	LOB_2	0.508	0.358	0	0.102	0.139	0.443	0.389	0.375	0.093	0.155
	MOA_2	0.130	0	0	0.219	0.184	0	0	0	0.218	0.175
	MOA_1	0.088	0	0	0.119	0.101	0	0	0	0.120	0.096
	MOB_1	0	0	0.088	0.119	0.101	0	0	0	0.120	0.096
	MOB_2	0	0	0.130	0.219	0.184	0	0	0	0.218	0.175
	NT	0.016	0.035	0.016	0.006	0.010	0.022	0.030	0.022	0.005	0.009
	E[Spread $\vert\cdot$]	0.205	0.190	0.205	0.280	0.264	0.221	0.217	0.221	0.300	0.284
	E[Depth A_2+A_1 $\vert\cdot$]	1.305	2.089	2.512	1.583	1.660	1.932	2.091	2.257	1.576	1.680
	E[Depth A_1 $\vert\cdot$]	0.194	0.451	0.740	0.301	0.333	0.346	0.414	0.442	0.262	0.290
	E[Depth B_1 $\vert\cdot$]	0.740	0.451	0.194	0.301	0.333	0.442	0.414	0.346	0.262	0.290
	E[Depth B_1+B_2 $\vert\cdot$]	2.512	2.089	1.305	1.583	1.660	2.257	2.091	1.932	1.576	1.680
	E[Welfare LO $\vert\cdot$]	0.119	0.086	0.119	0.052		0.060	0.064	0.060	0.050	
	E[Welfare MO $\vert\cdot$]	0.018	0	0.018	0.343		0	0	0	0.342	
	E[Welfare $\vert\cdot$]	0.137	0.086	0.137	0.394		0.060	0.064	0.060	0.392	

Our numerical analysis demonstrates this first result and also other facets of how adverse selection affects investor trading strategies. Consider again the directionally informed investors $I_{\bar{v}}$ and $I_{\underline{v}}$ with good or bad news. First, hold the informed-investor arrival probability α fixed and increase the amount of adverse selection by increasing the value-shock volatility δ. In a low-volatility market in which value shocks Δ are small relative to the tick size, informed traders with good and bad news are unwilling to pay a large tick size to trade on their information and instead act as liquidity providers who supply liquidity asymmetrically depending on the direction of their information. This can be seen in Table 1 where in both of the two parameter cells on the right (with $\alpha = 0.8$ and 0.2 and a small $\delta = 0.02$) informed investors $I_{\bar{v}}$ and $I_{\underline{v}}$ at time t_1 use limit orders at the outside quotes A_2 and B_2 exclusively. In contrast, in a high-volatility market where value shocks are large relative to the tick size, informed investors with good or bad news trade more aggressively. This can be seen in the left two parameterization cells (with $\alpha = 0.8$ and 0.2 and a large $\delta = 0.16$) where informed investors $I_{\bar{v}}$ and $I_{\underline{v}}$ use limit orders at both the inside quotes A_1 and B_1 as well at the outside quotes with positive probability at time t_1.

Now compare this to the effect of a change in the amount of adverse selection due to a change in the informed-investor arrival probability α while holding the value-shock size δ fixed. In this case, changing the amount of adverse selection does not affect which orders informed investors with good and bad news use at time t_1. This can be seen by comparing the lower two parameter cells (with $\delta = 0.02$ and 0.16 and a small α) with the upper two parameter cells (with the same δs and a larger α).

The average order-submission probabilities at times t_2 through t_4 in Table 2 are qualitatively similar to those for time t_1. When δ is small, informed investors with good and bad news tend to supply liquidity via limit orders following strategies that are somewhat asymmetric on the two sides of the market given the direction of their small amount of private information $I_{\bar{v}}$ and $I_{\underline{v}}$.

In contrast, when the value-shock volatility δ is larger in a high-volatility market, informed investors with good or bad news at times t_2 to t_4 switch from providing liquidity on both sides of the market to using a mix of taking liquidity via market orders and supplying liquidity via limit orders on the same side of the market as their information. Thus, once again, the trading strategies for informed investors $I_{\bar{v}}$ and $I_{\underline{v}}$ are qualitatively similar holding δ fixed and changing α, but their trading strategies change qualitatively when α is held fixed and δ is changed.

Next, consider informed investors I_0 who know that the value shock Δ is 0 and, thus, that the unconditional prior v_0 is correct. Tables 1 and 2 show that their liquidity provision trading strategies are qualitatively the same at time t_1 and on average over times t_2 through t_4. In constrast, uninformed investors U become less willing to provide liquidity via limit orders at the inside quotes as the adverse selection problem they face using limit orders worsens. Rather, they increasingly take liquidity via market orders or supply liquidity by less aggressive limit orders at the outside quotes. The reduction in liquidity provision at the inside quotes by uninformed investors is true at time t_1 (Table 1) and at times t_2 through t_4 (Table 2) both when the value shocks become larger and when the arrival probability of informed investors increases.

In this context, there are two noteworthy equilibrium effects. First, while the uninformed U investors reduce their liquidity provision at the inside quotes as adverse selection increases, the I_0 informed investors increase their liquidity provision at the inside quotes. This is because I_0 informed investors have an advantage in liquidity provision over the uninformed U investors in that there is no adverse selection risk for them. These results are qualitatively consistent with the intuition of Bloomfield, O'Hara and Saar (BOS, 2005). Informed traders are more likely to use limit orders than market orders when the value-shock volatility is low (and, thus, the profitability from trading on information asymmetries is low), and to use market orders when the reverse is true.

Second, uninformed U investors are unwilling to use aggressive limit orders at the inside quotes when the adverse selection risk is sufficiently high as in the upper left parametrization ($\alpha = 0.8$ and $\delta = 0.16$). This explains the fact that informed investors $I_{\bar{v}}$ and $I_{\underline{v}}$ use aggressive limit orders at the inside quotes with a higher probability in the lower left parametrization ($\alpha = 0.2$ and $\delta = 0.16$) than in the upper left parameterization. At first glance this might seems odd since competition from future informed investors (and the possibility of being undercut by later limit orders) is greater when the informed-investor arrival probability is large ($\alpha = 0.8$) than when α is smaller. However, in equilibrium there is camouflage from the uninformed U investors limit orders at the inside quotes in the lower left parametization, whereas limit orders at the inside quotes are fully revealing in the upper right parametrization.

2.1.2 Market quality

Market liquidity changes when the amount of adverse selection in a market changes. The standard intuition, as in Kyle (1985), is that liquidity deteriorates given more adverse selection. For example, Roşu (2016b) also finds worse liquidity (a wider bid-ask spread) given higher value volatility. However, we find that the standard intuition is not always true.

Result 2 Liquidity need not always deteriorate when adverse selection increases.

Markets can become more liquid given greater value-shock volatility if, given the tick size, high volatility makes the value shock Δ large relative the price grid. In addition, different measures of market liquidity — expected spreads, inside depth, and total depth — can respond differently to changes in adverse selection.

The impact of adverse selection on market liquidity follows directly from the trading strategy effects discussed above. Two intuitions are useful in understanding our market liquidity results. First, different investors affect liquidity differently. Informed traders with neutral news (I_{v_0}) are natural liquidity providers. Thus, their impact on liquidity comes from whether they supply liquidity at the inside (A_1 and B_1) or outside (A_2 and B_2) prices. In contrast, informed traders with directional news ($I_{\bar{v}}$ and $I_{\underline{v}}$) and uninformed traders (U) can have a large impact on liquidity depending on whether they opportunistically take or supply liquidity. Second, the most aggressive way to trade (both on directional information and private values) is via market orders, which takes liquidity. However, the next most aggressive way to trade is via limit orders at the inside prices. Thus, changes in market conditions (i.e., δ and α) that make investors trade more aggressively (i.e., that reduce their use of limit orders at the outside prices A_2 and B_2) can potentially improve liquidity if this stronger trading interest migrates to aggressive limit orders at the inside quotes, A_1 and B_1, rather than to market orders.

Our analysis shows that the standard intuition that adverse selection reduces market liquidity depends on the relative magnitudes of asset value shocks and the tick size. In Table 1, the expected spread narrows at time t_1 (markets become more liquid) when the value-shock volatility δ increases (comparing parameterizations horizontally so that α is kept fixed). Liquidity improves in these cases because the informed traders $I_{\overline{v}}$ and $I_{\underline{v}}$ submit limit orders at the inside quotes in these high-volatility markets, whereas they only use limit orders at the outside quotes in low-volatility markets. In constrast, the expected spread at time t_1 widens when the informed-investor arrival probability α increases holding the value-shock size δ constant, as predicted by the standard intuition. The evidence against the standard intuition is even stronger in Table 2. At times t_2 through t_4, the expected spread narrows both when information becomes more volatile (δ is larger) and when there are more informed traders (when α is larger). The qualitative results for the expected depth at the inside quotes goes in the same direction as the results for the expected spread. This is because both results are driven by limit-order submissions at the inside quotes. The results for adverse selection and total depth at both the inside and outside quotes are mixed. For example, Table 1 shows that total depth at time t_1 increases when value-shock volatility δ increases when the informed-investor arrival probability α is high (comparing horizontally the two parametrizations on the top), but decreases in δ when the informed α is low. In contrast, average total depth at times t_2 through t_4 in Table 2 is decreasing in the value-shock volatility (comparing parameterizations horizontally). This is opposite the effect on the inside depth. Thus, these different liquidity results are mixed.

The main result in this section is that the relation between adverse selection and market liquidity is more subtle than the standard intuition. Increased adverse selection can improve liquidity. The ability of investors to choose endogenously whether to supply or demand liquidity and at what limit prices is what can overturn the standard intuition. The results from this specification are comparable with Goettler et al. (2009). Goettler et al. (2009) have endogenous information acquisition and therefore they have no regimes with both informed and uninformed traders having an intrinsic motive to trade. However, they have a regime with informed traders having no private-value trading motive and uninformed having only private-value motives. In this regime, when volatility increases, informed traders reduce their provision of liquidity and increase their demand of liquidity; with the opposite holding for uninformed traders. Our results are more nuanced.

Increased value-shock volatility is associated with increased liquidity supply in some cases and with decreased liquidity in others. This is because the tick size of the price grid constrains the prices at which liquidity can be supplied and demanded.

2.1.3 Information content of orders

Traders in real-world markets and empirical researchers are interested in the information content of different types of arriving orders.[11] A necessary condition for an order to be informative is that informed investors use it. However, the magnitude of order informativeness is determined by the mix of equilibrium probabilities with which both informed and uninformed traders use an order. If uninformed traders use the same orders as informed investors, they add noise to the overall price discovery process, and orders become less informative. In our model, the mix of information-based and noise-based orders depends on the underlying proportion of informed investors α and and the value-shock volatility δ.

We expect different market and limit orders to have different information content. A natural conjecture is that the sign of the information revision associated with an order should agree with the direction of the order (e.g, buy market and limit orders should lead to positive valuation revisions). Another natural conjecture is that the magnitude of information revisions should be greater for more aggressive orders. However, while the sign conjecture is true in our first model specification, the order aggressiveness conjecture does not alway hold here.

Result 3 Order informativeness is not always increasing in the aggressiveness of an order.

This, at-first-glance surprising, result is another consequence of the impact of the tick size on how informed investors trade on their information. As a result, the relative informativeness of different market and limit orders can flip in high-volatility and low-volatility markets given a fixed tick size. The result is immediate for market orders versus (less aggressive) limit orders in low-volatility markets in which informed investors avoid market orders (see Table 1). However, we show here

[11] Fleming et al. (2017) extend the VAR estimation approach of Hasbrouck (1991) to estimate the price impacts of limit orders as well as market orders. See also Brogaard et al. (2016).

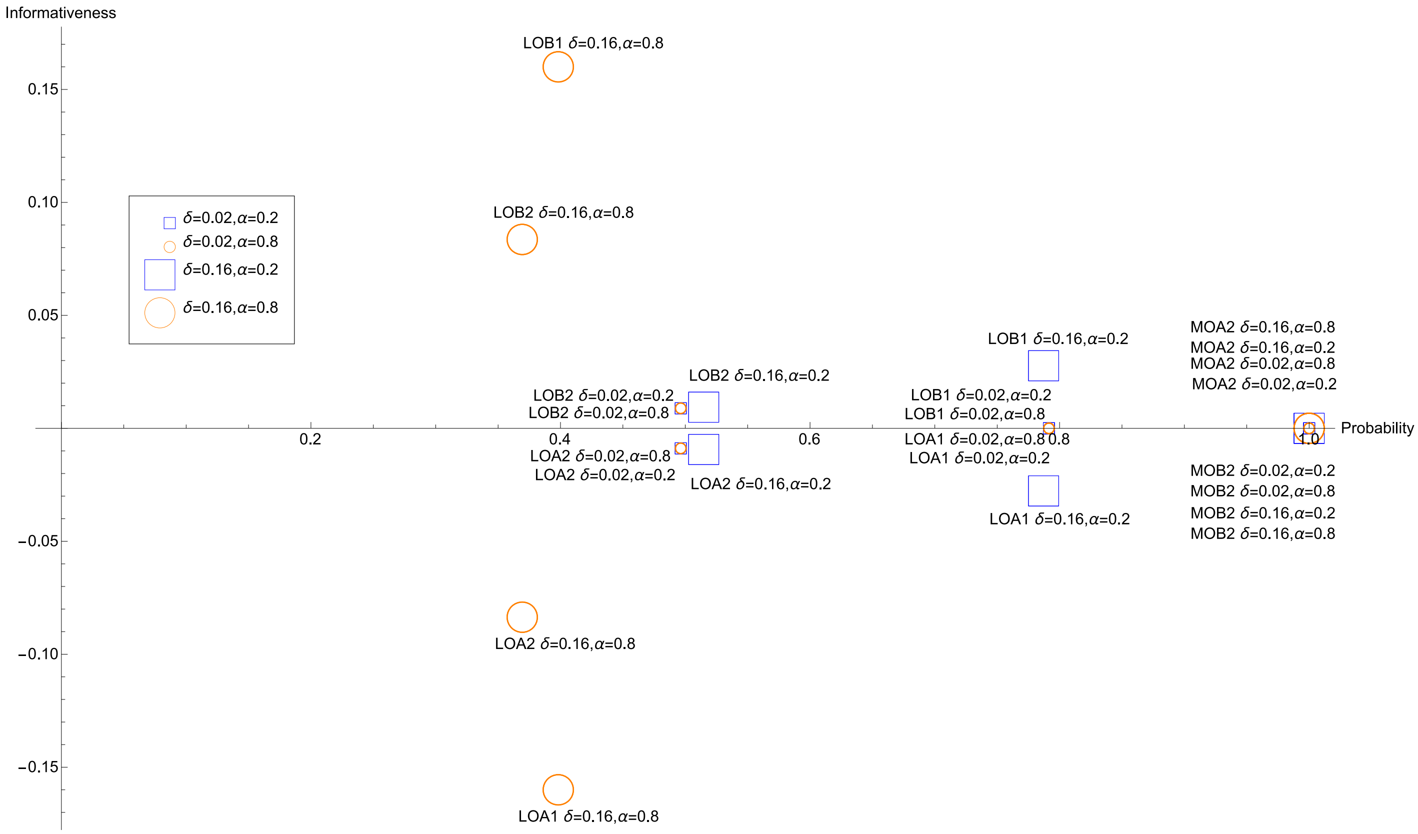
that it can also hold for aggressive limit orders at the inside quotes A_1 and B_1 versus less aggressive limit orders at the outside quotes A_2 and B_2.

Figure 4 shows the informativeness of different types of orders at time t_1. Informativeness at time t_1 is measured here as the Bayesian revision $E[v|\, x_{t_1}] - E[v]$ in the uninformed investor's expectation of the terminal value v after observing different types of orders x_{t_1} at time t_1. The informational revisions for the different orders are plotted against the respective order-execution probabilities on the horizontal axis. Orders with higher execution probabilities are statistically more aggressive than orders with low execution probabilities. The results for the four parameterizations are indicated using different symbols: high vs low informed-investor arrival probabilities (circles vs squares), and high vs low value-shock volatility (large vs small symbols). These are described in the figure legend. For example, in the low α and high δ scenario (large squares), the informativeness of a limit buy order at B_1 at time t_1 is 0.026 and the order-execution probability is 78.9 percent (see Table B1 in the Numerical Appendix).

Consider first the cases with high informed-investor arrival probabilities. The case with a high informed-investor arrival probability and high value-shock volatility is denoted with large circles. Informed investors in this case use limit orders at both the outside quotes (LOA_2 and LOB_2) and inside quotes (LOA_1 and LOB_1) at time t_1, so these are therefore the only informative orders. Since uninformed investors also use the outside limit orders, they are not fully revealing, however the inside limit orders are fully revealing. Thus, the price impacts for the inside and outside limit orders here are consistent with the order aggressiveness conjecture. The market orders (MOB_2 and MOA_2) are also used in equilibrium, but only by uninformed investors (U). Thus, they are not informative. While market orders would be profitable for the informed investors, the potential price improvement with the limit orders leads informed investors to use the limit orders despite the zero price impact and guaranteed execution probability of the market orders. Since both inside and outside limit orders have larger price impacts than the market orders, this case is inconsistent with the aggressiveness conjecture.

Next, consider the case of low value-shock volatility and high informed-investor arrival probability, denoted here with small circles. Once again, the order-aggressiveness conjecture is not

Figure 4: Informativeness of Orders after Trading at Time t_1 for the Model with Informed Traders with $\beta = 1$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This figure plots the *Informativeness* of the equilibrium orders at the end of t_1 against the probability of order execution. Four different combinations of informed-investor arrival probabilities and value-shock volatilities are considered. The informativeness of an order is measured as $E[v|x_{t_1}] - E[v]$, where x_{t_1} denotes one of the different possible orders that can arrive at time t_1.



true. The most informative orders are now, not the most aggressive orders, but rather the most patient limit orders posted at A_2 and B_2 (since these are the only orders used by informed investors). The market orders and more aggressive inside limit orders are non-informative here (since only uninformed investors with extreme βs use them). In this case, this — again at first glance perhaps counterintuitive — result is a consequence of the fact that the tick size is large relative to the informed trader's potential information. Low-volatility makes market orders unprofitable for informed traders given good and bad news, and it increases the price improvement attainable through limit orders deeper in the book relative to limit orders at the inside quotes.

Similar results hold when the proportion of insiders is low ($\alpha = 0.2$). When the asset-value volatility is high (large squares), the most aggressive orders (LOB_1 and LOA_1) are again the most informative ones in contrast to the market orders. However, when volatility is low (small squares), the most informative orders, as before, are the least aggressive orders (LOB_2 and LOA_2). Therefore, the potential failures of the order-aggressiveness conjecture are robust to variation in informed-investor arrival probabilities and value-shock volatility.

2.1.4 Non-Markovian learning

This section investigates the role of the order history on Bayesian learning. The first question we consider is whether the prior order history has information about the asset value v in excess of the information in the current limit order book.

The candlestick plots in Figure 5 measure the incremental information content of order histories as the difference $E[v|\,\mathscr{L}_{t_j}(L_{t_j})] - E[v|\,L_{t_j}]$, which is the uninformed investors' expected asset value conditional on an order history path $\mathscr{L}_{t_j}(L_{t_j})$ ending with a particular limit order book L_{t_j} at time t_j net of the corresponding expectation conditional on just the ending book L_{t_j}. In particular, we are interested in books L_{t_j} that are preceded in equilibrium by more than one different prior history. If learning is Markov, then order histories $\mathscr{L}_{t_j}(L_{t_j})$ preceding a book L_{t_j} should convey no additional information beyond L_{t_j}; in which case the difference in expectations should be zero. The candlestick plots show the maximum and minimum values, the interquartile range, and the median of the incremental information of the prior history. The horizontal axis in the plots shows the times t_1 through t_4 at which different orders x_{t_j} are submitted. Time t_1 is included in the plot because books at t_1 can potentially be produced by different sequences of investor actions x_{t_j} and resulting crowd responses. Each plot is for a different combination of adverse-selection parameters.

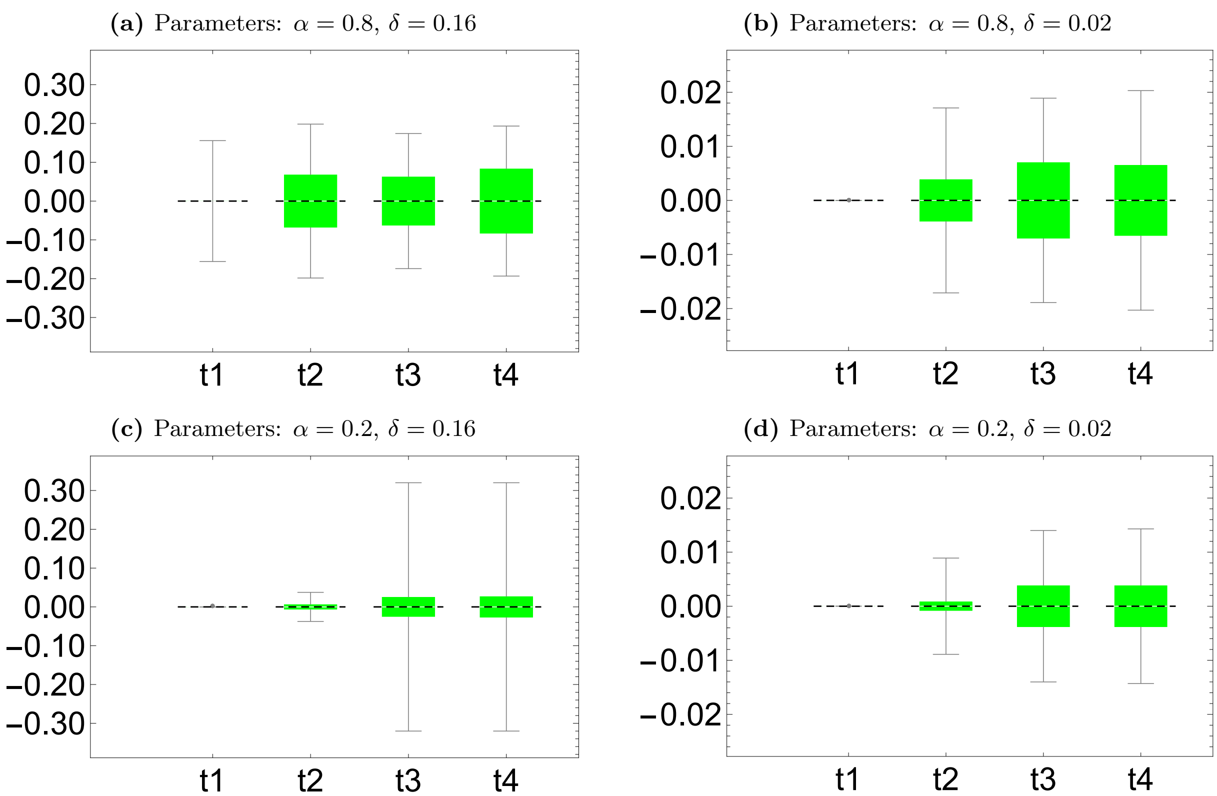
The main result from Figure 5 is that there is substantial informational variation in the Bayesian revisions conditional on different trading histories. Thus, we have

Result 4 The price discovery dynamics can be significantly non-Markovian.

As expected, the variation in the incremental information content of the prior trading history in Figure 5 is greater when the shock volatility δ is greater (note differences in vertical scales).

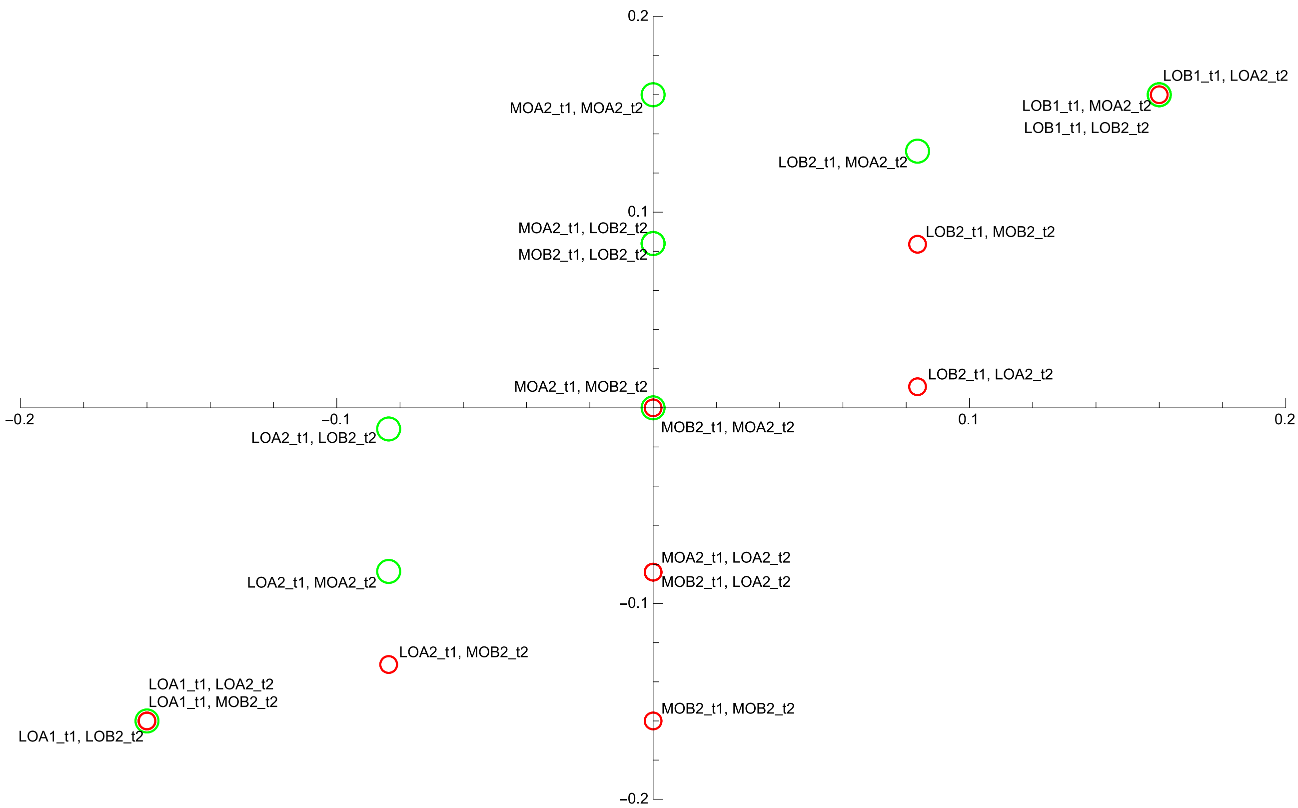
Given that learning is non-Markov, our next question is about how the size of the valuation

Figure 5: Informativeness of the Order History for the Model with Informed Traders with $\beta = 1$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for Times t_1 through t_4. This Figure shows the incremental information content of the past order history in excess of the information in the current limit order book observed at the end of time t_j as measured by $E[v|\mathscr{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ where $\mathscr{L}_{t_j}(L_{t_j})$ is a history ending in the limit order book L_{t_j}. We only consider books L_{t_j} when they occur in equilibrium in the different trading periods. The candlesticks indicate for each of these two metrics the maximum, the minimum, the median and the 75^{th} (and 25^{th}) percentile respectively as the top (bottom) of the bar.



revisions depends on the prior trading history. In Figure 6, the horizontal axis shows the price impact of different equilibrium orders at t_1, and the vertical axis gives the corresponding cumulative price impact of the sequence of a given action at time t_1 and different subsequent equilibrium actions at time t_2. Consistently with our previous analysis, the size of the valuation revision crucially depends on the insiders' equilibrium strategies. As informed investors do not use market orders at t_1 (see Table 1), market orders do not have a price impact at t_1 which is also the reason why the price impact of any order at t_2 conditional on a market order at t_1 lays on the vertical axis.

Figure 6: Order Informativeness for the Model with Informed Traders with $\beta = 1$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for times t_1 to t_2 and parameters $\alpha = 0.8$, $\delta = 0.16$. The horizontal axis reports $E(v|x_{t_1}) - E(v)$ which shows how the uninformed traders' Bayesian value-forecast changes with respect to the unconditional expected value of the fundamental when uninformed traders observe at t_1 an equilibrium order x_{t_1}. The vertical axis reports $E(v|x_{t_2}, x_{t_1}) - E(v)$ which shows how the uninformed traders' Bayesian value-forecast changes with respect to the unconditional expected value of the fundamental when uninformed traders observe at x_{t_2} at t_2. We consider all the equilibrium strategies at t_1 and t_2 which are symmetrical. Red (green) circles show equilibrium sell (buy) orders at t_2.



Interestingly, there are no observations in the second and fourth quadrants in our model, which means there are no sign reversals in the direction of the cumulative price impacts. The first and third quadrants (which are perfectly symmetrical) show the duplets of orders which have a positive and a negative price impact, respectively. The duplets with the highest price impact are driven by the insiders' equilibrium strategies at t_1 and are limit orders at the inside quotes followed any other order. In fact, Table 1 shows that insiders' limit orders at the inside quotes at t_1 are fully revealing. So once more the price impact does not depend on the aggressiveness of the orders but on the informed investors' orders choice. Overall, Figure 6 also confirms that the price impact is non-Markovian: for example the price impact of MOB_2 at t_2 may be either positive or negative depending on whether it is preceded by LOB_2 or LOA_2 at t_1.

2.1.5 Summary

The analysis of our first model specification has identified a number of empirically testable predictions. First, liquidity and the relative information content of different orders differ in high-volatility markets in which value shocks are large relative to the tick size vs. in low-volatility markets in which value shocks are small relative to the tick size. Second, the price impact of order flow should vary conditional on different trading histories and the current book at the time new orders are submitted.

2.2 Informed and uninformed traders both have private-value motives

Our second model specification generalizes our earlier analysis so that now informed investors also have random private-valuation factors β with the same truncated Normal distribution $\beta \sim Tr[\mathscr{N}(\mu, \sigma^2)]$ as the uninformed investors. Hence, informed traders not only speculate on their information, but they also have a private-value motive to trade. As a result, informed investors with the same signal may end up buying and selling from each other. We use this second model specification to show the robustness of the results in Section 2.1 and to extend them.

2.2.1 Trading strategies

Tables 3 and 4 report numerical results for our second model specification for time t_1 by itself and for averages over times t_2 through t_4. Since all investors have private-value motives to trade, we see that now all investors use all of the possible limit orders at time t_1 and that directionally informed and uninformed investors also use market orders. Over times t_2 through t_4, all investors again use all types of limit orders and also market orders. In particular, directionally informed investors trade sometimes opposite their asset-value information because their private-value motive adds non-informational randomness to their orders. Informed investor with neutral news I_{v_0} no longer

Table 3: Trading Strategies, Liquidity, and Welfare at Time t_1 in an Equilibrium with Informed and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu,\, \sigma^2)]$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and two different value-shock volatilities δ (0.16 and 0.02). The private-value factor parameters are $\mu = 1$ and $\sigma = 1.5$, and the tick size is $\kappa = 0.10$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market at time t_1 as well as the welfare expectation of market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column (*Uncond.*) reports unconditional results for the market.

		$\delta = 0.16$					$\delta = 0.02$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*
	LOA_2	0.118	0.054	0.031	0.064	0.067	0.054	0.048	0.042	0.048	0.048
	LOA_1	0.314	0.446	0.282	0.426	0.363	0.438	0.452	0.466	0.452	0.452
	LOB_1	0.282	0.446	0.314	0.426	0.363	0.466	0.452	0.438	0.452	0.452
	LOB_2	0.031	0.054	0.118	0.064	0.067	0.042	0.048	0.054	0.048	0.048
	MOA_2	0.256	0	0	0.009	0.070	0	0	0	0	0
	MOA_1	0	0	0	0	0	0	0	0	0	0
	MOB_1	0	0	0	0	0	0	0	0	0	0
	MOB_2	0	0	0.256	0.009	0.070	0	0	0	0	0
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.8$											
	E[Spread $\lvert\cdot$]	0.240	0.211	0.240	0.215	0.227	0.210	0.210	0.210	0.210	0.210
	E[Depth A_2+A_1 $\lvert\cdot$]	1.432	1.500	1.312	1.491	1.430	1.492	1.500	1.508	1.500	1.500
	E[Depth A_1 $\lvert\cdot$]	0.314	0.446	0.282	0.426	0.363	0.438	0.452	0.466	0.452	0.452
	E[Depth B_1 $\lvert\cdot$]	0.282	0.446	0.314	0.426	0.363	0.466	0.452	0.438	0.452	0.452
	E[Depth B_1+B_2 $\lvert\cdot$]	1.312	1.500	1.432	1.491	1.430	1.508	1.500	1.492	1.500	1.500
	E[Welfare LO $\lvert\cdot$]	0.259	0.445	0.259	0.410		0.446	0.446	0.446	0.446	
	E[Welfare MO $\lvert\cdot$]	0.187	0	0.187	0.015		0	0	0	0	
	E[Welfare $\lvert\cdot$]	0.446	0.445	0.446	0.425		0.446	0.446	0.446	0.446	
	LOA_2	0.063	0.051	0.042	0.051	0.051	0.049	0.048	0.046	0.048	0.048
	LOA_1	0.356	0.449	0.476	0.449	0.445	0.441	0.452	0.464	0.452	0.452
	LOB_1	0.476	0.449	0.356	0.449	0.445	0.464	0.452	0.441	0.452	0.452
	LOB_2	0.042	0.051	0.063	0.051	0.051	0.046	0.048	0.049	0.048	0.048
	MOA_2	0.063	0	0	0	0.004	0	0	0	0	0
	MOA_1	0	0	0	0	0	0	0	0	0	0
	MOB_1	0	0	0	0	0	0	0	0	0	0
	MOB_2	0	0	0.063	0	0.004	0	0	0	0	0
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.2$											
	E[Spread $\lvert\cdot$]	0.217	0.210	0.217	0.210	0.211	0.210	0.210	0.210	0.210	0.210
	E[Depth A_2+A_1 $\lvert\cdot$]	1.419	1.500	1.518	1.500	1.496	1.490	1.500	1.510	1.500	1.500
	E[Depth A_1 $\lvert\cdot$]	0.356	0.449	0.476	0.449	0.445	0.441	0.452	0.464	0.452	0.452
	E[Depth B_1 $\lvert\cdot$]	0.476	0.449	0.356	0.449	0.445	0.464	0.452	0.441	0.452	0.452
	E[Depth B_1+B_2 $\lvert\cdot$]	1.518	1.500	1.419	1.500	1.496	1.510	1.500	1.490	1.500	1.500
	E[Welfare LO $\lvert\cdot$]	0.394	0.445	0.394	0.442		0.447	0.446	0.447	0.446	
	E[Welfare MO $\lvert\cdot$]	0.059	0	0.059	0		0	0	0	0	
	E[Welfare $\lvert\cdot$]	0.453	0.445	0.453	0.442		0.447	0.446	0.447	0.446	

Table 4: Averages for Trading Strategies, Liquidity, and Welfare across Times t_2 through t_4 for Informed and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and for two different asset-value volatilities δ (0.16 and 0.02). The private-value factor parameters are $\mu = 1$ and $\sigma = 1.5$, and the tick size is $\kappa = 0.10$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market at time t_1 as well as the welfare expectation of market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column (*Uncond.*) reports unconditional results for the market.

		$\delta = 0.16$					$\delta = 0.02$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*
	LOA_2	0.140	0.121	0.090	0.114	0.117	0.127	0.123	0.119	0.123	0.123
	LOA_1	0.108	0.058	0.050	0.067	0.071	0.057	0.053	0.048	0.053	0.053
	LOB_1	0.050	0.058	0.108	0.067	0.071	0.048	0.053	0.057	0.053	0.053
	LOB_2	0.090	0.121	0.140	0.114	0.117	0.119	0.123	0.127	0.123	0.123
	MOA_2	0.275	0.192	0.113	0.195	0.194	0.207	0.194	0.181	0.194	0.194
	MOA_1	0.158	0.127	0.062	0.122	0.117	0.133	0.128	0.124	0.129	0.128
	MOB_1	0.062	0.127	0.158	0.122	0.117	0.124	0.128	0.133	0.129	0.128
	MOB_2	0.113	0.192	0.275	0.195	0.194	0.181	0.194	0.207	0.194	0.194
	NT	0.003	0.003	0.003	0.005	0.004	0.004	0.003	0.004	0.004	0.004
$\alpha = 0.8$											
	E[Spread $\|\cdot$]	0.253	0.259	0.253	0.274	0.259	0.268	0.269	0.268	0.269	0.268
	E[Depth A_2+A_1 $\|\cdot$]	1.599	1.600	1.537	1.563	1.576	1.590	1.593	1.596	1.593	1.593
	E[Depth A_1 $\|\cdot$]	0.301	0.339	0.338	0.314	0.324	0.324	0.333	0.344	0.333	0.334
	E[Depth B_1 $\|\cdot$]	0.338	0.339	0.301	0.314	0.324	0.344	0.333	0.324	0.333	0.334
	E[Depth B_1+B_2 $\|\cdot$]	1.537	1.600	1.599	1.563	1.576	1.596	1.593	1.590	1.593	1.593
	E[Welfare LO $\|\cdot$]	0.089	0.071	0.089	0.072		0.067	0.067	0.067	0.067	
	E[Welfare MO $\|\cdot$]	0.328	0.332	0.328	0.331		0.336	0.336	0.336	0.336	
	E[Welfare $\|\cdot$]	0.418	0.403	0.418	0.404		0.403	0.403	0.403	0.403	
	LOA_2	0.131	0.123	0.114	0.122	0.122	0.124	0.123	0.122	0.123	0.123
	LOA_1	0.059	0.054	0.049	0.053	0.054	0.053	0.053	0.052	0.053	0.053
	LOB_1	0.049	0.054	0.059	0.053	0.054	0.052	0.053	0.053	0.053	0.053
	LOB_2	0.114	0.123	0.131	0.122	0.122	0.122	0.123	0.124	0.123	0.123
	MOA_2	0.257	0.194	0.137	0.196	0.196	0.202	0.194	0.186	0.194	0.194
	MOA_1	0.160	0.127	0.090	0.127	0.127	0.133	0.128	0.124	0.128	0.128
	MOB_1	0.090	0.127	0.160	0.127	0.127	0.124	0.128	0.133	0.128	0.128
	MOB_2	0.137	0.194	0.257	0.196	0.196	0.186	0.194	0.202	0.194	0.194
	NT	0.004	0.003	0.004	0.004	0.004	0.004	0.003	0.004	0.004	0.004
$\alpha = 0.2$											
	E[Spread $\|\cdot$]	0.266	0.267	0.266	0.269	0.269	0.269	0.269	0.269	0.269	0.269
	E[Depth A_2+A_1 $\|\cdot$]	1.547	1.595	1.636	1.591	1.591	1.587	1.593	1.599	1.592	1.592
	E[Depth A_1 $\|\cdot$]	0.288	0.334	0.378	0.332	0.332	0.327	0.333	0.339	0.333	0.333
	E[Depth B_1 $\|\cdot$]	0.378	0.334	0.288	0.332	0.332	0.339	0.333	0.327	0.333	0.333
	E[Depth B_1+B_2 $\|\cdot$]	1.636	1.595	1.547	1.591	1.591	1.599	1.593	1.587	1.592	1.592
	E[Welfare LO $\|\cdot$]	0.068	0.068	0.068	0.067		0.067	0.067	0.067	0.067	
	E[Welfare MO $\|\cdot$]	0.348	0.334	0.348	0.335		0.336	0.336	0.336	0.336	
	E[Welfare $\|\cdot$]	0.416	0.403	0.416	0.402		0.403	0.403	0.403	0.403	

just provide liquidity using limit orders. Now, due to their private-value motive, they sometimes also take liquidity using market orders.

Consider next the impact of the amount of adverse selection on trading behavior. Tables 3 and 4 show for time t_1 and for trading averages over t_2 through t_4 respectively that the effects of an increase in value-shock volatility on the strategies of informed traders with good or bad news differs if we consider traders' own or opposite side of the market. In particular, the "own" side of the market for an informed investor with good news is the bid (buy) side of the limit order book. The effect on the informed trader's own-side behavior is similar to the previous model specification in Section 2.1. With higher value-shock volatility, the private information about the asset value is more valuable, and both $I_{\bar{v}}$ and $I_{\underline{v}}$ investors change some of their aggressive limit orders into market orders. Table 3 shows that, at time t_1 when $\alpha = 0.8$, the $I_{\bar{v}}$ investors reduce the strategy probability for LOB_1 orders from 0.466 to 0.282 and increase the strategy probability for MOA_2 orders from 0 to 0.256, and symmetrically $I_{\underline{v}}$ investors shifts from LOA_1 to MOB_2.

The effects of higher volatility on uninformed traders slightly differs if we consider t_1 as opposed to times t_2 through t_4. At t_1 uninformed traders post slightly more aggressive orders when they demand liquidity (the strategy probabilities for MOA_2 and MOB_2 increase from 0 to 0.009), and more patient orders when they supply liquidity (the strategy probabilities for LOB_2 and LOA_2 increase slightly from 0.048 to 0.064). This change in order-submission strategies is the consequence of uninformed traders now perceiving higher adverse selection costs. They feel safer hitting the trading crowd at A_2 and B_2 and offering liquidity at more profitable price levels to make up for the increased adverse selection costs. In later periods t_1 through t_4, as uninformed traders learn about the fundamental value of the asset, they still take liquidity at the outside quotes (the probabilities of MOA_2 and MOB_2 increase slightly to 0.195 in Table 4), but move to the inside quotes to supply liquidity (LOA_1 and LOB_1 increase to 0.067 for times t_2 through t_4). As they learn about the future value of the asset, uninformed traders perceive less adverse selection costs and can afford to offer liquidity at more aggressive quotes. In contrast, the effects of increased value-shock volatility on the trading behavior of I_{v_0} investors are relatively modest both at time t_1 and at times t_2 through t_4.

The effects of an increase in volatility on the opposite side is different than on the own side. For example, when asset-value volatility δ increases from 0.02 to 0.16, $I_{\bar{v}}$ investors at time t_1 switch on the own side from LOB_1 limit orders to aggressive MOA_2 market orders and at the same time they switch on the opposite side from aggressive limit orders to more patient limit orders. The reason why $I_{\bar{v}}$ investors with low private-values become more patient when selling via limit orders on the opposite side is that they know that the execution probability of limit sells at A_2 is higher because other $I_{\bar{v}}$ investors in future periods will hit limit sell orders at A_2 more aggressively given that $\bar{v}$ is much bigger (see the increased order submission probabilities for MOA_2 in Table 4).

2.2.2 Market quality

The effect of value-shock volatility on market liquidity is mixed in our second model specification. This is not surprising given the nuanced effect of increased volatility on investor trading behavior, particularly on informed trading behavior on the own and opposite sides of the market. At time t_1, holding the informed-investor arrival probability α fixed, increased value-shock volatility leads to wider spreads and less total depth. However, the average effects over times t_2 through t_4 is the opposite with increased asset-value volatility leading to narrower spreads and smaller depth. This is due — in particular in the high α framework — to uninformed traders perceiving greater adverse selction costs and therefore being less willing to supply liquidity. Interestingly, the effects of an increase in the proportion of informed investors (α) on the equilibrium strategies of market participants is qualitatively similar to that of an increase in volatility (δ) in this model.

Lastly, our model shows how an increase in volatility and in the proportion of insiders affect the welfare of market participants. When volatility increases, directional informed investors are generally better off as their signal is stronger and hence more profitable: At t_1 their welfare is unchanged with high proportion of insiders (0.446), whereas it increases in all the other scenarios, with low proportion of insiders (0.453) and in later periods with both high and low α (0.418 and 0.416). At t_1 uninformed traders are worse off because liquidity deteriorates with higher volatility. At later periods the result is ambiguous: there are cases in which the uninformed investors are better off and cases in which they are worse off.

2.2.3 Information content of orders

Figure 7 plots the Bayesian revisions for different orders at time t_1 against the corresponding order-execution probabilities for our second model specification. Once again, the magnitudes and signs of the Bayesian updates depends on the mix of informed and uninformed investors who submit these orders. Consider, for example, the market with both high value-shock volatility and a high informed-investor arrival probability (large circles). The most informative orders are the market orders MOA_2 and MOB_2 as they are chosen much more often by informed investors than by uninformed investors (See Table 3). However, the next most aggressive orders are the inside limit orders LOB_1 and LOA_1, and they are less informative than the LOB_2 and LOA_2 limit orders. Even though the aggressive limit orders LOB_1 and LOA_1 are posted with a relatively high probability (0.282 and 0.314) by informed investors $I_{\bar{v}}$ and $I_{\underline{v}}$, they are also submitted with a high probability by uninformed investors (0.426), and an even higher submission probability by I_{v_0} investors (0.446). As a result, they are less informative.[12] Thus, this is another example in which order informativeness is not increasing in order aggressiveness.

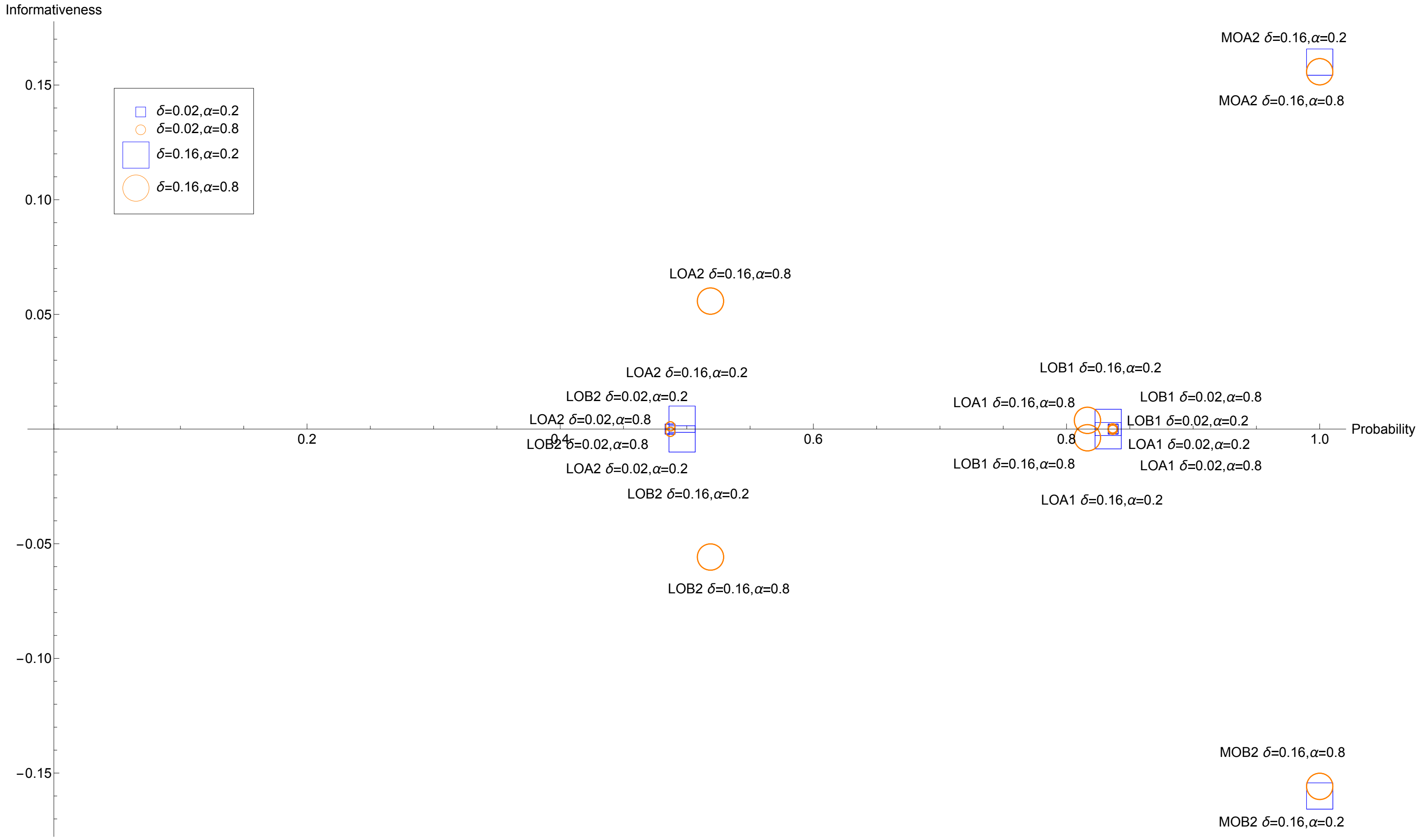
Perhaps more surprising, the order-sign conjecture need not hold in our second model specification:

Result 5 The Bayesian value expectation revision can be opposite the direction of an order.

This is to say that the direction of orders is sometimes different from the sign of their information content. For example, a limit sell LOA_1 is informative of good news (rather than bad news as one might expect) because limit sells at A_1 are used by informed investor to trade on the opposite side of their information (i.e., due to their private-value β factors) more frequently than these orders are used to trade on the same side of their information. In particular, $I_{\underline{v}}$ investors usually sell using market orders at MOB_2 rather than using limit sells. This goes back to our previous discussion of how informed investors trade differently on the own side of their information (when their private value β reinforces the trading direction from their information) and on the opposite side of their information (when their β reverses the trading incentive from their information).

[12]The informativeness of limit orders LOA_1 and LOB_1 in Table B2 in the Numerical Appendix are 0.004 and -0.004 respectively, whereas the informativeness of limit orders LOA_2 and LOB_2 are 0.056 and -0.056 respectively.

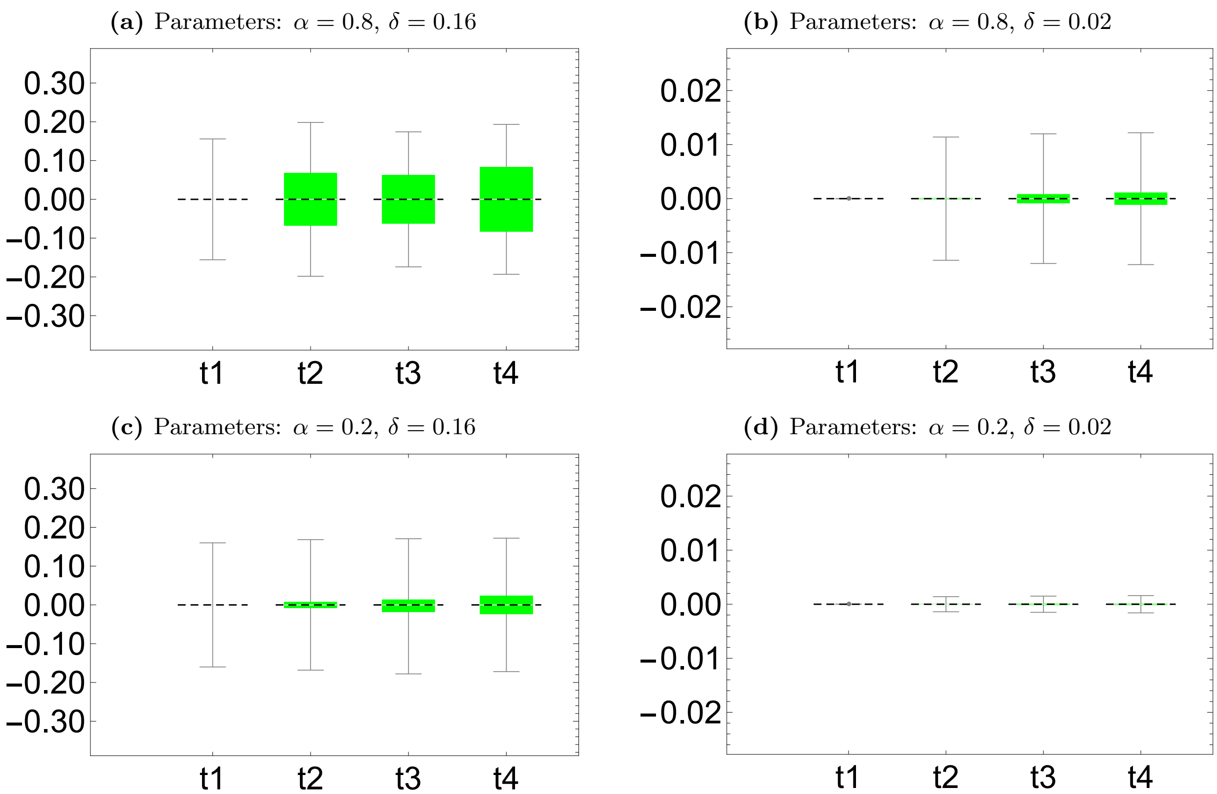
Figure 7: Informativeness of Orders at the End of t_1 for the Model with Informed and Uninformed Traders both with $\boldsymbol{\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]}$. This figure plots the *Informativeness* of the equilibrium orders at the end of t_1 against the probability of execution. We consider four different combinations of informed investors arrival probability. The informativeness of an order is measured as $E[v|x_{t_1}] - E[v]$, where x_{t_1}denotes one of the different possible orders that can arrive at time t_1.



2.2.4 Non-Markovian price discovery

This section continues our investigation of the importance of non-Markovian effects in information aggregation. Figure 8 shows once again the variation in the incremental information $E[v|\mathscr{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ of the prior order history $\mathscr{L}_{t_j}(L_{t_j})$. The plots here confirm our earlier results about non-Markovian learning.

Figure 8: History Informativeness for Informed and Uninformed Traders both with $\beta \sim Tr[\mathscr{N}(\mu, \sigma^2)]$ for times t_2 through t_4. This Figure shows the incremental information content of the past order history in excess of the information in the current limit order book observed at the end of time t_j as measured by $E[v|\mathscr{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ where $\mathscr{L}_{t_j}(L_{t_j})$ is a history ending in the limit order book L_{t_j}. We only consider books L_{t_j} when they occur in equilibrium in the different trading periods. The candlesticks indicate for each of these two metrics the maximum, the minimum, the median and the 75^{th} (and 25^{th}) percentile respectively as the top (bottom) of the bar.



Goettler et al. (2009) and Roşu (2016b) assume that information dynamics are Markovian and that the current limit order book is a sufficient statistic for the information content of the prior trading history. Figure 8 shows the uninformed investor's expectation of the asset value conditional on the path and various books. It also shows the expectation of these expectations across the paths, which, by iterated expectation, is the expectation conditional on the book. Again, we see that the trading history has substantial information content above and beyond the information in the book alone. The figure also shows the standard deviation of the valuation forecast errors. Here again, the results are non-Markovian.

2.3 Summary

The results for our second model specification — with a richer specification of the informed investors' trading motives — confirm and extent the analysis from our first model specification. The main findings are

- When all market participants trade not only to speculate on their signal but also to satisfy their private-value motive, all investors use both market and limit orders in equilibrium.
- Increased value-shock volatility and an increased informed-investor arrival probability can affect informed investor trading behavior differently when they trade with their information or (because of private-value shocks) against their information.
- The effect of asset-value volatility and informed investor arrival probability on market liquidity is mixed.
- The informativeness of an order can again be opposite the order direction and aggressiveness.
- The information content of order arrivals is history-dependent.
- Both order informativeness and the dispersion of believes increase with volatility and the proportion of insiders. With higher volatility the insiders' signal becomes stronger, whereas with a higher proportion of insiders uninformed traders have more opportunities to learn.

3 Robustness

Our analysis makes a number of simplifying assumptions for tractability, but we conjecture that our qualitative results are robust to relaxing these assumptions. We consider two of these assumptions here. First, our model of the trading day only has five periods. Relatedly, our analysis abstracts from limit orders being carried over from one day to the next. However, our results about the impact of adverse selection on investor trading strategies and about order informativeness are driven in large part by the relative size of information shocks and the tick size rather than by the number of rounds of trading. In addition, increasing the trading horizon just leads to longer histories that are potentially even more informative. Second, arriving investors are only allowed to submit single orders that cannot be cancelled or modified subsequently. However, it seems likely that order flow histories will still be informative if orders at different points in time are correlated due to correlated actions of returning investors.

4 Conclusions

This paper has studied the information aggregation and liquidity provision processes in dynamic limit order markets. We show a number of interesting theoretical properties in our model. First, informed investors switch between endogenously demanding liquidity via market orders and supplying liquidity via limit orders. Second, the information content/price impact of orders is non-monotone in the direction of the order and in the aggressiveness of their orders. Third, the information aggregation process is non-Markovian. In particular, the prior trading history has information content beyond that in the current limit order book. We also show that the price impact of orders depends on the prior trading history. In other words, a given order may have a very different price impact following one trading history and another.

Our model suggests several interesting directions for future research. First, the model can be enriched by allowing investors to trade dynamically over time (rather than just submitting an order one time). In addition, if traders had a quantity decision, they might want to use multiple orders. Second, the model could be extended to allow for trading in multiple co-existing limit order markets. This would be a realistic representation of current equity trading in the US. Third, the model could be used to study high frequency trading and the effect of different investors being able to process and trade on different types of information at different latencies.

5 Appendix A: Algorithm for computing equilibrium

The computational problem to solve for a Perfect Bayesian Nash equilibrium in our model is complex. Given investors' equilibrium beliefs, the optimal order-submission problems in (5) and (6) require computing limit-order execution probabilities and stock-value expectations conditional on both the past trading history and on future state-contingent limit-order execution at each time t_j at each node of the trading game. For an informed trader (who knows the future value of the asset), there is no uncertainty about the payoff of a market order. However, the payoff of a market order for an uninformed trader entails uncertainty about the future asset value and therefore computing the optimal order requires computing the expected stock value conditional on the prior trading history up to time t_j. For limit orders, the expected payoff depends on the future execution probability of that limit order, which depends, in turn, on the optimal order-submission probabilities for future informed and uninformed traders. In addition, the uninformed investors have a learning problem. They extract information about the expected future stock value from both the past trading history and also from state-contingent future order execution given that the future states in which limit orders are executed are correlated with the stock value. Thus, optimal actions at each date t depend on past and future actions where future actions also depend on the prior histories at future dates (which included the action at date t) as traders dynamically update their equilibrium beliefs as the trading process unfolds. In addition, rational expectations involves finding a fixed point so that the equilibrium beliefs underlying the optimal order-submission strategies are consistent with the execution probabilities and value expectations that those optimal strategies produce in equilibrium.

Our numerical algorithm uses backwards induction to solve for optimal order strategies given a set of asset-value beliefs for all dates and nodes in the trading game and an iterative recursion to solve for RE asset-value beliefs. The backwards induction makes order-execution probabilities consistent with optimal future behavior by later arriving investors. It also takes future state-contingent execution into account in the uninformed investors' learning problem. We start at time t_5 — when traders only use market orders which allows us to compute the execution probabilities of limit orders at t_4 — and recursively solve the model for optimal trading strategies back to time t_1. We then embed the optimal order strategy calculation in an iterative recursion to solve for a fixed point for the RE asset-value beliefs. In this recursion, the asset-value probabilities $\pi_t^{v,r-1}$ from round $r-1$ are used iteratively as the asset-value beliefs in round r. In particular, these beliefs are used in the learning problem of the uninformed investor to extract information about the ending stock value v from the prior trading histories. They also affect the behavior of informed investors whose order-execution probability beliefs depend in part on the behavior of uninformed traders. We iterate this recursion to find a RE fixed point in investor beliefs.

In a generic round r of our recursion, investors' asset-value beliefs are set to be the asset-value probabilities from the previous recursive round $r-1$. In particular, at each time t_j in each node of the trading process, the round $r-1$ probabilities are used as priors in computing traders' conditional payoffs in round r when computing expected order payoffs and optimal orders:

$$\max_{x \in X_{t_j}} \varphi^{I,r}(x \,|\, v, \mathscr{L}_{t_{j-1}}) \quad = \quad [\beta\, v_0 + \Delta - p(x)]\, Pr^{r-1}(\theta^x_{t_j} \,|\, v, \mathscr{L}_{t_{j-1}}) \tag{12}$$

and

$$\max_{x \in X_{t_j}} \varphi^{U,r}(x \,|\mathscr{L}_{t_{j-1}}) \quad = \quad [\beta\, v_0 + E^{r-1}[\Delta \,|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] - p(x)]\, Pr^{r-1}(\theta^x_{t_j} |\mathscr{L}_{t_{j-1}}) \tag{13}$$

where

$$E^{r-1}[\Delta|\mathscr{L}_{t_{j-1}}, \theta^x_{t_j}] = (\hat{\pi}^{\bar{v},\, r-1}_{t_j}\, \bar{v} + \hat{\pi}^{v_0,\, r-1}_{t_j}\, v_0 + \hat{\pi}^{\underline{v},\, r-1}_{t_j}\, \underline{v}) - v_0 \tag{14}$$

$$\hat{\pi}^{v,\, r-1}_{t_j} \quad = \quad \frac{\mathrm{Pr}^{r-1}(\theta^x_{t_j}|v, \mathscr{L}_{t_j})}{Pr^{r-1}(\theta^x_{t_j}|\mathscr{L}_{t_j})} \pi^{v,\, r-1}_{t_j} \tag{15}$$

The resulting order-submission strategies $x_{t_j,r}$ in round r are then used to to compute new asset-value asset value beliefs for the next recursive round $r+1$.

The recursion is started in round $r = 1$ by setting the beliefs of uninformed traders about the fundamental value of the asset at each time t_j in the backwards induction to be the unconditional priors given in (1). In particular, the algorithm starts by ignoring conditioning on history in the initial round $r = 1$. Hence traders' expected payoffs on an order x in round $r = 1$ simplify to:

$$\max_{x \in X_{t_j}} \varphi^U_{r=1}(x \,|\mathscr{L}_{t_{j-1}}) \quad = \quad [\beta\, v_0 + E[\Delta] - p(x)]\, Pr(\theta^x_{t_j}) \tag{16}$$

$$\max_{x \in X_{t_j}} \varphi^I_{r=1}(x \,|\, v, \mathscr{L}_{t_{j-1}}) \quad = \quad [\beta\, v_0 + \Delta - p(x)]\, Pr(\theta^x_{t_j} |\, v) \tag{17}$$

In each round r given the asset-value beliefs in that round, we solve for investors' optimal trading strategies by backward induction. Starting at t_5, the execution probability of new limit orders is zero, and therefore optimal order-submission strategies only use market orders. Given the linearity of the expected payoffs in the private-value factor β (equations (16) and (17)), the optimal trading strategies for an informed trader at t_5 are[13]

$$x_{t_5,I,r}(\beta|\mathscr{L}_{t_4}, v) = \begin{cases} MOB_{i,t_5} & if \;\; \beta \in [0, \beta^{MOB_{i,t_5},NT}_{t_5,I,r}) \\ NT & if \;\; \beta \in [\beta^{MOB_{i,t_5},NT}_{t_5,I,r},\, \beta^{NT,MOA_{i,t_5}}_{t_5,I,r}) \\ MOA_{i,t_5} & if \;\; \beta \in [\beta^{NT,MOA_{i,t_5}}_{t_5,I,r},\, 2] \end{cases} \tag{18}$$

where

$$\begin{aligned} \beta^{MOB_{i,t_5},NT}_{t_5,I,r} &= \frac{B_{i,t_5} - \Delta}{v} \\ \beta^{NT,MOA_{i,t_5}}_{t_5,I,r} &= \frac{A_{i,t_5} - \Delta}{v} \end{aligned} \tag{19}$$

are the critical thresholds that solve $\varphi_{t_5,r}(MOB_{i,t_5}) = \varphi_{t_5,r}(NT)$ and $\varphi_{t_5,r}(NT) = \varphi_{t_5,r}(MOA_{i,t_5})$, respectively. The optimal trading strategies and β thresholds for an uninformed traders are similar

[13]For instance, an informed trader would post a MOA_1 only if the payoff is positive and thus outperforms the NT payoff of zero, i.e, $\beta v + \Delta - A_1 > 0$ or $\beta > \frac{A_1 - \Delta}{v}$.

but the conditioning set does not include the signal on v:

$$x_{t_5,U,r}(\beta|\mathscr{L}_{t_4}) = \begin{cases} MOB_{i,t_5} & if\ \beta \in [0, \beta_{t_5,U,r}^{MOB_{i,t_5},NT}) \\ NT & if\ \beta \in [\beta_{t_5,U,r}^{MOB_{i,t_5},NT},\ \beta_{t_5,U,r}^{NT,MOA_{i,t_5}}) \\ MOA_{i,t_5} & if\ \beta \in [\beta_{t_5,U,r}^{NT,MOA_{i,t_5}},\ 2] \end{cases} \tag{20}$$

where

$$\begin{aligned} \beta_{t_5,U,r}^{MOB_{i,t_5},NT} &= \frac{B_{i,t_5} - E^{r-1}[\Delta|\mathscr{L}_{t_4}]}{v} \\ \beta_{t_5,U,r}^{NT,MOA_{i,t_5}} &= \frac{A_{i,t_5} - E^{r-1}[\Delta|\mathscr{L}_{t_4}]}{v} \end{aligned} \tag{21}$$

Once we know the β ranges associated with each strategy, we compute the submission probabilities associated with each optimal order at t_5 using the distribution of β. At time t_4 these probabilities are the execution probabilities for limit orders at the best bid and ask, B_{i,t_4} and A_{i,t_4} respectively at time t_5:

$$Pr_r(\theta^{LOB_{i,t_4}}|\mathscr{L}_{t_3}, v) = \begin{cases} \int_{\beta \in \left[0, \beta_{t_5,I,r}^{MOB_{i,t_4},NT}\right)} \mathfrak{n}(\beta)\, d\beta & \text{where } i \text{ indexes the best bid and if } q_{t_3}^{B_{i,t_4}} = 0 \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

$$Pr_r(\theta^{LOA_{i,t_4}}|\mathscr{L}_{t_3}, v) = \begin{cases} \int_{\beta \in \left[\beta_{t_5,I,r}^{NT,MOA_{i,t_4}}, 2\right]} \mathfrak{n}(\beta)\, d\beta & \text{where } i \text{ indexes the best ask and if } q_{t_3}^{A_{i,t_4}} = 0 \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

where $q_{t_3}^{B_{i,t_4}} = 0$ and $q_{t_3}^{A_{i,t_4}} = 0$ means that the incoming limit order book from time t_3 is empty at the best bid and ask at time t_4. The execution probabilities of uninformed at the best bid and the best ask:

$$Pr_r(\theta^{LOB_{i,t_4}}|\mathscr{L}_{t_3}) = \begin{cases} \int_{\beta \in \left[0, \beta_{t_5,U,r}^{MOB_{i,t_4},NT}\right)} \mathfrak{n}(\beta)\, d\beta & \text{where } i \text{ indexes the best bid and if } q_{t_3}^{B_{i,t_4}} = 0 \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

$$Pr_r(\theta^{LOA_{i,t_4}}|\mathscr{L}_{t_3}) = \begin{cases} \int_{\beta \in \left[\beta_{t_5,U,r}^{NT,MOA_{i,t_4}},2\right]} \mathfrak{n}(\beta)\, d\beta & \text{where } i \text{ indexes the best ask and if } q_{t_3}^{A_{i,t_4}} = 0 \\ 0 & otherwise \end{cases} \tag{25}$$

where $\mathfrak{n}(\cdot)$ is the truncated normal density function. At t_4 there is only one period before the end of the trading game. Thus, the execution probability of a limit order is positive if and only if the order is posted at the best price on its own side of the market (P_{i,t_j}), and if there are no limit orders already standing in the limit order book at that price at the time the limit order is posted ($q_{t_3}^{B_{i,t_4}} = 0$ and $q_{t_3}^{A_{i,t_4}} = 0$).

Having obtained the execution probabilities for limit orders at t_4, we next derive the optimal order-submission strategies at t_4. The book can open in many different ways at t_4 depending on the prior path of the trading game. As the payoffs of both limit and market orders are functions of β, we rank all the payoffs of adjacent optimal strategies in terms of β and equate them to determine the β thresholds at time t_4.[14]

Consider for example, a path of the game such that the book opens empty; so both limit and market orders are optimal strategies at t_4. For an informed trader, these strategies are:

$$x_{t_4,I,r}(\beta|\mathscr{L}_{t_3}, v) = \begin{cases} MOB_{2,t_4} & if\ \beta \in [0,\ \beta_{t_4,I,r}^{MOB_{2,t_4},LOA_{1,t_4}}) \\ LOA_{1,t_4} & if\ \beta \in [\beta_{t_4,I,r}^{MOB_{2,t_4},LOA_{1,t_4}},\ \beta_{t_4,I,r}^{LOA_{1,t_4},LOA_{2,t_4}}) \\ LOA_{2,t_4} & if\ \beta \in [\beta_{t_4,I,r}^{LOA_{1,t_4},LOA_{2,t_4}},\ \beta_{t_4,I,r}^{LOA_{2,t_4},NT}) \\ NT & if\ \beta \in [\beta_{t_4,I,r}^{LOA_{2,t_4},NT},\ \beta_{t_4,I,r}^{NT,LOB_{2,t_4}}) \\ LOB_{2,t_4} & if\ \beta \in [\beta_{t_4,I,r}^{NT,LOB_{2,t_4}},\ \beta_{t_4,I,r}^{LOB_{2,t_4},LOB_{1,t_{4},t_4}}) \\ LOB_{1,t_4} & if\ \beta \in [\beta_{t_4,I,r}^{LOB_{2,t_4},LOB_{1,t_4}},\ \beta_{t_4,I,r}^{LOB_{1,t_4},MOA_{2,t_4}}) \\ MOA_{2,t_4} & if\ \beta \in [\beta_{t_4,I,r}^{LOB_{1,t_4},MOA_{2,t_4}},\ 2] \end{cases} \tag{26}$$

and for an uninformed trader the optimal strategies are qualitatively similar but with different values for the β thresholds given the uninformed investor's different information.[15] As the payoffs of both limit and market orders are functions of β, we can rank all the payoffs of adjacent optimal

[14] Recall that the upper envelope only includes strategies that are optimal.

[15] If the book opened with some liquidity on any level of the book, the equilibrium strategies would be different. For example, if the book opened with a LOA_1 then no limit orders on the ask side would be equilibrium strategies.

strategies in terms of β and equate them to determine the β thresholds at t_4. For example, for the first threshold we have:

$$\beta_{t_4,I,r}^{MOB_{2,t_4},LOA_{1,t_4}} = \beta \in \mathbb{R} \text{ s.t. } \varphi_{t_4,r}^{I}\left(MOB_{2,t_4} \,|\, v, \beta, \mathscr{L}_{t_3}\right) = \varphi_{t_4,r}^{I}\left(LOA_{1,t_4} \,|\, v, \beta, \mathscr{L}_{t_3}\right) \tag{27}$$

and we obtain the other thresholds similarly.

The next step is to use the β thresholds together with the truncated Normal cumulative distribution $\mathbb{N}(.)$ for β to derive the probabilities of the optimal order-submission strategies at each possible node of the extensive form of the game at t_4. For example, the submission probability of LOA_{i,t_4} is:

$$Pr_r[LOA_{1,t_4} \,|\, \mathscr{L}_{t_3}, v] = \mathbb{N}(\beta_{t_4,I,r}^{LOA_{1,t_4},LOA_{2,t_4}} \,|\, \mathscr{L}_{t_3}, v) - \mathbb{N}(\beta_{t_4,I,r}^{MOB_{2,t_4},LOA_{1,t_4}} \,|\, \mathscr{L}_{t_3}, v) \tag{28}$$

and the submission probabilities of the equilibrium strategies can be obtained in a similar way. Next, given the market-order submission probabilities at t_4 (which are the execution probabilities of limit orders at t_3), we can solve the optimal orders at t_3 and recursively we can then solve the model by backward induction back to time t_1.

At each node of the trading game, the algorithm considers all feasible orders that traders may choose. Off-equilibrium orders are those that are never chosen as part of the optimal trading strategies. Suppose that in round r an order that is off-equilibrium in round $r-1$ is considered for time t_j. For example, consider in round r the path of the trading game ending with LOA_{1,t_3} formed by the sequence of orders: $\{MOA_{2,t_1}, MOB_{2,t_2}, LOA_{1,t_3}\}$, where LOA_{1,t_3} was not an equilibrium strategy at t_3 in round $r-1$ and where MOA_{2,t_1} and MOB_{2,t_2} are equilibrium strategies at times t_1 and t_2 respectively. Within the convergence process, for each strategy which is reconsidered in the subsequent round, uninformed traders generally use their previous round beliefs. For an off-equilibrium strategy at t_j in $r-1$, however, they cannot use their $r-1$ updated belief for that time and therefore they use their most recent equilibrium belief up to t_j still for round $r-1$. Considering the example above, uninformed traders cannot use their updated belief conditional on the sequence of orders $\{MOA_{2,t_1}, MOB_{2,t_2}, LOA_{1,t_3}\}$ at t_3 for round $r-1$ as LOA_{1,t_3} was not an equilibrium strategy. Therefore we assume that for this off-equilibrium belief, uninformed traders use the most updated equilibrium belief before t_3, formed by using the sequence of orders $\{MOA_{2,t_1}, MOB_{2,t_2}\}$. If instead in round $r-1$, MOB_{2,t_2} is still not an equilibrium strategy at t_2, we assume that uninformed traders use their belief at t_1 conditional on MOA_{2,t_1}. Finally, if neither MOA_{2,t_1} were an equilibrium strategy at t_1 we assume that traders use their unconditional prior belief.

We allow for both pure and mixed strategies in our Perfect Bayesian Nash equilibrium. When different orders have equal expected payoffs, we assume that traders randomize with equal probabilities across all such optimal orders. By construction, the expected payoffs of two different strategies are the same in correspondence of the β thresholds; however because we are considering single points in the support of the β distribution, the probability associated with any strategy that corresponds to those specific points is equal to zero. This means that mixed strategies that emerge in correspondence of the β thresholds, although feasible, have zero probability. Mixed strategies may also emerge in the framework in which informed traders have a fixed neutral private-value factor $\beta = 1$ (section 2.1). More specifically it may happen that the payoffs of two perfectly symmetrical strategies of I_{v_0} are the same, and in this case I_{v_0} randomizes between these two strategies.

RE beliefs for a Perfect Bayesian Nash equilibrium are obtained by solving the model recursively for multiple rounds. In particular, the asset-value probabilities from round $r = 1$ above are used as the priors to solve the model in round $r = 2$ (i.e., the round 1 probabilities are used in place of the unconditional priors used in round 1).[16] The asset-value probabilities from round $r = 2$ are then used as the priors in round $r = 3$ and so on. We continue the iteration until the updating process converges to a fixed point, which are the REE beliefs. In particular, the recursive process has converged to the RE beliefs when uninformed traders do not revise their asset-value beliefs. Operationally, we consider convergence to the RE beliefs to have occurred when the execution-contingent conditional probabilities $\hat{\pi}_{t_j}^{\overline{v},r}$, $\hat{\pi}_{t_j}^{v_0,r}$ and $\hat{\pi}_{t_j}^{\underline{v},r}$ in round r are almost equal to the corresponding

[16]In the second round of solutions we again solve the full 5-period model.

probabilities from round $r-1$:

$$\begin{aligned}
&\hat{\pi}_{t_j}^{\overline{v},*} \text{ when } \left|\hat{\pi}_{t_j}^{\overline{v},r} - \hat{\pi}_{t_j}^{\overline{v},r-1}\right| < 10^{-7} \\
&\hat{\pi}_{t_j}^{v_0,*} \text{ when } \left|\hat{\pi}_{t_j}^{v_0,r} - \hat{\pi}_{t_j}^{v_0,r-1}\right| < 10^{-7} \\
&\hat{\pi}_{t_j}^{\underline{v},*} \text{ when } \left|\hat{\pi}_{t_j}^{\underline{v},r} - \hat{\pi}_{t_j}^{\underline{v},r-1}\right| < 10^{-7}
\end{aligned} \tag{29}$$

The fixed point is such that conditional on the most recent pieces of information, uninformed traders can extract from the limit order book, they do not wish to revise their beliefs on $\hat{\pi}_{t_j}^{\overline{v}}$, $\hat{\pi}_{t_j}^{v_0}$ and $\hat{\pi}_{t_j}^{\underline{v}}$. A fixed-point solution to this recursive algorithm is an equilibrium in our model.

6 Appendix B: Additional numerical results

The tables is this section provide additional information on the execution probabilities of limit orders for informed investor with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders. The tables also report the asset value expectations of the uninformed investor at time t_2 after observing all the possible buy orders submissions at time t_1 (the expectations for sell orders are symmetric with respect to 1). Table B1 reports results for the model specification in which only uninformed traders have a random private value factor, Table B2 instead reports results for the model in which both the informed and the uniformed traders have private-value motives.

Table B1: Order Execution Probabilities and Asset-Value Expectation for Informed Traders with $\boldsymbol{\beta = 1}$ and Uninformed Traders with $\boldsymbol{\beta \sim Tr[\mathcal{N}(\mu,\, \sigma^2)]}$. This table reports results for two different values of the informed-investor arrival probability α (0.8 and 0.2) and for two different values of the asset-value volatility δ (0.16 and 0.02). $\sigma = 1.5$. For each set of parameters, the first four columns report the equilibrium limit order probabilities of executions for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports the unconditional order-execution probabilities in the market. Next, the columns report conditional and unconditional future order execution probabilities and the asset-value expectations of an uniformed investor at time t_2 after observing different order submissions at time t_1.

		$\delta = 0.16$					$\delta = 0.02$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	$P^{EX}(LOA_2\|\cdot)$	0.955	0.175	0.055	0.395	0.395	0.180	0.229	0.170	0.193	0.193
	$P^{EX}(LOA_1\|\cdot)$	0.989	0.125	0.078	0.397	0.397	0.323	0.323	0.323	0.323	0.323
	$P^{EX}(LOB_1\|\cdot)$	0.078	0.125	0.989	0.397	0.397	0.323	0.323	0.323	0.323	0.323
	$P^{EX}(LOB_2\|\cdot)$	0.055	0.175	0.955	0.395	0.395	0.170	0.229	0.180	0.193	0.193
$\alpha = 0.8$											
	E[v$\|LOB_1\ \|\cdot$]					1.160					1.000
	E[v$\|LOB_2\ \|\cdot$]					1.083					1.013
	E[v$\|MOA_1\ \|\cdot$]										
	E[v$\|MOA_2\ \|\cdot$]					1.000					1.000
	$P^{EX}(LOA_2\|\cdot)$	0.651	0.487	0.394	0.511	0.511	0.514	0.499	0.476	0.496	0.496
	$P^{EX}(LOA_1\|\cdot)$	0.886	0.766	0.717	0.789	0.789	0.792	0.792	0.790	0.791	0.791
	$P^{EX}(LOB_1\|\cdot)$	0.717	0.766	0.886	0.789	0.789	0.790	0.792	0.792	0.791	0.791
	$P^{EX}(LOB_2\|\cdot)$	0.394	0.487	0.651	0.511	0.511	0.476	0.499	0.514	0.496	0.496
$\alpha = 0.2$											
	E[v$\|LOB_1\ \|\ \cdot$]					1.026					1.000
	E[v$\|LOB_2\ \|\ \cdot$]					1.013					1.009
	E[v$\|MOA_1\ \|\ \cdot$]										
	E[v$\|MOA_2\ \|\ \cdot$]					1.000					1.000

Table B2: Order Execution Probabilities and Asset-Value Expectation for Informed and Uninformed Traders both with $\boldsymbol{\beta \sim Tr[\mathcal{N}(\mu,\, \sigma^2)]}$. This table reports results for two different values of the informed-investor arrival probability α (0.8 and 0.2) and for two different values of the asset-value volatility δ (0.16 and 0.02). $\sigma = 1.5$. For each set of parameters, the first four columns report the equilibrium limit order probabilities of executions for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports the unconditional order-execution probabilities in the market. Next, the columns report conditional and unconditional future order execution probabilities and the asset-value expectations of an uniformed investor at time t_2 after observing different order submissions at time t_1.

		$\delta = 0.16$					$\delta = 0.02$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	$P^{EX}(LOA_2\|\cdot)$	0.644	0.502	0.410	0.519	0.135	0.502	0.487	0.472	0.487	0.116
	$P^{EX}(LOA_1\|\cdot)$	0.913	0.834	0.702	0.817	0.392	0.849	0.837	0.824	0.836	0.470
	$P^{EX}(LOB_1\|\cdot)$	0.702	0.834	0.913	0.817	0.392	0.824	0.837	0.849	0.836	0.470
	$P^{EX}(LOB_2\|\cdot)$	0.410	0.502	0.644	0.519	0.135	0.472	0.487	0.502	0.487	0.116
$\alpha = 0.8$											
	E[v$\|LOB_1$ $\|\cdot$]					0.996					1.000
	E[v$\|LOB_2$ $\|\cdot$]					0.944					0.999
	E[v$\|MOA_1$ $\|\cdot$]										
	E[v$\|MOA_2$ $\|\cdot$]					1.156					
	$P^{EX}(LOA_2\|\cdot)$	0.525	0.494	0.470	0.496	0.402	0.490	0.487	0.483	0.487	0.394
	$P^{EX}(LOA_1\|\cdot)$	0.853	0.833	0.813	0.833	0.737	0.839	0.837	0.834	0.837	0.745
	$P^{EX}(LOB_1\|\cdot)$	0.813	0.833	0.853	0.833	0.737	0.834	0.837	0.839	0.837	0.745
	$P^{EX}(LOB_2\|\cdot)$	0.470	0.494	0.525	0.496	0.402	0.483	0.487	0.490	0.487	0.394
$\alpha = 0.2$											
	E[v$\|LOB_1$ $\|\cdot$]					1.003					1.000
	E[v$\|LOB_2$ $\|\cdot$]					0.996					1.000
	E[v$\|MOA_1$ $\|\cdot$]										
	E[v$\|MOA_2$ $\|\cdot$]					1.160					

References

Aït-Sahalia, Yacine, and Mehmet Saglam, 2013, High frequency traders: Taking advantage of speed, Technical report, National Bureau of Economic Research.

Biais, Bruno, Thierry Foucault, and Sophie Moinas, 2015, Equilibrium fast trading, *Journal of Financial Economics* 116, 292–313.

Bloomfield, Robert, Maureen O'Hara, and Gideon Saar, 2005, The "make or take" decision in an electronic market: Evidence on the evolution of liquidity, *Journal of Financial Economics* 75, 165–199.

Brogaard, Jonathan, Terrence Hendershott, and Ryan Riordan, 2016, Price discovery without trading: Evidence from limit orders.

Fleming, Michael J, Bruce Mizrach, and Giang Nguyen, 2017, The microstructure of a us treasury ecn: The brokertec platform, *Journal of Financial Markets* 1–52.

Foucault, Thierry, 1999, Order flow composition and trading costs in a dynamic limit order market, *Journal of Financial Markets* 2, 99–134.

Foucault, Thierry, Johan Hombert, and Ioanid Roşu, 2016, News trading and speed, *Journal of Finance* 71, 335–382.

Foucault, Thierry, Ohad Kadan, and Eugene Kandel, 2005, Limit order book as a market for liquidity, *Review of Financial Studies* 18, 1171–1217.

Gencay, Ramazan, Soheil Mahmoodzadeh, Jakub Rojcek, and Michael C Tseng, 2016, Price impact and bursts in liquidity provision.

Glosten, Lawrence R, and Paul R Milgrom, 1985, Bid, ask and transaction prices in a specialist market with heterogeneously informed traders, *Journal of Financial Economics* 14, 71–100.

Goettler, Ronald L, Christine A Parlour, and Uday Rajan, 2005, Equilibrium in a dynamic limit order market, *Journal of Finance* 60, 2149–2192.

Goettler, Ronald L, Christine A Parlour, and Uday Rajan, 2009, Informed traders and limit order markets, *Journal of Financial Economics* 93, 67–87.

Jain, Pankaj K, 2005, Financial market design and the equity premium: Electronic versus floor trading, *Journal of Finance* 60, 2955–2985.

Kaniel, Ron, and Hong Liu, 2006, So what orders do informed traders use?, *The Journal of Business* 79, 1867–1913.

Kumar, Praveen, and Duane J Seppi, 1994, Limit and market orders with optimizing traders.

Kyle, Albert S, 1985, Continuous auctions and insider trading, *Econometrica* 1315–1335.

Milgrom, Paul, and Nancy Stokey, 1982, Information, trade and common knowledge, *Journal of Economic Theory* 26, 17–27.

O'Hara, Maureen, 2015, High frequency market microstructure, *Journal of Financial Economics* 116, 257–270.

Parlour, Christine A, 1998, Price dynamics in limit order markets, *Review of Financial Studies* 11, 789–816.

Parlour, Christine A, and Duane J Seppi, 2008, Limit order markets: A survey, *Handbook of Financial Intermediation and Banking* 5, 63–95.

Rindi, Barbara, 2008, Informed traders as liquidity providers: Anonymity, liquidity and price formation, *Review of Finance* 497–532.

Roşu, Ioanid, 2016a, Fast and slow informed trading.

Roşu, Ioanid, 2016b, Liquidity and information in order driven markets.