Smart TWAP Trading in Continuous-Time Equilibria*

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ABSTRACT: This paper presents a continuous-time equilibrium model of TWAP trading and liquidity provision in a market with multiple strategic investors with intraday trading targets. We demonstrate analytically that there are infinitely many Nash equilibria. We solve for the welfare-maximizing equilibrium and the competitive equilibrium, and we show that these equilibria are different. The model is computationally tractable, and we provide a number of numerical illustrations.

KEYWORDS: Dynamic trading, TWAP, rebalancing, liquidity, Nash equilibria, market microstructure

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1 Introduction

Intraday trading targets play an important role in the dynamics of liquidity provision and demand over the trading day in financial markets. For example, high-frequency trading (HFT) market makers — who are the dominate source of intraday liquidity on most exchanges[1] — usually seek to maintain a neutral (e.g., zero) inventory level. In addition, trade execution by large institutional investors is widely benchmarked relative to a *time-weighted average price (TWAP)* or *volume-weighted average price (VWAP)* reference price. Compensation schemes tied to such benchmarks can lead to strategies with intraday targets to trade a constant amount per unit time (TWAP) or an amount indexed to the average daily volume curve (VWAP). Deviations from the intraday trading targets are then penalized. However, traders sometimes intentionally deviate from their ideal trading targets in order to achieve trading profits (such as HFT market makers) and price improvement (such as large institutions). We call optimized trading strategies that trade off trading profits and target-deviation penalties *smart TWAP* or *smart VWAP* strategies.

Our paper solves for continuous-time equilibria in a market in which multiple strategic investors with different trading targets follow optimal dynamic trading strategies. To the best of our knowledge, our paper is the first to model the equilibrium impact of intraday smart TWAP strategies on intraday trading and market liquidity. Our results are as follows:

- An infinite number of equilibria exist with each equilibrium pinned down by a continuous function giving the impact of the strategic investors' orders on prices.

- There is no predatory trading in equilibrium in our model. This is because all intraday liquidity is provided by rational optimizing agents.

- The welfare-maximizing equilibrium differs from the competitive equilibrium. For example, one functional difference is that the private trading target is irrelevant for the investors' expectation of their target deviation in the competitive equilibrium whereas for the welfare-maximizing equilibrium, the private target plays a dominate role for how investors plan on deviating from it.

[1]Hagströmer and Nordén (2013) estimate that HFT market makers participate in the best bid or ask quote 57% of the time on Nasdaq-OMX.

The market in our model is incomplete in that there is only one stock and two stochastic processes driving its price. We focus our discussion on preferences that are linear in investor ending wealth and a penalty for deviations from target holdings over the day. Proofs are in Appendix A. Appendix B extends the analysis to exponential utilities.[2]

Our model is most closely related to previous research by Brunnermeier and Pedersen (2005) and Carlin, Lobo, and Viswanathan (2007) on optimal rebalancing and predatory trading. There are two main differences: First, our strategic agents are subject to penalties tied to intraday trading targets. Second, there is no group of outside ad hoc intraday liquidity providers in our model. As a result, in our model, all intraday market liquidity is provided endogenously by strategic agents with trading targets close to zero. We call these liquidity-providing investors HFT market makers. We conjecture that there is no predatory trading in our model because all liquidity is provided endogenously.[3] In current work in progress, Capponi and Menkveld are exploring liquidity provision when arriving strategic agents have different time horizons over which their portfolio must be rebalanced. In contrast, the strategic investors in our model all have the same terminal rebalancing horizon. Lastly, there is no asymmetric information about future asset cash-flow fundamentals in our model. Thus, the analysis here on trading and the non-informational component of market liquidity is complimentary to Choi, Larsen, and Seppi (2018), which studies order-splitting and dynamic rebalancing in a Kyle (1985) style market in which a strategic informed investor with long-lived private information and a strategic rebalancer with a terminal trading target both follow dynamic trading strategies.

[2] Existence of continuous-time Radner equilibria with exponential utilities in an incomplete competitive market has been proved in various levels of generality in Christensen, Larsen, and Munk (2012), Žitković (2012), Christensen and Larsen (2014), Choi and Larsen (2015), Larsen and Sae-Sue (2016), and Xing and Žitković (2018). To the best of our knowledge, there is no extension of these models to a continuous-time incomplete market equilibrium with price impact (Vayanos (1999) proves existence in a discrete-time model with an exogenous constant interest rate). Appendix B contains such a continuous-time extension in which investors have trading targets.

[3] In Brunnermeier and Pedersen (2005), ad hoc liquidity providers do not rationally anticipate future price changes given early trading.

2 Model

We develop a continuous-time equilibrium model with a unit horizon in which trade takes place at each time point $t \in [0,1]$. This can be interpreted as one trading day. Our model has two securities: a money market account with a constant unit price (i.e., the account pays a zero interest rate) and a stock with an endogenously determined price process $S = (S_t)_{t\in[0,1]}$. The stock pays an exogenously specified random dividend D_1 at maturity that is generated by a publicly observable and exogenous Brownian motion $D = (D_t)_{t\in[0,1]}$ with a given constant initial value $D_0 \in \mathbb{R}$. We view D_t as a sufficient statistic at time $t < 1$ for the terminal dividend D_1. In our continuous setting, with an exogenous terminal dividend as in, e.g., Grossman and Stiglitz (1980), a natural terminal stock price restriction is

$$S_1 = D_1. \tag{2.1}$$

Ohasi (1991, 1992) shows that the validity of (2.1) relies on continuity of information sets, which holds in our Brownian setting. While (2.1) is natural in our Brownian setting, we also allow for a more general structure in (3.6) below that relaxes the assumption of an exogenously specified terminal liquidating dividend.

Two types of investors trade in our model:

- There are $M \in \mathbb{N}$ strategic investors who each have three pieces of private information: their initial money market account holdings $(\theta^{(0)}_{1,-}, ..., \theta^{(0)}_{M,-})$, their initial stock holdings $(\theta_{1,-}, ..., \theta_{M,-})$, and their terminal stock-holding targets $(\tilde{a}_1, ..., \tilde{a}_M)$. At this point we make no distributional assumptions about these variables. The strategic investors incur a penalty if the intraday trajectory of their cumulative stock trades $\theta_{i,t} - \theta_{i,-}$ at time $t \in [0,1]$ deviates from a target trajectory $\gamma(t)(\tilde{a}_i - \theta_{i,-})$. Here $\gamma(t)$ is a positive continuous function for $t \in [0,1]$. The target trajectory $\gamma(t)(\tilde{a}_i - \theta_{i,-})$ is the amount of the total target trading $\tilde{a}_i - \theta_{i,-}$ the investor would ideally like to have completed by time $t \in [0,1]$. For a TWAP target, for example, we have $\gamma(t) := t$. Alternatively, $\gamma(t)$ for a VWAP investor might follow the shape of the average cumulative volume curve over the trading day.

The penalty process for investor $i \in \{1, ..., M\}$ is[4]

$$L_{i,t} := \int_0^t \kappa(s)\Big(\gamma(s)(\tilde{a}_i - \theta_{i,-}) - (\theta_{i,s} - \theta_{i,-})\Big)^2 ds, \quad t \in [0,1]. \tag{2.2}$$

The severity of the penalty is controlled by $\kappa(t)$ which is a deterministic strictly positive function. Intuitively, the penalty severity for deviations from the target trajectory is likely to be increasing over the trading day. Our results below allow both for the possibility of penalty-severity functions $\kappa(t)$ that explode towards the end of the trading day as $t \to 1$ as well as for a bounded penalty severity. To keep the model as simple as possible, we assume that all strategic investors are subject to the same deterministic functions $\gamma(t)$ and $\kappa(t)$, and we refer to (5.4) below for specific examples. We differentiate between two types of investors based on their realized trading targets $\tilde{a}_i$. We refer to investors with targets $\tilde{a}_i \neq \theta_{i,-}$ as *smart TWAP investors* (see Chapter 5 in Johnson (2010) for more about TWAP trading). Traders with $\tilde{a}_i = \theta_{i,-}$ do not need to trade, but can provide liquidity. We call these traders *HFT market makers* or strategic liquidity providers. Thus, smart TWAP traders and HFT market makers differ in the target amount they want to trade but face the same penalties for diverging from their trading target.

- There are *noise traders* whose trading motives we do not model. We assume that the stock supply that the strategic investors must absorb (i.e., the stock's fixed shares outstanding minus the aggregate noise traders' holdings) is given by w_t at time $t \in [0,1]$. Consequently, the stock market clears at time $t \in [0,1]$ when the strategic investors' holdings $(\theta_i)_{i=1}^M$ satisfy

$$w_t = \sum_{i=1}^M \theta_{i,t}. \tag{2.3}$$

[4]Quadratic penalization schemes constitute a cornerstone in research related to mean-variance analysis and dates back to Markowitz (1952). We note that (2.2) penalizes the aggregate holdings $\theta_{i,t}$ and not the buying/selling rates as in, e.g., Almgren (2012) and Gârleanu and Pedersen (2016) whereas Bouchard et all (2018) penalize both rates and holdings. Penalizing the buying/selling rate forces the optimal stock holdings to be given in terms of rates (i.e., the optimal holding process is a finite variation process). Optimal buying/selling rates also exist in the continuous-time model in Kyle (1985) as well as in its non-Gaussian extension in Back (1992). In contrast, as we shall see, optimal holding processes in our model have infinite first variation (and only finite second variation). This property allows our strategic investors to absorb the below noise trader orders (2.4).

We assume that the stock supply has dynamics (Gaussian Ornstein-Uhlenbeck)

$$dw_t := (\alpha - \pi w_t)dt + \sigma_w dW_t, \quad w_0 \in \mathbb{R}. \tag{2.4}$$

Gaussian noise traders have been widely used; see, e.g., Grossman and Stiglitz (1980) and Kyle (1985). In (2.4), the parameters σ_w, α, and π are all constants and W is another Brownian motion that is independent of the dividend Brownian motion D. The specification (2.4) includes cumulative noise-trader order-flow imbalances that follow an arithmetic Brownian motion ($\pi = 0$) as well as possible mean-reverting dynamics ($\pi \neq 0$).

The information structure of our model is as follows: For tractability, we assume that the strategic investors have homogeneous beliefs in the sense that they all believe the processes (D, W) are the same independent Brownian motions. Over time, the realized dividend factor D_t and the noise trader orders w_t are publicly observed (i.e., either directly or by inference from observations of S_t). At time $t \in [0, 1]$, investor i chooses a cumulative stock-holding position $\theta_{i,t}$ that satisfies the measurability requirement[5]

$$\theta_{i,t} \in \mathcal{F}_{i,t} := \sigma(\tilde{a}_i, \tilde{a}_\Sigma, W_u, D_u)_{u\in[0,t]}, \tag{2.5}$$

where the total target for all M strategic investors is defined by

$$\tilde{a}_\Sigma := \sum_{i=1}^{M} \tilde{a}_i. \tag{2.6}$$

It might seem unclear why $\tilde{a}_\Sigma$ is included in investor i's information set. One possibility is that $\tilde{a}_\Sigma$ may be directly observable in the market. However, public observability is not necessary. As we shall see in the next section, in the equilibria we construct, we have

$$\mathcal{F}_{i,t} = \sigma(\tilde{a}_i, S_u, D_u)_{u\in[0,t]}. \tag{2.7}$$

In other words, although investors only know their own target $\tilde{a}_i$ directly, they can

[5]As usual in continuous-time models, we also need to impose an integrability condition to ensure that certain stochastic integrals are martingale; see Definition 2.1 below.

infer the aggregate net target $\tilde{a}_\Sigma$ in equilibrium from the initial stock price so that

$$\sigma(S_0) = \sigma(\tilde{a}_\Sigma). \tag{2.8}$$

Thus, our model lets us investigate the effects of known or inferable aggregate trading targets on intraday trading and pricing.

Next, we turn to the strategic investors' individual optimization problems. For a strategy $\theta_{i,t} \in \mathcal{F}_{i,t}$, we let $X_{i,t}$ denote investor i's wealth process, which has dynamics

$$dX_{i,t} := \theta_{i,t} dS_t, \quad X_{i,0} := \theta_{i,-} S_0 + \theta_{i,-}^{(0)}. \tag{2.9}$$

The set of admissible strategies $\mathcal{A}_i$ for investor i is defined as follows:

Definition 2.1. We deem a jointly measurable and $\mathcal{F}_i$ adapted process $\theta_i = (\theta_{i,t})_{t \in [0,1]}$ to be *admissible* and write $\theta_i \in \mathcal{A}_i$ if the following integrability condition holds

$$\mathbb{E}\left[\int_0^1 \theta_{i,t}^2 dt\right] < \infty. \tag{2.10}$$

$\diamondsuit$

It is well-known that an integrability condition like (2.10) rules out doubling strategies (see, e.g., Chapters 5 and 6 in Duffie (2001) for a discussion of such conditions). While all bounded processes satisfy the integrability condition (2.10), the optimal stock holding process in (3.12) below is not a bounded process but does, however, satisfy (2.10).

For simplicity, we assume that all strategic investors have linear utility functions: $U_i(x) := x$ for all $i \in \{1, ..., M\}$. For a given stock price process S, investor i seeks an expected utility-maximizing holding strategy $\hat{\theta}_i \in \mathcal{A}_i$ which attains

$$V(X_{i,0}, w_0, \tilde{a}_i, \tilde{a}_\Sigma) := \sup_{\theta_i \in \mathcal{A}_i} \mathbb{E}\left[X_{i,1} - L_{i,1} \,\middle|\, \sigma(\theta_{i,-}^{(0)}, \theta_{i,-}, \tilde{a}_i)\right]. \tag{2.11}$$

In (2.11), the variable $L_{i,1}$ is the terminal penalty value from the penalty process in (2.2), and the terminal wealth $X_{i,1}$ is defined from the wealth process in (2.9). Section 6 and Appendix B extend our analysis to homogeneous exponential utilities $U_i(x) := -e^{-x/\tau}$, $x \in \mathbb{R}$, for a common risk-tolerance parameter $\tau > 0$.

3 Existence of Nash equilibria

We start by discussing the clearing conditions. First, the initial stock holdings (also private information variables) satisfy in aggregate

$$w_0 = \sum_{i=1}^{M} \theta_{i,-}. \tag{3.1}$$

This means that the amount of the total initial outstanding supply not held by the noise traders is held by the strategic traders. Furthermore, at all later times $t \in (0,1]$, the stock holdings $(\theta_{1,t}, ..., \theta_{M,t})$ must also satisfy the intraday clearing condition (2.3).

Next, we turn to the stock-price dynamics. The price dynamics we consider are described in terms of smooth deterministic functions

$$\mu_0, ..., \mu_5, \sigma_{SW} : [0,1] \to \mathbb{R}. \tag{3.2}$$

The price process that investor $i \in \{1, ..., M\}$ faces is given by

$$dS_t = \mu_{i,t} dt + \sigma_{SW}(t) dW_t + dD_t, \tag{3.3}$$

where

$$\mu_{i,t} := \mu_0(t)\tilde{a}_\Sigma + \mu_1(t)\theta_{i,t} + \mu_2(t)\tilde{a}_i + \mu_3(t)w_t + \mu_4(t)w_0 + \mu_5(t)\theta_{i,-}. \tag{3.4}$$

These price dynamics can be interpreted as follows: First, the prices move one-to-one with the changes in the dividend factor D_t. Second, random shocks to the noise-trader order imbalance dW_t move prices linearly as given by the (deterministic) function $\sigma_{SW}(t)$. Third, the drift $\mu_{i,t}$ adjusts prices predictably (to each investor i) over time to clear markets given public information ($\tilde{a}_\Sigma, w_t$, and w_0) and private information for investor i ($\tilde{a}_i, \theta_{i,t}$, and $\theta_{i,-}$).

Definition 3.1 (Nash)**.** The deterministic functions $(\mu_0, ..., \mu_5, \sigma_{SW})$ constitute a *Nash equilibrium* if, given the stock price dynamics (3.3), the resulting optimal controls $(\hat{\theta}_i)_{i=1}^M$ satisfy (i) the clearing condition

$$w_t = \sum_{i=1}^{M} \hat{\theta}_{i,t} \tag{3.5}$$

at all times $t \in [0,1]$, (ii) the terminal stock-price condition holds

$$S_1 = D_1 + \varphi_0 \tilde{a}_\Sigma + \varphi_1 w_1 \tag{3.6}$$

for given constants $\varphi_0, \varphi_1 \in \mathbb{R}$, and (iii) when we replace $\theta_{i,t}$ by the maximizer $\hat{\theta}_{i,t}$ in (3.4), the resulting drift $\mu_{i,t}$ does not depend on the investor-specific private information variables $(\tilde{a}_i, \theta_{i,-}\theta_{i,-}^{(0)})$.

$\diamondsuit$

Requirement (iii) in Definition 3.1 means that equilibrium prices only depend on individual variables $\tilde{a}_i, \theta_{i,-}$, and $\theta_{i,t}$ via their impact on the corresponding aggregate variables. To understand the terminal requirement (3.6), we consider first the simpler case in (2.1). In this case, it is clear that for S_t to converge to $S_1 = D_1$ that $\sigma_{SW}(t)$ in (3.3) must converge to zero as time approaches maturity (i.e., as $t \uparrow 1$). However, as we prove below, only one such smooth function $\sigma_{SW}(t)$ will work.[6] An additional difficulty in constructing an equilibrium is that investors are penalized whenever they deviate from their targets but that, because of (2.1), the price dynamics leave little wiggle room to provide the investors with compensation (i.e., non-zero expected returns when $S_t \neq D_t$) to entice them to hold w_1 in aggregate.

Condition (3.6) allows for a more general terminal requirement than (2.1). When our model is applied to a short time horizon (e.g., a trading day), then the end-of-day stock valuation is not the value of an exogenously specified terminal dividend D_1, but rather the valuation attached to future stock cash flows by a group of unmodeled overnight liquidity providers.[7] The $\varphi_0 \tilde{a}_\Sigma$ term in (3.6) with $\varphi_0 \geq 0$ represents the amount the terminal stock valuation S_1 is moved by price pressure from the net trading-target imbalance of the strategic agents, and $\varphi_1 w_1$ with $\varphi_1 \leq 0$ is the amount S_1 is moved by the ending cumulative noise-trader order imbalance.[8]

Theorem 3.4 below gives restrictions on the pricing functions in (3.2) for existence of a Nash equilibrium. However, as we shall see, there is one degree of freedom, and

[6]The ODE for the equilibrium volatility function $\sigma_{SW}(t)$ in (3.3) that achieves market clearing is given below in (3.14).

[7]Our overnight liquidity providers are different from the ad hoc residue liquidity providers who trade continuously over the day as in the Brunnermeier and Pedersen (2005) predatory trading model.

[8]A natural restriction here is $|\varphi_0| < |\varphi_1|$. Noise traders trade inelastically, so the full noise-trader order imbalance w_1 must be held by the overnight liquidity-providers and the strategic investors. In contrast, the strategic investors do not demand to achieve their aggregate ideal holdings $\tilde{a}_\Sigma$ inelastically.

so there are multiple (indeed, infinitely many) equilibria. We find that leaving the price-impact function $\mu_1(t)$ as the free parameter produces the simplest expressions. Next, we consider two possible examples of how one might choose the function $\mu_1(t)$ and thereby uniquely pin down the equilibrium.

Example 3.2 (Radner)**.** In a fully competitive Radner equilibrium, the realization of the price process S is unaffected by the choice of investor i's strategy θ_i. This case is recovered by setting the investor price-impact function in (3.3) to

$$\mu_1(t) := 0, \quad \text{for all } t \in [0, 1]. \tag{3.7}$$

$\diamondsuit$

Example 3.3 (Welfare maximization)**.** The certainty equivalent $\mathrm{CE}_i \in \mathbb{R}$ for TWAP investor i is defined as

$$\mathrm{CE}_i := V(X_{i,0}, w_0, \tilde{a}_i, \tilde{a}_\Sigma) \tag{3.8}$$

where V is the value function defined in (2.11). The certainty equivalent (3.8) follows from our assumption that all TWAP investors are risk neutral, (i.e., their utilities are $U_i(x) := x$). We are interested in finding the price-impact function $\mu_1^* : [0, 1] \to \mathbb{R}$ that maximizes the objective equal to total welfare

$$\sum_{i=1}^{M} \mathbb{E}[\mathrm{CE}_i]. \tag{3.9}$$

The objective (3.9) is an ex ante perspective in sense that the expectation $\mathbb{E}$ is taken over the private information variables $(\theta^{(0)}_{1,-}, ..., \theta^{(0)}_{M,-}), (\theta_{1,-}, ..., \theta_{M,-})$, and $(\tilde{a}_1, ..., \tilde{a}_M)$. This example is discussed in detail in the next section. $\diamondsuit$

As mentioned above, our equilibria are parameterized in terms of an arbitrary continuous (deterministic) price-impact function $\mu_1 : [0, 1] \to \mathbb{R}$. The proof of Theorem 3.4 establishes that the second-order condition ensuring optimality of the investors' conjectured optimal controls is given by

$$\mu_1(t) < \kappa(t), \quad t \in (0, 1]. \tag{3.10}$$

Our main existence result is the following:

Theorem 3.4. *Let $\gamma : [0,1] \to [0,\infty)$ be a continuous function, and let $\mu_1, \kappa : [0,1) \to (0,\infty)$ be continuous and square integrable functions; i.e.,*

$$\int_0^1 \big(\kappa(t)^2 + \mu_1(t)^2\big) dt < \infty, \tag{3.11}$$

that satisfy the second-order condition (3.10). *Then a Nash equilibrium exists in which (i) investor optimal holdings $\hat{\theta}_i$ in equilibrium are given by*

$$\hat{\theta}_{i,t} = \frac{2\kappa(t)\gamma(t)}{2\kappa(t)-\mu_1(t)}\Big(\tilde{a}_i - \frac{\tilde{a}_\Sigma}{M}\Big) + \frac{w_t}{M} + \frac{2\kappa(t)\big(1-\gamma(t)\big)}{2\kappa(t)-\mu_1(t)}\Big(\theta_{i,-} - \frac{w_0}{M}\Big), \tag{3.12}$$

(ii) the equilibrium stock price is given by

$$S_t = g_0(t) + g(t)\tilde{a}_\Sigma + \sigma_{SW}(t)w_t + D_t, \tag{3.13}$$

where the deterministic functions g_0, g, and $\sigma_{SW} : [0,1] \to \mathbb{R}$ are the unique solutions of the following linear ODEs:

$$\begin{aligned} \sigma'_{SW}(t) &= \frac{2\kappa(t)-\mu_1(t)}{M} + \pi\sigma_{SW}(t), \quad \sigma_{SW}(1) = \varphi_1, \\ g'(t) &= -\frac{2\gamma(t)\kappa(t)}{M}, \quad g(1) = \varphi_0, \\ g_0'(t) &= \frac{2w_0(\gamma-1)\kappa}{M} - \alpha\sigma_{SW}(t), \quad g_0(1) = 0, \end{aligned} \tag{3.14}$$

and (iii) the functions $\mu_0, \mu_2, \mu_3, \mu_4$, and μ_5 in the price-impact relation (3.4) *are given in terms of μ_1 by* (A.11)-(A.15) *in Appendix A.*

Remark 3.1.

1. From (3.13), the initial stock price is

$$S_0 = g_0(0) + g(0)\tilde{a}_\Sigma + \sigma_{SW}(0)w_0 + D_0. \tag{3.15}$$

   Therefore, whenever the solution function $g(t)$ in (3.14) satisfies $g(0) \neq 0$ we have $\sigma(S_0) = \sigma(\tilde{a}_\Sigma)$. This identity is behind the measurability properties (2.7) and (2.8) which means that the investors can infer $\tilde{a}_\Sigma$ from the initial price S_0 (recall that w_0 and D_0 are constants). From the ODE for g in (3.14), a sufficient condition for the property $g(0) \neq 0$ is $\gamma(t) > 0$ and $\kappa(t) > 0$ for some $t \in (0,1]$, and $\varphi_0 \geq 0$.

2. The terminal ODE values in (3.14) produce the terminal stock price (3.6) at $t = 1$. Because the ODEs in (3.14) are linear, their solutions are unique. Consequently, there is a unique function $\sigma_{SW}(t)$ that produces (3.6).

3. The difference $S_t - D_t$ is the price effect of imbalances in liquidity supply and demand. We call this the *liquidity premium* and note that it can be positive or negative depending on whether the noise traders are buying or selling. From (3.13), liquidity has a deterministic component $g_0(t)$ (due to predictable imbalances from the noise traders), and effects due to the net investor imbalance $\tilde{a}_\Sigma$ and the current noise-trader imbalance w_t.

4. Investor holdings in (3.12) have an intuitive structure. First, strategic investors share the noise-trader order imbalance w_t equally. Second, over time there is some persistence in imbalances in the strategic investors' initial holdings. In particular, given (3.10) and $\gamma(t) < 1$, the coefficient $\frac{2\kappa(t)(1-\gamma(t))}{2\kappa(t)-\mu_1(t)}$ is positive. Third, the coefficient $\frac{2\kappa(t)\gamma(t)}{2\kappa(t)-\mu_1(t)}$ is positive (given the second order condition (3.10)) and so an investor with an above-average target $\tilde{a}_i > \tilde{a}_\Sigma/M$ holds more of the stock.

5. Investor i places a discrete order at time $t = 0$ (but trades continuously thereafter). From (3.12), investor i's initial trade is

$$\hat{\theta}_{i,0} - \theta_{i,-} = \frac{2\kappa(0)\gamma(0)}{2\kappa(0) - \mu_1(0)}\Big(\tilde{a}_i - \frac{\tilde{a}_\Sigma}{M}\Big) + \frac{2\kappa(0)\big(1 - \gamma(0)\big)}{2\kappa(0) - \mu_1(0)}\Big(\theta_{i,-} - \frac{w_0}{M}\Big), \quad (3.16)$$

which is non-zero when $\tilde{a}_i \neq \tilde{a}_\Sigma/M$ and/or $\theta_{i,-} \neq w_0/M$.

6. From (3.13), the equilibrium price dynamics are given by

$$dS_t = \Big(g_0'(t) + g'(t)\tilde{a}_\Sigma + \sigma_{SW}'(t)w_t\Big)dt + \sigma_{SW}(t)dw_t + dD_t, \quad (3.17)$$

where the drift of S is given by

$$\begin{aligned}\mathrm{drift}(S)_t &= g_0'(t) + g'(t)\tilde{a}_\Sigma + \sigma_{SW}'(t)w_t + \sigma_{SW}(t)(\alpha - \pi w_t)\\ &= \frac{2\kappa(t)-\mu_1(t)}{M}w_t + \frac{2\kappa(t)\big(\gamma(t)-1\big)}{M}w_0 - \frac{2\kappa(t)\gamma(t)}{M}\tilde{a}_\Sigma \\ &=: \hat{\mu}_t.\end{aligned} \tag{3.18}$$

The second equality in (3.18) follows from substitution of (3.14) into the first line of (3.18). The quadratic variation of S, and the quadratic cross-variations between S and (D, w) are given by

$$d\langle S\rangle_t = \big(\sigma_{SW}^2(t)\sigma_w^2 + 1\big)dt, \quad d\langle S, D\rangle_t = dt, \quad d\langle S, w\rangle_t = \sigma_{SW}(t)\sigma_w^2 dt. \tag{3.19}$$

7. In the case of an arithmetic Brownian motion imbalance process ($\pi := 0$) and equal initial sharing $\theta_{i,-} := \frac{1}{M}w_0$ in (3.12), we get the conditional expectation

$$\mathbb{E}[\hat{\theta}_{i,t}|\sigma(\tilde{a}_i, \tilde{a}_\Sigma)] = \frac{2\kappa(t)\gamma(t)}{2\kappa(t)-\mu_1(t)}\Big(\tilde{a}_i - \frac{\tilde{a}_\Sigma}{M}\Big) + \frac{1}{M}(w_0 + \alpha t). \tag{3.20}$$

The conditional variance is given by

$$\mathbb{V}[\hat{\theta}_{i,t}|\sigma(\tilde{a}_i, \tilde{a}_\Sigma)] = \frac{1}{M^2}\sigma_w^2 t, \tag{3.21}$$

which is independent of the private target $\tilde{a}_i$.

8. The presence of w_t in (3.12) prevents the $\hat{\theta}_i$ holding paths from being differentiable. Consequently, there is no dt-rate at which buying and selling occur. However, when κ, γ, and μ_1 are smooth functions, the equilibrium holding paths have Itô dynamics

$$\begin{aligned}d\hat{\theta}_{i,t} = &\left(\frac{2\kappa(t)\gamma(t)}{2\kappa(t)-\mu_1(t)}\right)'\Big(\tilde{a}_i - \frac{\tilde{a}_\Sigma}{M}\Big)dt + \frac{1}{M}\Big((\alpha - \pi w_t)dt + \sigma_w dW_t\Big)\\ &+ \left(\frac{2\kappa(t)\big(1-\gamma(t)\big)}{2\kappa(t)-\mu_1(t)}\right)'\Big(\theta_{i,-} - \frac{w_0}{M}\Big)dt.\end{aligned} \tag{3.22}$$

Consequently, when $\pi := 0$ and $\theta_{i,-} := \frac{w_0}{M}$, the drift in investor i's holdings is

$$\left(\frac{2\kappa(t)\gamma(t)}{2\kappa(t)-\mu_1(t)}\right)'\left(\tilde{a}_i - \frac{\tilde{a}_\Sigma}{M}\right) + \frac{\alpha}{M}, \tag{3.23}$$

which is a deterministic function of investor i's private target $\tilde{a}_i$ and the public aggregate variable $\tilde{a}_\Sigma$ defined in (2.6). Additionally, in the Radner equilibrium in Example 3.2 where $\mu_1 := 0$, the drift (3.23) of $\hat{\theta}_{i,t}$ simplifies even further to

$$\gamma'(t)\left(\tilde{a}_i - \frac{\tilde{a}_\Sigma}{M}\right) + \frac{\alpha}{M}. \tag{3.24}$$

4 Welfare analysis

This section examines the welfare-maximizing equilibrium in Example 3.3 and provides an objective to determine the associated function $\mu_1(t)$. The following result guarantees the existence of a welfare maximizer $\mu_1^*(t)$. Furthermore, its proof shows that $\mu_1^*(t)$ is linear in the value of $\kappa(t)$ controlling the penalty severity in (2.2).

Theorem 4.1. *We assume that $\mathbb{E}[\tilde{a}_i^2] < \infty$ for $i \in \{1, ..., M\}$. For the parameters*

$$\pi := 0, \quad \theta_{i,-} := \frac{w_0}{M},$$

and under the two parameter restrictions

$$4\gamma(t)^2\left(\mathbb{E}[\tilde{a}_\Sigma^2] - M\sum_{i=1}^M \mathbb{E}[\tilde{a}_i^2]\right) < t(\sigma_w^2 + \alpha^2 t + \alpha w_0) < 0, \quad t \in (0,1], \tag{4.1}$$

there exists a unique continuous price-impact function $\mu_1^*(t) \in \big(0, \kappa(t)\big)$, $t \in (0,1]$, which attains

$$\sup_{\mu_1(t)} \sum_{i=1}^M \mathbb{E}[CE_i], \tag{4.2}$$

where the supremum is taken over all continuous functions $\mu_1 : [0,1] \to \mathbb{R}$ satisfying the second-order condition (3.10)*. Furthermore, the maximizer $\mu_1^*(t)$ is linear in $\kappa(t)$.*

The two restrictions in (4.1) are sufficient conditions for a maximizer $\mu_1^*(t)$ to exist. The first inequality ensures that $\mu_1^*(t)$ stays strictly below $\kappa(t)$ which is needed for the second-order condition (3.10). The second inequality in (4.1) is a coercivity condition which ensures that very negative values of $\mu_1^*(t)$ can never be optimal. While the standard TWAP target trajectory $\gamma(t) := t$ is included in Theorem 3.4, the first restriction in (4.1) prevents it from being included in Theorem 4.1.

5 Numerics

This section compares model outcomes of the welfare-maximizing equilibrium (see Example 3.3 and Theorem 4.1) and the fully competitive Radner equilibrium (see Example 3.2). The objects of interest are first, the welfare-maximizing price-impact function $\mu_1(t)$; second, properties of the equilibrium prices S_t that equate the aggregate strategic investor demand $\sum_{i=1}^{M} \theta_{i,t}$ with the available inelastic supply w_t from the noise traders; third, how the smart TWAP traders and the HFT market makers share the available supply given their individual target holdings $\tilde{a}_i$; and four, welfare.

Our analysis uses the terminal stock price restriction (2.1) with an initial dividend factor $D_0 := 0$ and $M := 10$ strategic investors. For the strategic investors' private information variables, we use

$$\theta_{i,-}^{(0)} := 0, \quad \theta_{i,-} := \frac{w_0}{M}, \quad \tilde{a}_i \perp \tilde{a}_j \text{ for } i \neq j, \quad \mathbb{E}[\tilde{a}_i] = 0, \quad \mathbb{E}[\tilde{a}_i^2] = 1. \tag{5.1}$$

Under these assumptions, the aggregate variable $\tilde{a}_\Sigma$ in (2.5) has the moments

$$\mathbb{E}[\tilde{a}_\Sigma] = 0, \quad \mathbb{E}[\tilde{a}_\Sigma^2] = M. \tag{5.2}$$

For the dynamics of the noise-trader process $w = (w_t)_{t\in[0,1]}$ in (2.4), we use the parameter values

$$w_0 := 10, \quad \alpha := -1, \quad \pi := 0, \quad \sigma_w := 1. \tag{5.3}$$

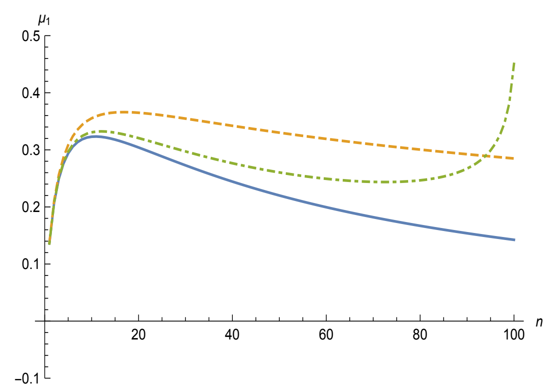
Finally, in the penalty process $L_{i,t}$ in (2.2), we use the deterministic functions

$$\kappa_1(t) := 1, \quad \kappa_2(t) := 1 + t, \quad \kappa_3(t) := \frac{1}{(1-t)^{0.25}}, \quad \gamma(t) := 0.1 + 0.9t, \tag{5.4}$$

for $t \in [0, 1]$. The target ratio function $\gamma(t)$ in (5.4) can be interpreted as a modified TWAP target trajectory in which traders are initially impatient to get part of their trading done quickly, but then become more patient during the rest of the day.

Figure 1 below shows the welfare-maximizing function μ_1^* for three different penalty-severity functions $\kappa(t)$. Comparing $\mu_1^*(t)$ for penalties $\kappa_1(t)$ and $\kappa_2(t)$, we note that the stronger the penalty, the larger is the welfare-maximizing μ_1^* function. This suggests a reason for why the welfare-maximizing equilibrium differs from the competitive equilibrium. In the competitive equilibrium with $\mu_1 = 0$, the strategic investors act like price-takers. Put differently, they act as if there is infinite liquidity. However, with only M strategic investors, liquidity is actually limited rather than infinite. The welfare-maximizing equilibrium forces investors to recognize that liquidity is finite via a positive personal price-impact function $\mu_1^*(t)$. As the penalty-severity function $\kappa(t)$ increases, investors, all things the same, want to trade more aggressively on their own personal targets and, hence, become less willing to provide liquidity to other investors. Thus, the welfare-maximizing price-impact function $\mu_1^*(t)$ increases to make investors recognize the reduced liquidity available in the market. The positive slopes of $\kappa_2(t)$ and $\kappa_3(t)$ in (5.4) mean that the penalty intensity is greater later in the day. As a result, we see that the welfare-maximizing $\mu_1^*(t)$ gets larger later in the day, the steeper the slopes are of the penalty-severity function. This effect is most apparent for $\kappa_3(t)$ which explodes toward the end of the day.

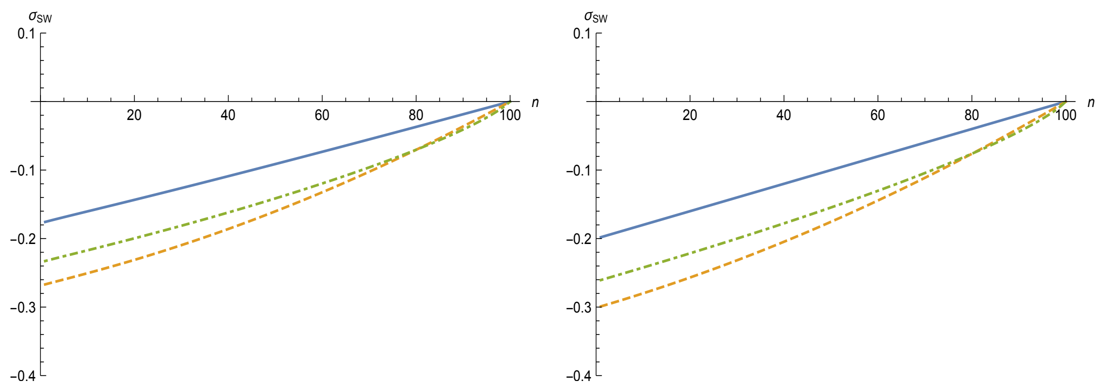
Figure 1: Plot of welfare-maximizing $\mu_1^*(t)$ for the objective (4.2). The parameters are given by (5.1)-(5.4), and the discretization uses 100 trading rounds.



κ_1 (———), κ_2 (— — —), κ_3 (— · — · —).

Our second topic is pricing. Figure 2 shows the price-loading function $\sigma_{SW}(t)$ in (3.13). The sign of $\sigma_{SW}(t)$ is negative because a larger value of w_t means that the strategic investors need to buy more (i.e., our sign convention is that w_t is the amount that noise traders want to sell). The greater the penalty severity $\kappa(t)$ is, the more sensitive prices are to shocks from the amount w_t that the strategic investors must absorb to accommodate the inelastic trades from the noise traders. For example, larger imbalances w_t depress prices more (in order to induce the strategic traders to buy), and the amount prices need to be depressed is larger, the greater the penalty is for the strategic traders to deviate from their intraday target trading trajectory.

Figure 2: Plot of equilibrium price loading $\sigma_{SW}(t)$ on noise-trader imbalance in the maximizing-welfare equilibrium (Plot A) and in the competitive Radner equilibrium with $\mu_1 := 0$ (Plot B). The parameters are given by (5.1)-(5.4) and the discretization uses 100 trading rounds.

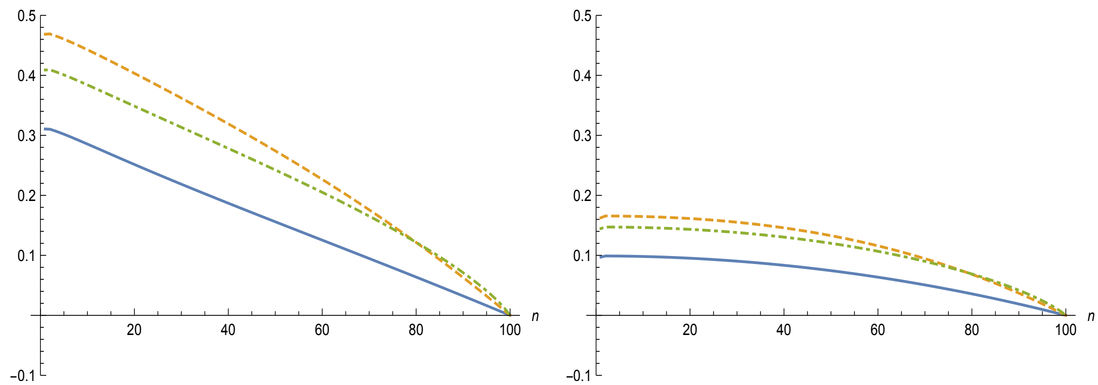


A: [Welfare] κ_1 (———), κ_2 $(-\ -\ -)$, κ_3 $(- \cdot - \cdot -)$.

B: [Radner] κ_1 (———), κ_2 $(-\ -\ -)$, κ_3 $(- \cdot - \cdot -)$.

The liquidity-premium $S_t - D_t$ is the impact of liquidity and order-flow imbalances on prices. Figure 3 shows the expected liquidity premium over the trading day, where the expectation is taken over the noise-trader imbalance w_t paths. The expected liquidity premium is positive here because it is common knowledge in this numerical example that the noise traders will be buying over the course of the day (i.e., the drift α of the available supply w_t of stock for the strategic investors to own is negative) whereas the strategic investors are on average content with their initial positions $\sum_{i=1}^{M} \theta_{i,-} = \tilde{a}_\Sigma = 10$. As is intuitive, the expected premium is larger when the penalty severity $\kappa(t)$ is greater since the strategic investors require more compensation (i.e., larger price discounts for buying and price premiums when selling) for deviating from their target trajectory. However, as the end of the day approaches, the terminal price constraint (2.1) forces the expected liquidity premium to converge to 0. In particular, this is even true for the exploding penalty $\kappa_3(t)$.

Figure 3: Plots of the expected liquidity premium $\mathbb{E}[S_t - D_t|\sigma(\tilde{a}_\Sigma)]$ with the welfare-maximizing $\mu_1^*(t)$ (Plot A) and the competitive equilibrium with $\mu_1 := 0$ (Plot B). The parameters are given by (5.1)-(5.4), and the discretization uses 100 trading rounds.

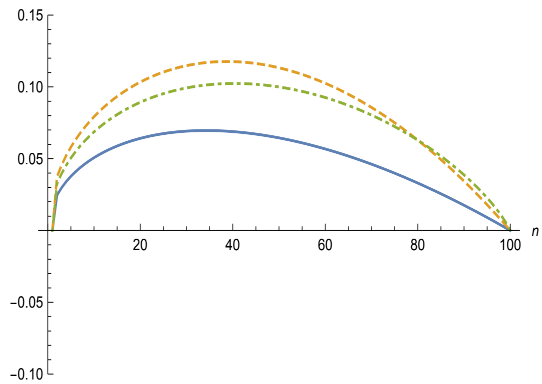


A: [Welfare] κ_1 (———), κ_2 (− − −), κ_3 (− · − · −).

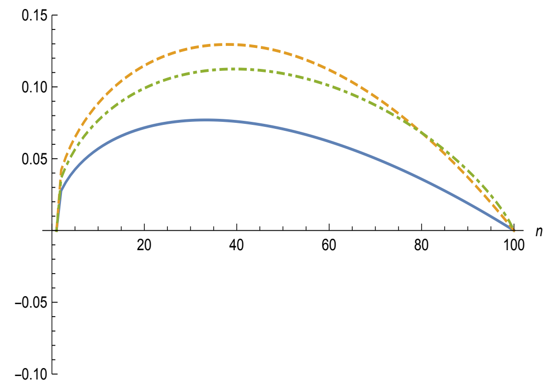
B: [Radner] κ_1 (———), κ_2 (− − −), κ_3 (− · − · −).

Figure 4 shows the volatility of the intraday liquidity premium induced by the random noise-trader imbalances. Initially, as expected, when the penalty severity $\kappa(t)$ is greater, prices need to move more to compensate investors for deviating from their target trajectories, which magnifies the effect of randomness in the noise-trader imbalances w_t. The liquidity premium volatility initially increases due to the growing volatility of w_t, but eventually the terminal price condition (2.1) causes the liquidity premium volatility to converge to zero.

Figure 4: Plots of the liquidity-premium volatility $\mathrm{SD}[S_t - D_t|\sigma(\tilde{a}_\Sigma)]$ with the welfare-maximizing $\mu_1^*(t)$ (Plot A) and the competitive equilibrium with $\mu_1 := 0$ (Plot B). The parameters are given by (5.1)-(5.4), and the discretization uses 100 trading rounds.



A: [Welfare] κ_1 (———), κ_2 (− − −), κ_3 (− · − · −).



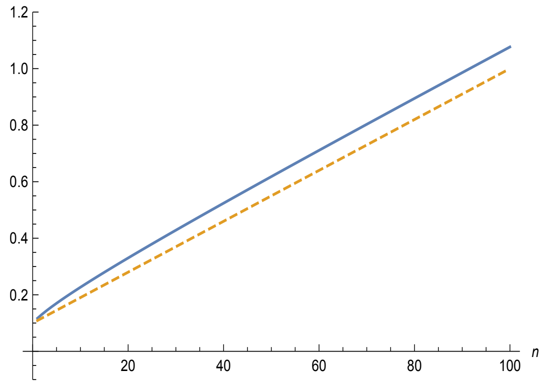
B: [Radner] κ_1 (———), κ_2 (− − −), κ_3 (− · − · −).

Our third topic is the strategic investor holdings. Their aggregate holdings are, in equilibrium, constrained by market clearing to equal the inelastic supply w_t from the noise traders. However, there is heterogeneity in individual investors' holdings given imbalances in their initial holdings $\theta_{i,-}$ and differences in their trading targets $\tilde{a}_i$. Figure 5 shows numerical values for the two coefficients in the strategic trader holdings in equation (3.12). Plot 5A shows the magnitude of the difference between the sensitivity of investor holdings to relative target imbalances in the two equilibria. As is intuitive, the difference increases over time as the penalty for target deviations increases. Plot 5B shows the corresponding difference for imbalances in the initial investor holdings. As expected, this difference decreases over time. Theorem 4.1 ensures that all plots in Figure 5 are independent of the severity function.

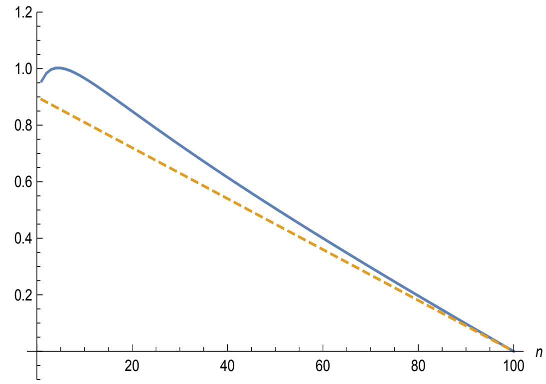
Next, we turn to investor i's expected trades. By combining (3.20) with the initial position $\theta_{i,-} := \frac{w_0}{M}$ from (5.3) we get

$$\mathbb{E}[\hat{\theta}_{i,t}|\sigma(\tilde{a}_i, \tilde{a}_\Sigma)] - \theta_{i,-} = \frac{2\kappa(t)\gamma(t)}{2\kappa(t) - \mu_1(t)}\Big(\tilde{a}_i - \frac{\tilde{a}_\Sigma}{M}\Big) + \frac{\alpha t}{M}. \tag{5.5}$$

Figure 5: Plots of $\frac{2\gamma(t)\kappa(t)}{2\kappa(t)-\mu_1(t)}$ (Plot A) and $\frac{2\kappa(t)(1-\gamma(t))}{2\kappa(t)-\mu_1(t)}$ (Plot B). The parameters are given by (5.1)-(5.4), and the discretization uses 100 trading rounds.



A: [Welfare] (———)
[Radner] (- - - -)



B: [Welfare] (———),
[Radner] (- - - -)

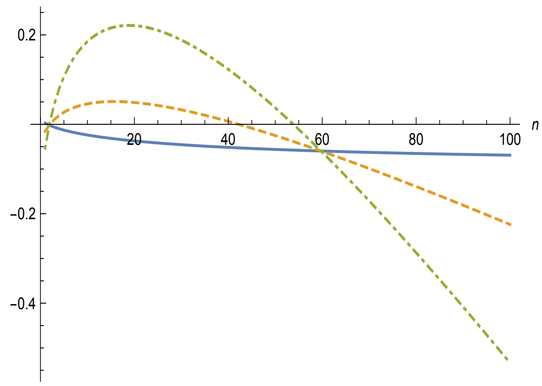
Consequently, trader i expects to deviate from the target holding path according to

$$\begin{aligned}\mathbb{E}[\hat{\theta}_{i,t}|\sigma(\tilde{a}_i,\tilde{a}_\Sigma)] - \Big(\theta_{i,-} + \gamma(t)\big(\tilde{a}_i - \hat{\theta}_{i,-}\big)\Big)\\ = \frac{2\kappa(t)\gamma(t)}{2\kappa(t)-\mu_1(t)}\Big(\tilde{a}_i - \frac{\tilde{a}_\Sigma}{M}\Big) + \frac{\alpha t}{M} - \gamma(t)\big(\tilde{a}_i - \frac{w_0}{M}\big).\end{aligned} \tag{5.6}$$

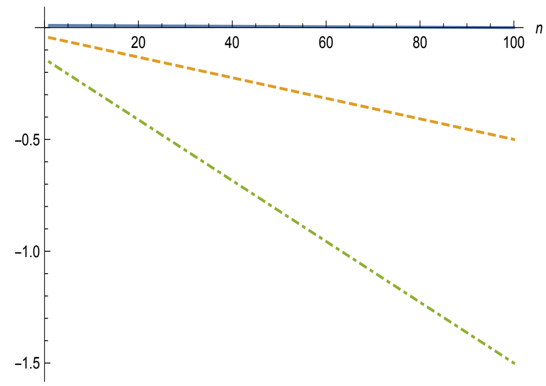
In the fully competitive equilibrium from Example 3.2 where $\mu_1 := 0$, the difference (5.6) does not depend on the target $\tilde{a}_i$ when $\tilde{a}_\Sigma$ is fixed and does not depend on the severity $\kappa(t)$ of the penalty. Remarkably, for the welfare-maximizing function $\mu_1^*(t)$, Theorem 4.1 ensures that the difference (5.6) also remains independent of $\kappa(t)$. Figure 6 shows the expected deviation between a strategic investor's cumulative trading up through time t and their corresponding target trading. In this figure, we change the target $\tilde{a}_i$ of a particular individual investor i while holding the targets of the other $M-1=9$ investors fixed at $\tilde{a}_j := 1$. Thus, both $\tilde{a}_i$ and $\tilde{a}_\Sigma = \tilde{a}_i + 9$ change in these plots. The figure shows that if investor i wants to hold a large target quantity (e.g., $\tilde{a}_i = 5$ or 15), she trades ahead of her target early in the day but then eventually falls behind.

Lastly, we turn to the sub-optimality of the choice of $\mu_1(t) := 0$ (Radner) relative to the welfare maximizer $\mu_1^*(t)$. Table 1 illustrates how the Radner equilibrium performs in the welfare objective (4.2).

Figure 6: Plots of expected trade deviation (5.6). The parameters are given by (5.1)-(5.4), $\tilde{a}_j := 1$ for $j \neq i$, $\tilde{a}_\Sigma := 9 + \tilde{a}_i$, and the discretization uses 100 trading rounds. In the competitive Radner equilibrium, the realization of $\tilde{a}_i$ is irrelevant.



A: [Welfare] $\tilde{a}_i := 0$ (———), $\tilde{a}_i := 5$ (− − −), $\tilde{a}_i := 15$ (− · − · −).



B: [Radner] $\tilde{a}_i := 0$ (———), $\tilde{a}_i := 5$ (− − −), $\tilde{a}_i := 15$ (− · − · −).

Table 1: Expected welfare objective (4.2). The parameters are given by (5.1)-(5.4) and $D_0 := 0$.

$\kappa(t)$	Welfare maximizer $\mu_1^*(t)$	Example 3.2 (Radner) $\mu_1(t) := 0$
1	-4.00155	-4.04341
$1+t$	-6.91678	-6.98449
$(1+t)^{-0.25}$	-6.28596	-6.34491

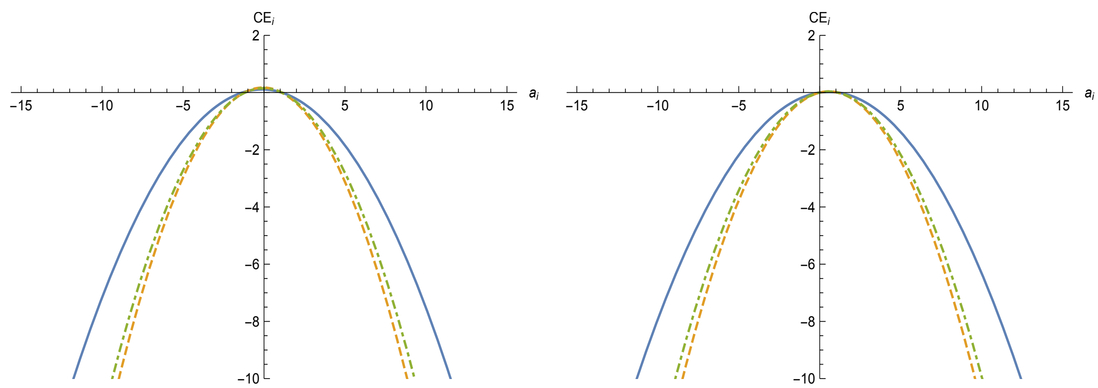
As discussed above, we conjecture that the welfare losses in the Radner equilibrium are due to failure of investors to recognize that liquidity is actually limited in equilibrium.

Figure 7 shows the certainty equivalents CE_i for the strategic investors as a function of their trading target $\tilde{a}_i$. The figure shows that there are meaningful differences in the welfare of investors in the Radner and welfare-maximizing equilibria when their targets are large. The figure also shows that the sensitivity of an investor's CE_i to their target $\tilde{a}_i$ is increasing in the penalty intensity $\kappa(t)$.

6 Extension to exponential utilities

Appendix B extends our model with linear preferences for the strategic investors as in (2.11) to exponential preferences. The analysis there shows that prices and stock

Figure 7: Plots of certainty equivalents for investor i with the welfare-maximizing $\mu_1^*(t)$ (Plot A) and the competitive equilibrium with $\mu_1 := 0$ (Plot B) seen as a function of the trading target $\tilde{a}_i$. The parameters are given by (5.1)-(5.4), $\tilde{a}_\Sigma := 10$, and the discretization uses 100 trading rounds.



A: [Welfare] κ_1 (———), κ_2 (— — —), κ_3 (— · — · —).

B: [Radner] κ_1 (———), κ_2 (— — —), κ_3 (— · — · —).

holdings are linear and that there are again an infinite number of equilibria associated with different price-impact functions $\mu_1(t)$. One new feature of our exponential preference model is that it distinguishes trading risk aversion — as reflected by the penalty severity $\kappa(t)$ for divergences from the target trading trajectory — and general risk aversion to both wealth and trading risk — as captured by the exponential risk tolerance parameter $\tau > 0$.

7 Conclusion

This paper has solved for continuous-time equilibria with endogenous liquidity provision and intraday trading targets. We show how intraday target trajectories in trading induce intraday patterns in investor positions and in the liquidity premium in prices. There are also potential extensions of our model. For example, it would be interesting to extend the model to allow for heterogeneity in the strategic investors' $\gamma(t)$ and $\kappa(t)$ penalty functions. Perhaps the most pertinent extension would be to allow for randomness in w_0 appearing in (2.4) in which case the initial equilibrium stock price S_0 cannot fully reveal the aggregate target $\tilde{a}_\Sigma$.

A Proofs

We start with a technical lemma, which will be used in the proof of Theorem 3.4. The arguments used in lemma's proof are standard and can be found in, e.g., Chapter 7 in Lipster and Shiryeav (2001) as well as in the appendix of Cheridito, Filipović, and Kimmel (2007). We include the lemma for completeness.

Lemma A.1. *Let the functions* γ, κ, μ_1*, and* σ_{SW} *be as in Theorem 3.4. The strictly positive local martingale* $N = (N_t)_{t\in[0,1]}$ *defined by*

$$N_t := e^{-\int_0^t \lambda_u dZ_u - \frac{1}{2}\int_0^t \lambda_t^2 dt}, \quad t \in [0,1], \tag{A.1}$$

is a martingale with respect to the filtration $\sigma(\tilde{a}_\Sigma, D_u, W_u)_{u\in[0,t]}$ *where* $\hat{\mu}$ *is defined by* (3.18) *and*

$$\lambda_t := \frac{\hat{\mu}_t}{\sqrt{\sigma_{SW}(t)^2+1}}, \tag{A.2}$$

$$dZ_t := \frac{\sigma_{SW}(t)\sigma_w dW_t + dD_t}{\sqrt{\sigma_{SW}(t)^2\sigma_w^2+1}}, \quad Z_0 := 0. \tag{A.3}$$

Proof. For $(t,x,a) \in [0,1]\times\mathbb{R}^2$ we start by defining the linear function

$$H(t,x,a) := \frac{2\kappa(t)-\mu_1(t)}{M}x + \frac{2\kappa(t)\big(\gamma(t)-1\big)}{M}w_0 - \frac{2\kappa(t)\gamma(t)}{M}a, \tag{A.4}$$

and then we note that $\hat{\mu}_t = H(t, w_t, \tilde{a}_\Sigma)$ from (3.18). We define the process

$$dv_t := (\alpha - \pi v_t)dt + \sigma_w dW_t - \frac{H(t, v_t, \tilde{a}_\Sigma)}{\sigma_{SW}(t)^2\sigma_w^2+1}\sigma_{SW}(t)\sigma_w^2 dt, \quad v_0 := w_0, \tag{A.5}$$

which is an Ornstein-Uhlenbeck process (Gaussian). Inserting H from (A.4) into (A.5) produces the various drift-coefficient functions in dv_t to be

$$\begin{aligned} \text{constant}: &\quad \alpha - \frac{2\kappa(t)\big(\gamma(t)-1\big)}{M}\frac{\sigma_{SW}(t)\sigma_w^2}{\sigma_{SW}(t)^2\sigma_w^2+1}w_0,\\ v_t: &\quad -\frac{2\kappa(t)-\mu_1(t)}{M}\frac{\sigma_{SW}(t)\sigma_w^2}{\sigma_{SW}(t)^2\sigma_w^2+1} - \pi,\\ \tilde{a}_\Sigma: &\quad \frac{2\kappa(t)\gamma(t)}{M}\frac{\sigma_{SW}(t)\sigma_w^2}{\sigma_{SW}(t)^2\sigma_w^2+1}. \end{aligned} \tag{A.6}$$

Because these deterministic coefficient functions (A.6) are integrable (indeed, the functions are square integrable), the SDE (A.5) has a (unique) non-exploding strong solution.

For $n \in \mathbb{N}$, we define the stopping times

$$\tau_n^w := \inf\{t > 0 : \int_0^t H(t, w_s, \tilde{a}_\Sigma)^2 ds \geq n\} \wedge 1, \tag{A.7}$$

$$\tau_n^v := \inf\{t > 0 : \int_0^t H(s, v_s, \tilde{a}_\Sigma)^2 ds \geq n\} \wedge 1. \tag{A.8}$$

Because the functions (μ_1, κ) are assumed to be square integrable, we have

$$\lim_{n\to\infty} \mathbb{P}(\tau_n^v = 1) = 1.$$

By Novikov's condition, the processes $N_{t\wedge\tau_n^w}$ are martingales for $n \in \mathbb{N}$, and so we can define on $\mathcal{F}_{\tau_n^w}$ the $\mathbb{P}$-equivalent probability measure $\mathbb{Q}^{(n)}$ by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{(n)}}{d\mathbb{P}} := N_{\tau_n^w}. \tag{A.9}$$

For each $n \in \mathbb{N}$, Girsanov's theorem produces a $\mathbb{Q}^{(n)}$ Brownian motion $W_t^{(n)}$ such that

$$dw_t = (\alpha - \pi w_t)dt + \sigma_w dW_t^{(n)} - \frac{H(t, w_t, \tilde{a}_\Sigma)}{\sigma_{SW}(t)^2\sigma_w^2 + 1}\sigma_{SW}(t)\sigma_w^2 dt, \quad t \in [0, \tau_n^w]. \tag{A.10}$$

Because

$$\mathbb{Q}^{(n)}(\tau_n^w \leq x) = \mathbb{P}(\tau_n^v \leq x), \quad x > 0.$$

we have

$$\mathbb{E}[N_1] = \lim_{n\to\infty} \mathbb{E}[N_{\tau_n^w} 1_{\tau_n^w = 1}] = \lim_{n\to\infty} \mathbb{Q}^{(n)}(\tau_n^w = 1) = \lim_{n\to\infty} \mathbb{P}(\tau_n^v = 1) = 1.$$

Consequently, N defined in (A.1) is a positive supermartingale with constant expectation and is therefore also a martingale.

$\diamondsuit$

Proof of Theorem 3.4: We conjecture (and verify) the following equilibrium drift functions in (3.4) defined in terms of a continuous function $\mu_1(t)$ satisfying (3.10):

$$\mu_0(t) := \frac{4\kappa(t)\gamma(t)\big(\mu_1(t)-\kappa(t)\big)}{M\big(2\kappa(t)-\mu_1(t)\big)}, \tag{A.11}$$

$$\mu_2(t) := -\frac{2\kappa(t)\gamma(t)\mu_1(t)}{2\kappa(t)-\mu_1(t)}, \tag{A.12}$$

$$\mu_3(t) := \frac{2\big(\kappa(t)-\mu_1(t)\big)}{M}, \tag{A.13}$$

$$\mu_4(t) := \frac{4\big(\gamma(t)-1\big)\kappa(t)\big(\kappa(t)-\mu_1(t)\big)}{M\big(2\kappa(t)-\mu_1(t)\big)}, \tag{A.14}$$

$$\mu_5(t) := \frac{2\big(\gamma(t)-1\big)\kappa(t)\mu_1(t)}{2\kappa(t)-\mu_1(t)}. \tag{A.15}$$

We split the proof into two steps.

Step 1 (individual optimality): Given the price-impact function $\mu_1(t)$ and the conjectured associated functions (A.11)-(A.15) for the price-impact relation (3.3), we derive the individual investor's value function V for the maximization problem (2.11) as well as the corresponding optimal control $\hat{\theta}_{i,t}$. To this end, for $a_i, a_\Sigma, X_i, w \in \mathbb{R}$, $t \in [0,1]$, and $L_i \geq 0$, we define the quadratic function

$$\begin{aligned} V(t,&X_i, w, L_i, a_i, a_\Sigma) \\ &:= X_i - L_i - \Big(\beta_0(t) + \beta_1(t)a_i^2 + \beta_2(t)a_i a_\Sigma + \beta_3(t)a_\Sigma^2 + \beta_4(t)w^2 \\ &\quad + \beta_5(t)wa_i + \beta_6(t)a_\Sigma w + \beta_7(t)w + \beta_8(t)a_i + \beta_9(t)a_\Sigma\Big), \end{aligned} \tag{A.16}$$

where the deterministic coefficient functions $(\beta_j)_{j=0}^{9}$ are given by the ODEs

$$\begin{aligned}
\beta_0' &= -\alpha\beta_7 - \beta_4\sigma_w^2 + \frac{(\gamma-1)^2\kappa^2(\kappa-\mu_1)(4w_0^2 - 8Mw_0\theta_{i,-}) - M^2(\gamma-1)^2\theta_{i,-}^2\kappa\mu_1^2}{M^2(\mu_1-2\kappa)^2},\\
\beta_1' &= -\frac{\gamma^2\kappa\mu_1^2}{(\mu_1-2\kappa)^2},\\
\beta_2' &= \frac{8\gamma^2\kappa^2(\mu_1-\kappa)}{M(\mu_1-2\kappa)^2},\\
\beta_3' &= \frac{4\gamma^2\kappa^2(\kappa-\mu_1)}{M^2(\mu_1-2\kappa)^2},\\
\beta_4' &= \frac{\kappa-\mu_1}{M^2} + 2\beta_4\pi,\\
\beta_5' &= \frac{4\gamma\kappa(\kappa-\mu_1)}{M(2\kappa-\mu_1)} + \beta_5\pi,\\
\beta_6' &= \frac{4\gamma\kappa(\mu_1-\kappa)}{M^2(2\kappa-\mu_1)} + \beta_6\pi,\\
\beta_7' &= -2\alpha\beta_4 + \beta_7\pi + \frac{4(\gamma-1)(w_0 - M\theta_{i,-})\kappa(\kappa-\mu_1)}{M^2(2\kappa-\mu_1)},\\
\beta_8' &= -\alpha\beta_5 + \frac{2(\gamma-1)\gamma\kappa(4w_0\kappa(\kappa-\mu_1) + M\theta_{i,-}\mu_1^2)}{M(\mu_1-2\kappa)^2},\\
\beta_9' &= -\alpha\beta_6 + \frac{8(\gamma-1)\gamma\kappa^2(M\theta_{i,-} - w_0)(\kappa-\mu_1)}{M^2(\mu_1-2\kappa)^2},
\end{aligned} \tag{A.17}$$

together with the terminal conditions $\beta_j(1) = 0$ for $j \in \{0, ..., 9\}$. We start by showing that V is investor i's value function. The terminal conditions for the ODEs describing $(\beta_j)_{j=0}^{9}$ produce the terminal condition

$$V(1, X_i, w, L_i, a_i, a_\Sigma) = X_i - L_i, \quad a_i, a_\Sigma, X_i, w \in \mathbb{R}, L_i \geq 0. \tag{A.18}$$

For an arbitrary strategy $\theta_i \in \mathcal{A}_i$, Itô's lemma produces the dynamics

$$\begin{aligned}
dV &= V_t dt + V_X\theta_{i,t}(\mu_{i,t}dt + \sigma_{SW}dW_t + dD_t) + \frac{1}{2}V_{XX}\theta_{i,t}^2(\sigma_{SW}^2 + 1)dt\\
&\quad + V_{Xw}\theta_{i,t}\sigma_{SW}\sigma_w dt + V_w\big((\alpha - \pi w_t)dt + \sigma_w dW_t\big) + \frac{1}{2}V_{ww}\sigma_w^2 dt\\
&\quad + V_L\kappa(t)\big(\gamma(t)(\tilde{a}_i - \theta_{i,-}) - (\theta_{i,t} - \theta_{i,-})\big)^2 dt\\
&\leq V_X\theta_{i,t}(\sigma_{SW}dW_t + dD_t) + V_w\sigma_w dW_t.
\end{aligned} \tag{A.19}$$

The inequality in (A.19) is from the HJB-equation

$$
\begin{aligned}
0 = \sup_{\theta_{i,t}\in\mathbb{R}} \Big(& V_t + V_X\theta_{i,t}\mu_{i,t} + \frac{1}{2}V_{XX}\theta_{i,t}^2(\sigma_{SW}^2+1) + V_{Xw}\theta_{i,t}\sigma_{SW}\sigma_w \\
& + V_w(\alpha-\pi w_t) + \frac{1}{2}V_{ww}\sigma_w^2 + V_L\kappa(t)\big(\gamma(t)(\tilde{a}_i-\theta_{i,-}) - (\theta_{i,t}-\theta_{i,-})\big)^2\Big),
\end{aligned}
\tag{A.20}
$$

which the function V defined in (A.16) satisfies. In integral form, (A.19) reads

$$
\begin{aligned}
& V(1, X_{i,1}, w_1, L_{i,1}, \tilde{a}_i, \tilde{a}_\Sigma) - V(0, X_{i,0}, w_0, L_{i,0}, \tilde{a}_i, \tilde{a}_\Sigma) \\
& \qquad \leq \int_0^1 \Big(V_X\theta_{i,t}(\sigma_{SW}dW_t + dD_t) + V_w\sigma_w dW_t\Big).
\end{aligned}
\tag{A.21}
$$

To see that the Brownian integral (which is always a local martingale) on the right-hand-side in (A.21) is indeed a martingale, we first compute the partial derivatives of V defined in (A.16). These derivatives are

$$
V_X = 1, \quad V_w = -\big(2\beta_4 w + \tilde{a}_i\beta_5 + \tilde{a}_\Sigma\beta_6 + \beta_7\big). \tag{A.22}
$$

Because the coefficient functions β_j are bounded, the integrability condition (2.10) in the definition of the admissible set $\mathcal{A}_i$ (see Definition 2.1) ensures the needed martingality. Consequently, the terminal condition (A.18) and the inequality in (A.21) produce

$$
\mathbb{E}[X_{i,1} - L_{i,1}] = \mathbb{E}[V(1, X_{i,1}, w_1, L_{i,1}, \tilde{a}_i, \tilde{a}_\Sigma)] \leq V(0, X_{i,0}, w_0, L_{i,0}, \tilde{a}_i, \tilde{a}_\Sigma). \tag{A.23}
$$

Therefore, because the right-hand side does not depend on $\theta_{i,t} \in \mathcal{A}_i$, we have

$$
\sup_{\theta_i\in\mathcal{A}_i} \mathbb{E}[X_{i,1} - L_{i,1}] \leq V(0, X_{i,0}, w_0, L_{i,0}, \tilde{a}_i, \tilde{a}_\Sigma). \tag{A.24}
$$

From this we see that V is an upper bound for the maximization problem (2.11). To get equality in (A.24), we show that $\hat{\theta}_{i,t}$ defined in (3.12) is optimal. To this end, we re-write (3.12) as

$$
\hat{\theta}_{i,t} = G_0(t)\tilde{a}_\Sigma + G_1(t)w_t + G_2(t)\tilde{a}_i + G_3(t)\theta_{i,-} + G_4(t)w_0, \tag{A.25}
$$

where we have defined the deterministic functions

$$\begin{aligned}
G_0(t) &:= -\frac{2\kappa(t)\gamma(t)}{2\kappa(t)-\mu_1(t)}\frac{1}{M},\\
G_1(t) &:= \frac{1}{M},\\
G_2(t) &:= \frac{2\kappa(t)\gamma(t)}{2\kappa(t)-\mu_1(t)},\\
G_3(t) &:= \frac{2\kappa(t)\big(1-\gamma(t)\big)}{2\kappa(t)-\mu_1(t)},\\
G_4(t) &:= -\frac{2\kappa(t)\big(1-\gamma(t)\big)}{2\kappa(t)-\mu_1(t)}\frac{1}{M}.
\end{aligned} \tag{A.26}$$

The second-order condition (3.10) comes from requiring negativity of the coefficient in front of $\theta_{i,t}^2$ in (A.20). Because $\mu_1(t)$ is assumed to satisfy (3.10), we see that $\hat{\theta}_{i,t}$ defined in (3.12) belongs to the admissible set $\mathcal{A}_i$ as defined in Definition (2.1). Furthermore, $\hat{\theta}_{i,t}$ produces equality in (A.20) and (A.21). Therefore, the upper bound (A.24) ensures that $\hat{\theta}_{i,t}$ is optimal.

Step 2 (equilibrium): This step of the proof establishes the equilibrium properties in Definition 3.1. We start with the clearing condition (2.3), which gives us the following three restrictions for the w_t-coefficients, the $\tilde{a}_\Sigma$-coefficients, and the constants:

$$1 = MG_1(t), \quad 0 = MG_0(t) + G_2(t), \quad 0 = G_3 + MG_4. \tag{A.27}$$

To ensure that the last restriction (iii) in Definition 3.1 holds, we define

$$\mu_t^* := \mu_0(t)\tilde{a}_\Sigma + \mu_1(t)\hat{\theta}_{i,t} + \mu_2(t)\tilde{a}_i + \mu_3(t)w_t + \mu_4(t)w_0 + \mu_5\theta_{i,-}. \tag{A.28}$$

The requirement in (iii) that the $\tilde{a}_i$ and $\theta_{i,-}$ coefficients in μ_t^* are zero in equilibrium can be stated as

$$0 = \mu_1(t)G_2(t) + \mu_2(t), \quad 0 = \mu_1 G_3 + \mu_5. \tag{A.29}$$

The formulas for μ_0, μ_2, μ_3, μ_4 and μ_5 in (A.11)-(A.15) ensure that the five requirements in (A.27) and (A.29) hold. In particular, by inserting $\hat{\theta}_{i,t}$ into μ_t^*, (A.28) becomes (3.18). Because the private information variables $(\tilde{a}_i, \theta_{i,-}, \theta_{i,-}^{(0)})$ do not appear in (3.18), we see that requirement (iii) in Definition 3.1 holds.

Finally, we need to establish the terminal condition (3.6). To this end, we need the ODEs for (g_0, g, σ_{SW}) in (3.14). Lemma A.1 above ensures that the minimal $\mathbb{P}$-equivalent martingale measure $\mathbb{Q}$ can be defined on $\sigma(\tilde{a}_\Sigma, w_u, D_u)_{u\in[0,1]}$ by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := e^{-\int_0^1 \lambda_u^2 dZ_u - \frac{1}{2}\int_0^1 \lambda_u^2 du},$$

where (λ, μ^*, Z) are defined by (3.18), (A.2), and (A.3). The $\mathbb{Q}$-dynamics of the $\mathbb{P}$-Brownian motions (D, W) can be found using Girsanov's theorem to be

$$\begin{aligned}
dD_t^{\mathbb{Q}} &:= dD_t + \frac{\mu_t^*}{\sigma_{SW}(t)^2\sigma_w^2 + 1}dt,\\
dW_t^{\mathbb{Q}} &:= dW_t + \frac{\mu_t^*}{\sigma_{SW}(t)^2\sigma_w^2 + 1}\sigma_{SW}(t)\sigma_w dt.
\end{aligned}$$

The $\mathbb{Q}$-dynamics of (D, w) then become

$$dD_t = dD_t^{\mathbb{Q}} - \frac{\mu_t^*}{\sigma_{SW}(t)^2\sigma_w^2 + 1}dt, \tag{A.30}$$

$$dw_t = (\alpha - \pi w_t)dt + \sigma_w dW_t^{\mathbb{Q}} - \frac{\mu_t^*}{\sigma_{SW}(t)^2\sigma_w^2 + 1}\sigma_{SW}(t)\sigma_w^2 dt. \tag{A.31}$$

These dynamics (A.30)-(A.31) ensure that the pair (D, w) remains a Markov process under $\mathbb{Q}$. We now have all the needed quantities to see

$$\mathbb{E}^{\mathbb{Q}}[D_1 + \varphi_0\tilde{a}_\Sigma + \varphi_1 w_1|\mathcal{F}_t] = g_0(t) + g(t)\tilde{a}_\Sigma + \sigma_{SW}(t)w_t + D_t, \quad t \in [0,1]. \tag{A.32}$$

The terminal conditions for the ODEs listed in (3.14) ensure that (A.32) holds for $t = 1$. Furthermore, the conditional expectation on the left-hand-side of (A.32) is a martingale under the minimal martingale measure $\mathbb{Q}$. Therefore, to see that (A.32) also holds for $t \in [0, 1)$, it suffices to show that the right-hand-side of (A.32) is a martingale under $\mathbb{Q}$. To this end, we apply Ito's lemma to the right-hand-side of (A.32) to produce the $\mathbb{P}$-dynamics

$$\begin{aligned}
&d\big(g_0(t) + g(t)\tilde{a}_\Sigma + \sigma_{SW}(t)w_t + D_t\big)\\
&\qquad = \Big(g_0'(t) + g'(t)\tilde{a}_\Sigma + \sigma_{SW}'(t)w_t\Big)dt + dD_t + \sigma_{SW}(t)dw_t.
\end{aligned}$$

The risk-neutral drift (i.e., the drift under the minimal martingale measure $\mathbb{Q}$) is given by

$$\begin{aligned}
&g_0'(t) + g'(t)\tilde{a}_\Sigma + \sigma_{SW}'(t)w_t - \frac{\mu_t^*}{\sigma_{SW}(t)^2\sigma_w^2 + 1} \\
&\quad + \Big(\alpha - \pi w_t - \frac{\mu_t^*}{\sigma_{SW}(t)^2\sigma_w^2 + 1}\sigma_{SW}(t)\sigma_w^2\Big)\sigma_{SW}(t) \\
&= g_0'(t) + g'(t)\tilde{a}_\Sigma + \sigma_{SW}'(t)w_t - \mu_t^* + (\alpha - \pi w_t)\sigma_{SW}(t) \\
&= g_0'(t) + g'(t)\tilde{a}_\Sigma + \sigma_{SW}'(t)w_t + (\alpha - \pi w_t)\sigma_{SW}(t) \\
&\quad - \Big(\mu_0(t)\tilde{a}_\Sigma + \mu_1(t)\hat{\theta}_{i,t} + \mu_2(t)\tilde{a}_i + \mu_3(t)w_t + \mu_4(t)w_0 + \mu_5\theta_{i,-}\Big) \\
&= 0,
\end{aligned}$$

where the last equality follows from inserting $\hat{\theta}_{i,t}$ from (3.12) and using the ODEs in (3.14).

$\diamondsuit$

Remark A.1. The above proof is that of a "backward engineer's". Instead of (A.11)-(A.15), we could alternatively let $\mu_j(t)$, $j \in \{0, 2, 3, 4, 5\}$, be arbitrary functions and adjust (A.26) appropriately. Then (A.27) and (A.29) would produce five restrictions which would in turn produce (A.11)-(A.15).

Proof of Theorem 4.1: We will write ... for terms that do not depend on μ_1. We first need

$$\begin{aligned}
\sum_{i=1}^M X_{i,0} &= S_0 \sum_{i=1}^M \theta_{i,0} \\
&= \Big(g_0(0) + g(0)\tilde{a}_\Sigma + \sigma_{SW}(0)w_0 + D_0\Big)w_0 \\
&= ... + \alpha w_0 \int_0^1 u\frac{\mu_1 - 2\kappa}{M}du + w_0^2\int_0^1 \frac{\mu_1 - 2\kappa}{M}du,
\end{aligned}$$

where the second equality follows from $\sum_{i=1}^{M} \theta_{i,0} = w_0$. Then we have

$$\begin{aligned}\sum_{i=1}^{M} \mathrm{CE}_i &= \sum_{i=1}^{M} X_{i,0} - \Big(M\beta_0(0) + \beta_1(0) \sum_{i=1}^{M} \tilde{a}_i^2 + \big(\beta_2(0) + M\beta_3(0)\big)\tilde{a}_\Sigma^2 \\ &\quad + M\beta_4(0)w_0^2 + \big(\beta_5(0) + M\beta_6(0)\big) w_0 \tilde{a}_\Sigma + M\beta_7(0)w_0 + (\beta_8(0) + M\beta_9(0))\tilde{a}_\Sigma \Big) \\ &= ... + \int_0^1 \Big\{ -u\big(\sigma_w^2 + \alpha^2 u + \alpha w_0\big)\frac{\mu_1}{M} - \frac{\gamma^2\kappa\mu_1^2}{(\mu_1 - 2\kappa)^2} \sum_{i=1}^{M} \tilde{a}_i^2 - \frac{4\gamma^2\kappa^2(\kappa - \mu_1)}{M(\mu_1 - 2\kappa)^2}\tilde{a}_\Sigma^2 \Big\} du.\end{aligned}$$

We define the constants

$$c_1 := \mathbb{E}[\tilde{a}_\Sigma^2] - M \sum_{i=1}^{M} \mathbb{E}[\tilde{a}_i^2], \quad c_2 := t(\sigma_w^2 + \alpha^2 t + \alpha w_0), \tag{A.33}$$

in which case the two conditions in (4.1) become

$$4\gamma(t)^2 c_1 < c_2 < 0, \quad t \in (0, 1]. \tag{A.34}$$

Based on the above, we seek to maximize

$$-c_2 \frac{\mu_1}{M} - \frac{\gamma^2\kappa\mu_1^2}{(2\kappa - \mu_1)^2} \sum_{i=1}^{M} \mathbb{E}[\tilde{a}_i^2] - \frac{4\gamma^2\kappa^2(\kappa - \mu_1)}{M(2\kappa - \mu_1)^2}\mathbb{E}[\tilde{a}_\Sigma^2] \tag{A.35}$$

By changing variables to $y := \frac{2\kappa\gamma}{2k - \mu_1}$ so that $\mu_1 = \frac{2(y-\gamma)\kappa}{y}$, we see that the maximization problem becomes

$$\max_{y \in (0,2\gamma)} G(y), \text{ where } G(y) := \frac{(c_1 y^2 - 2c_2)(y - 2\gamma) - 2c_2\gamma}{My}\kappa - \gamma^2\kappa \sum_{i=1}^{M} \mathbb{E}[\tilde{a}_i^2]. \tag{A.36}$$

The inequalities in (A.34) produce

$$G''(y) = \frac{2\kappa}{My^3}\big(c_1 y^3 + 2c_2\gamma\big) < 0, \quad y \in (0, 2\gamma). \tag{A.37}$$

Therefore, the first-order condition is sufficient. We observe that

$$G'(y) = \frac{2\kappa}{My^2}\big(c_1 y^3 - \gamma(c_1 y^2 + c_2)\big). \tag{A.38}$$

Then, (A.34) produces

$$G'(\gamma) = -\frac{2\kappa c_2}{M\gamma} > 0, \tag{A.39}$$

$$G'(2\gamma) = \frac{\kappa}{2M\gamma}(4\gamma^2 c_1 - c2) < 0. \tag{A.40}$$

By the intermediate value theorem and the strict concavity of G, we conclude that the unique solution of $G'(y) = 0$ satisfies $\gamma < \hat{y} < 2\gamma$. This $\hat{y}$ corresponds to $0 < \hat{\mu}_1 < \kappa$. Finally, (A.38) says that $\hat{y} = \frac{2\kappa\gamma}{2k - \hat{\mu}_1}$ is the solution of $c_1 y^3 - \gamma(c_1 y^2 + c_2) = 0$ and here κ does not appear.

$\diamondsuit$

B Equilibrium with exponential utilities

This appendix extends our equilibrium analysis to strategic investors with exponential utilities $U_i(x) := -e^{-x/\tau}$ with common risk tolerance parameter $\tau > 0$. In other words, we replace the risk-neutral objective (2.11) with

$$\inf_{\theta_i \in \mathcal{A}_i} \mathbb{E}\left[e^{-\frac{1}{\tau}(X_{i,1} - L_{i,1})} \middle| \sigma(\theta_{i,-}^{(0)}, \theta_{i,-}, \tilde{a}_i) \right]. \tag{B.1}$$

Here the processes (L_i, X_i) are still defined by (2.2) and (2.9); however, the admissible set $\mathcal{A}_i$ needs to be altered (see Definition B.1 below). Unlike risk-neutral utilities, exponential utilities produce coupled non-linear ODEs (see (B.4) and (B.5) below), which potentially explode in finite time. While it is possible to work out the exponential utility model without the parameter restrictions

$$\alpha := 0, \quad \pi := 0, \quad \theta_{i,-} := \frac{w_0}{M}, \tag{B.2}$$

these restrictions greatly simplify the following presentation.

We will consider continuous functions $\mu_1 : [0, 1] \to \mathbb{R}$ which satisfy the following two conditions. First, in the exponential case, the second-order condition (3.10) becomes

$$\mu_1(t) < \frac{1 + \sigma_{SW}(t)^2}{2\tau} + \kappa(t), \quad t \in [0, 1). \tag{B.3}$$

Second, the following coupled Riccati ODEs

$$\beta_4' = \frac{1+\sigma_{SW}^2+2\kappa\tau-2\tau(\mu_1+2M^2\beta_4^2\sigma_w^2\tau)}{2M^2\tau^2}, \quad \beta_4(1)=0, \tag{B.4}$$
$$\sigma_{SW}' = \frac{1+\sigma_{SW}^2+2\kappa\tau-\mu_1\tau-2M\beta_4\sigma_{SW}\sigma_w\tau}{M\tau}, \quad \sigma_{SW}(1)=\varphi_1, \tag{B.5}$$

must have non-exploding solutions for $t\in[0,1]$. Whenever the ODEs (B.4) and (B.5) have well-defined non-exploding solutions, we can define the function

$$\begin{aligned} &V(t,X_i,w,L_i,a_i,a_\Sigma) := \\ &\qquad e^{-\frac{1}{\tau}(X_i-L_i)+\beta_0(t)+\beta_1(t)a_i^2+\beta_2(t)a_ia_\Sigma+\beta_3(t)a_\Sigma^2+\beta_4(t)w^2+\beta_5(t)wa_i+\beta_6(t)a_\Sigma w+\beta_8(t)a_i}. \end{aligned} \tag{B.6}$$

This function V turns out to be the value function for the optimization problem (B.1) when the deterministic coefficient functions are given by the following linear ODEs

$$\begin{aligned}
\beta_0' &= -\beta_4\sigma_w^2 - \frac{w_0^2(\gamma-1)^2\kappa}{M^2\tau},\\
\beta_1' &= \frac{1}{2\tau(1+2\tau\kappa-\tau\mu_1+\sigma_{SW}^2)^2}\Big(4\tau\sigma_w\beta_5\gamma\kappa\sigma_{SW}(1+2\tau\kappa-2\tau\mu_1+\sigma_{SW}^2)\\
&\quad -\tau\sigma_w^2\beta_5^2\big((1+2\tau\kappa-\tau\mu_1)^2+(1+2\tau\kappa)\sigma_{SW}^2\big)\\
&\quad -2\gamma^2\kappa\big(2\tau\kappa(1+\sigma_{SW}^2)+(1-\tau\mu_1+\sigma_{SW}^2)^2\big)\Big),\\
\beta_2' &= -\sigma_w^2\beta_5\beta_6 - \frac{(2\gamma\kappa+\sigma_w\beta_5\sigma_{SW})^2(1+2\tau\kappa-2\tau\mu_1+\sigma_{SW}^2)}{M(1+2\tau\kappa-\tau\mu_1+\sigma_{SW}^2)^2},\\
\beta_3' &= -\frac{\sigma_w^2\beta_6^2}{2} + \frac{(2\gamma\kappa+\sigma_w\beta_5\sigma_{SW})^2(1+2\tau\kappa-2\tau\mu_1+\sigma_{SW}^2)}{2M^2(1+2\tau\kappa-\tau\mu_1+\sigma_{SW}^2)^2},\\
\beta_5' &= -2\sigma_w^2\beta_4\beta_5 + \frac{(2\gamma\kappa+\sigma_w\beta_5\sigma_{SW})(1+2\tau\kappa-2\tau\mu_1+\sigma_{SW}^2)}{\tau M(1+2\tau\kappa-\tau\mu_1+\sigma_{SW}^2)},\\
\beta_6' &= -2\sigma_w^2\beta_4\beta_6 - \frac{(2\gamma\kappa+\sigma_w\beta_5\sigma_{SW})(1+2\tau\kappa-2\tau\mu_1+\sigma_{SW}^2)}{\tau M^2(1+2\tau\kappa-\tau\mu_1+\sigma_{SW}^2)},\\
\beta_8' &= \frac{2w_0(\gamma-1)\gamma\kappa}{\tau M},
\end{aligned} \tag{B.7}$$

with zero terminal conditions (i.e., $\beta_j(1)=0$ for $j\in\{0,1,2,3,5,6,8\}$). Finally, we can adjust the notion of admissibility given in Definition 2.1 to the case of exponential utilities.

Definition B.1. We deem a jointly measurable and $\mathcal{F}_i$ adapted process $\theta_i=(\theta_{i,t})_{t\in[0,1]}$

admissible and write $\theta_i \in \mathcal{A}_i$ if the local martingale

$$\int_0^s \Big(V_X \theta_{i,t}(\sigma_{SW} dW_t + dD_t) + V_w \sigma_w dW_t\Big), \quad s \in [0,1], \tag{B.8}$$

is well-defined and is a martingale. In (B.8), the terms V_X and V_w denote the partial derivatives of the function V defined in (B.6).

$\diamondsuit$

We adjust the deterministic pricing coefficients (A.11)-(A.15) to

$$\begin{aligned}
\mu_0 &:= -\frac{(2\gamma\kappa + \sigma_{SW}\sigma_w\beta_5)(1 + 2\tau(\kappa - \mu_1) + \sigma_{SW}^2}{M\big(1 + \sigma_{SW}^2 + 2\kappa\tau - \mu_1\tau\big)} - \sigma_{SW}\sigma_w\beta_6,\\
\mu_2 &:= -\frac{\mu_1(2\kappa\gamma + \beta_5\sigma_{SW}\sigma_w)\tau}{1 + \sigma_{SW}^2 + 2\kappa\tau - \mu_1\tau},\\
\mu_3 &:= \frac{1 + \sigma_{SW}^2 + 2(\kappa - \mu_1 - M\beta_4\sigma_{SW}\sigma_w)\tau}{M\tau},\\
\mu_4 &:= \frac{2\kappa(\gamma - 1)}{M},\\
\mu_5 &:= 0.
\end{aligned} \tag{B.9}$$

The analogue of Theorem 3.4 for the case of exponential utilities is:

Theorem B.2. *Let the parameter restrictions* (B.2) *hold and let* $\gamma : [0,1] \to [0,\infty)$ *be a continuous function. Let* $\mu_1, \kappa : [0,1) \to (0,\infty)$ *be continuous and square integrable functions (i.e.,* (3.11) *holds), satisfy the second-order condition* (B.3)*, and ensure that the coupled Riccati ODEs* (B.4) *and* (B.5) *have well-defined non-explosive solutions on* $[0,1]$*. Define the deterministic functions* g_0 *and* g *as the unique solutions of the following linear ODEs:*

$$\begin{aligned}
g'(t) &= -\frac{2\gamma(t)\kappa(t) + \big(\beta_5(t) + M\beta_6(t)\big)\sigma_{SW}(t)\sigma_w}{M}, \quad g(1) = \varphi_0,\\
g_0'(t) &= \frac{2w_0(\gamma(t) - 1)\kappa(t)}{M}, \quad g_0(1) = 0.
\end{aligned} \tag{B.10}$$

Then the functions $\mu_0, \mu_2, \mu_3, \mu_4,$ *and* μ_5 *defined in* (B.9) *together with* σ_{SW} *defined in* (B.5) *form a Nash equilibrium in which (i) investor optimal holdings in equilibrium* *are given by*

$$\hat{\theta}_{i,t} = \frac{\big(2\kappa(t)\gamma(t) + \beta_5(t)\sigma_{SW}(t)\sigma_w\big)\tau}{\big(2\kappa(t) - \mu_1(t)\big)\tau + 1 + \sigma_{SW}(t)^2}\Big(\tilde{a}_i - \frac{\tilde{a}_\Sigma}{M}\Big) + \frac{w_t}{M}, \tag{B.11}$$

and (ii) the equilibrium stock price is given by (3.13)*.*

Proof. The proof of Theorem B.2 is similar to the proof of Theorem 3.4 and here we only outline the two needed changes. First, to verify that (B.11) is admissible in the sense of Definition B.1, we re-write the local martingale dynamics (A.19) appearing in (B.8) as

$$dV(t, X_i, w, L_i, \tilde{a}_i, \tilde{a}_\Sigma) = V(t, X_i, w, L_i, \tilde{a}_i, \tilde{a}_\Sigma)\big(J_t dW_t - \tfrac{1}{\tau}\hat{\theta}_{i,t} dD_t\big), \tag{B.12}$$

where $\hat{\theta}_{i,t}$ is defined in (B.11) and

$$J_t := \big(2\beta_4(t)w_t + \tilde{a}_i\beta_5(t) + \tilde{a}_\Sigma\beta_6(t) + \beta_7(t)\big)\sigma_w - \tfrac{1}{\tau}\hat{\theta}_{i,t}\sigma_{SW}(t). \tag{B.13}$$

Because the deterministic functions appearing in front of w_t, $\tilde{a}_i$, and $\tilde{a}_\Sigma$ in (B.11) and (B.13) are uniformly bounded, and because w_t defined in (2.4) is Gaussian, the representation (B.12) combined with Corollary 3.5.16 in Karatzas and Shreve (1991) produces the wanted martingality of V.

Second, we need to verify that the local martingale $N = (N_t)_{t\in[0,1]}$ in (A.1) is a martingale when $\hat{\mu}_t$ in (3.18) is replaced by

$$\begin{aligned}\hat{\mu}_t := &\frac{1 + \sigma_{SW}(t)^2 + \big(2\kappa(t) - \mu_1(t)\big)\tau - 2M\beta_4(t)\sigma_{SW}(t)\sigma_w\tau}{M\tau} w_t \\ &+ \frac{2\big(\gamma(t) - 1\big)\kappa(t)}{M} w_0 - \frac{2\gamma(t)\kappa(t) + \big(\beta_5(t) + M\beta_6(t)\big)\sigma_{SW}(t)\sigma_w}{M}\tilde{a}_\Sigma.\end{aligned} \tag{B.14}$$

To this end, we note that w_t remains a non-exploding Gaussian Ornstein-Uhlenbeck process under the $\mathbb{P}$-equivalent probability measures $(\mathbb{Q}^{(n)})_{n\in\mathbb{N}}$ defined in (A.9). Consequently, the proof of Lemma A.1 carries over to the current exponential utility case where $\hat{\mu}_t$ is defined in (B.14).

$\diamondsuit$

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