Liquidity and Asset Market Dynamics[*]

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Abstract

We study economies with an essential role for liquid assets in transactions. The model can generate multiple stationary equilibria, across which asset prices, market participation, capitalization, output and welfare are positively related. It can also generate a variety of nonstationary equilibria, even when fundamentals are deterministic and time invariant, including periodic, chaotic, and stochastic (sunspot) equilibria with recurrent market crashes. Some equilibria have asset price trajectories that resemble bubbles growing and bursting. We also analyze endogenous private and public liquidity provision. Sometimes it is efficient to have enough liquid assets to satiate demand; other times it is not.

JEL Classification: D82, D83, E40, E50

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1 Introduction

The last two recessions in the U.S., starting in 2001 and 2007, were preceded by rapid increases followed by abrupt collapses in some asset prices. These events have suggested to many observers, and reinforced the beliefs of others, that asset prices can rise above levels justified by fundamentals, and that price corrections can trigger or amplify fluctuations with dramatic consequences for the macroeconomy. Moreover, independent of the desire to understand recent crises, as a matter of theory one should want to know when and how economic models can generate this type of equilibrium asset price behavior – increases above fundamental values followed by collapses, or more generally, various types of complicated dynamics. This is a nontrivial issue, in the sense that generating such asset price dynamics is *not* easy in standard models.

We take the view that asset markets are better understood in the context of economies where, because of certain frictions, assets have a role in facilitating exchange. One version of this view is the claim that finance has something to learn from monetary theory, which is suggested (to us) by Kiyotaki and Moore (2005), Duffie et al. (2005,2008), and many of the other papers surveyed in Nosal and Rocheteau (2010) or Williamson and Wright (2010). We find this approach sensible because in monetary theory assets are valued not only for their rates of return, or dividends, but also for their liquidity services. As a result, price trajectories that seem anomalous from the perspective of standard asset pricing theory might emerge naturally in models with trading frictions like those studied in monetary economics. The most obvious example is that monetary models have equilibria with valued fiat currency, which is an asset with 0 dividend, or no fundamental value, and therefore one that should have a 0 price according to standard finance. In monetary economics, currency is valued for its liquidity. Our position is that liquidity considerations can potentially affect the price of any asset, especially as the distinction has blurred between monetary and non-monetary assets, and hence we propose a model where assets play a transactions role.[1]

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[1] Some people, including Krishnamurthy and Vissing-Jorgensen (2008), argue that certain assets have "convenience yields" – a loose term that might capture liquidity considerations. Our goal is to be more explicit about the transactions role of assets. The general idea has been discussed before. Cochrane (2002) e.g. describes the tech-stock bubble of the late 90's as "a mechanism much like the transactions demand for money [that] drove many stock prices above the 'fundamental value' they would have had in a frictionless market." Regarding the 2007-08 recession, Reinhart and Rogoff (2009) argue that securitization allowed consumers "to turn their previously illiquid housing assets into ATM machines." Ferguson (2008) also says it "allowed borrowers to treat their homes as cash machines," and reports that between 1997 and 2006, "US consumers withdrew an estimated $9 trillion in cash from the equity in their homes." Greenspan and Kennedy (2007) say home equity withdrawal financed about 3% of personal consumption from 2001 to 2005. Again, our goal is to model these phenomena more explicitly.
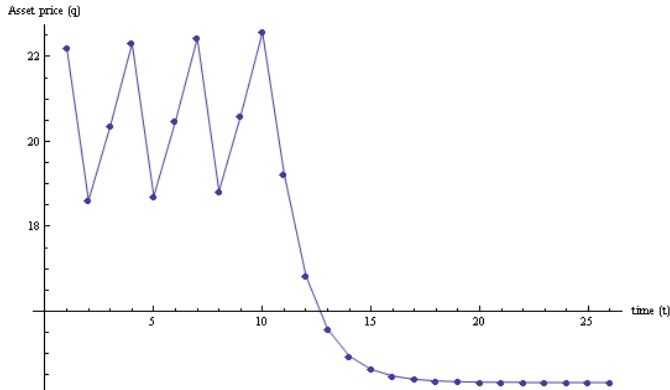
Figure 1: Sample price trajectory

While, as we show, the model can generate various, qualitatively different, outcomes, an example of what we think of as an interesting equilibrium asset price trajectory is depicted in Figure 1. It has two noteable features. First, asset prices fluctuate even though fundamentals (preferences, technologies and government policies) are deterministic and time invariant, and agents are fully rational. Second, the price ultimately crashes, which would typically be interpreted as a bubble bursting.[2] This type of asset price behavior can emerge easily in models that incorporate the frictions common in monetary economies, and can be interpreted as a liquidity premium that fluctuates over time. The potential importance of time-varying liquidity is discussed by Acharya and Pedersen (2005), who present an overlapping generations model of asset pricing where the illiquidity cost of an asset is exogenous. Here the liquidity premium emerges, and varies, endogenously over time as a self-fulfilling prophecy. This can happen because liquidity is at least in part a matter of beliefs. As any monetary economist knows, what agents are ready to accept in exchange can depend critically on what they think others readily accept. Kiyotaki and Wright (1989,1993) formalize this idea in a model of decentralized trade. But since that setup is crude on several dimensions (e.g. it has indivisible goods and assets), we instead build on the extension of the framework developed in Lagos and Wright (2005) and Rocheteau and Wright (2005).

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[2]We do not claim to propose, nor we did not try to find in the literature, a definitive definition of a bubble (but see the discussion in LeRoy 2004). To avoid purely semantic disagreement, for our purposes, a bubble is defined as an equilibrium outcome where the price of some asset is different from its fundamental price, where the fundamental price is the present value of its dividend stream.

In this model, agents trade periodically in both centralized and decentralized markets. The centralized markets are there mainly for technical convenience, as they help reduce the complexity of the state space. Much of the interesting action occurs in decentralized markets, characterized by bilateral matching and bargaining. There are gains from trade in these bilateral meetings, but this is hindered by a standard double-coincidence problem: a potential buyer cannot simply produce something in the match that a potential seller wants, ruling out barter. Also, because of imperfect commitment, enforcement and record keeping, in at least some matches, unsecured credit is not viable, because borrowers can renege with impunity. When unsecured credit is not available, assets have a role in transactions. There are two interpretations: (1) buyers directly give assets to sellers in the meetings; (2) buyers use assets to collateralize loans, to be repaid later, say in the next centralized market (which is convenient because in frictionless markets we can assume without loss of generality loans are repaid in numéraire goods). In case (1) there is finality when the assets are handed over. In case (2) there is delayed settlement, and a seller gets the assets if and only if a buyer reneges on an obligation. Aside from this detail, for our purposes (1) and (2) are equivalent. Also, although we often interpret transactions as payments or collateralized debt by consumers, one can equivalently interpret the model as having producer transactions or collateralized debt (see Section 7.3).

As a special case, we study an economy where credit works perfectly, say because there is perfect commitment/enforcement. In this economy, liquidity plays no role: buyers do not need to transfer assets to sellers or use them as collateral, since loan repayment is guaranteed by assumption. We prove in this case that assets must be priced fundamentally. When credit is imperfect, however, liquid assets are essential, and more interesting outcomes are possible. In our baseline model, the assets with a liquidity role are assumed to be in fixed supply, just like the claims to trees that give off fruit as dividends in the Lucas (1978) asset-pricing model. We are agnostic about the exact nature of assets, however, and they can alternatively be interpreted as representing land and/or housing. However one likes to think about the asset, we will assume that is has certain properties, including the fact that it is easily recognizable, that make it acceptable as means of payment or collateral. There can also be other assets in the model that are less liquid, and cannot be used in this way, for reasons discussed below. This allows us to talk about a premium on liquid assets.

An additional important ingredient in the model is the decision of potential sellers to participate in decentralized trade. This allows us to endogenize the frequency of trading opportunities, and hence the need for liquidity, which can generate multiple stationary equilibria and dynamic equilibria where asset prices follow paths that look like bubbles. The mechanism works in part through complementarities between buyers' asset holdings and sellers' participation decisions. When there are many sellers, it is a buyers' market, and hence buyers want to hold more liquid assets. This drives up asset prices, which gives sellers a greater incentives to participate. These complementarities can deliver multiple stationary equilibria, across which asset prices, output, stock market capitalization and welfare are positively related. An additional mechanism works through the intertemporal relationship between asset prices and liquidity. In equilibria where asset prices fluctuate, the liquidity premium depends negatively on the total value of liquid wealth, because a marginal asset is more useful in transactions when liquidity is scarce. Thus, in a boom, asset prices are high because agents anticipate they will be low and liquidity more valuable in the future when wealth falls.

The results about multiple stationary equilibria and dynamic equilibria with fluctuating or bubble-like asset prices depend on a positive liquidity premium, which depends on the supply of liquidity being scarce: such equilibria do not exist when supply satiates the demand for liquidity. The idea that the available supply of liquid assets is relatively scarce is arguably empirically relevant. For instance, Caballero (2006) champions the view that the world has a shortage of financial assets, and suggests that the recurrent emergence of speculative bubbles is easier to understand once one acknowledges this. In terms of the needs for liquid assets, Geanakoplos and Zame (2010) document that "the total [value] of collateralized lending is enormous: the value of U.S. residential mortgages alone exceeds $9.7 trillion (only slightly less than the $10.15 trillion total capitalization of S&P 500 firms)." So we think it is plausible to consider the case where demand is not satiated by the supply of liquid assets. In this case, the theory can be used to analyze the interaction between public and private liquidity.

Consider a situation where firms can be created by issuing equity that, like the Lucas trees, serves to satisfy transactions needs. A shortage of liquidity generates a decrease in the real interest rate, which provides an incentive to create new firms, and the new shares provide additional liquidity. Hence, in principle, the

market can generate its own liquidity. But perhaps not enough, in which case we can consider the public provision of liquidity, say the issuance of government bonds that also satisfy transactions needs. In our benchmark model, if government policy provides enough of these assets to satiate demand, we eliminate any liquidity premia and get a unique equilibrium where assets are priced fundamentally. Because of the nature of decentralized trade, however, this policy may not implement a constrained-efficient allocation. In one variant of the model, we show that if the government provides enough liquidity to satiate demand the economy may be characterized by excess participation in decentralized trade, in which case it is better from a welfare perspective to keep liquidity scarce.

While many of these results are novel, there is much related work. There is a large literature on pricing currency, as well as assets more generally, in overlapping generations economies, including Wallace (1980), Grandmont (1985), Tirole (1985) and Santos and Woodford (1997); see Azariadis (1993) for a textbook treatment. In contrast to the outcomes that can emerge in overlapping generations models, here the bubble-like component of an asset price is best interpreted as a liquidity premium. And, again in contrast to those models, this liquidity premium does not grow forever at the rate of interest. In our formulation, assets prices can increase above fundamental values, and liquidity premia can grow at the rate of interest for a while, but ultimately they collapse. An important difference between our infinitely-lived agent model and models with overlapping generations of finitely-lived agents is that we have to take the relevant transversality condition into account. LeRoy (2004) surveys the literature on bubbles more generally, some of which is based on irrationality or otherwise complicated belief structures.[3] By contrast, we can generate equilibria where asset prices rise and markets crash in bubble-like fashion in deterministic perfect foresight equilibrium, with no appeal to irrationality or complicated beliefs.

A somewhat related literature, including Kiyotaki and Moore (1997), Kiyotaki (1998), Kocherlakota (2000,2009), Ferraris and Watanabe (2008), and Mills and Reed (2009), discusses economies with assets playing dual roles as factors of production and collateral. Also related is work by Geanakopolos and Zame

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[3]We cannot discuss it all here, but by way of example, consider Abreu and Brunnermeier (2003). There bubbles occur because irrationally exuberant behavioral traders believe that asset prices will grow at a rate higher than the risk-free rate in perpetuity. Rational arbitrageurs sequentially become aware that the price has departed from fundamentals, and the bubble can burst only if there is a sufficient mass of arbitrageurs who have sold out. See also Allen and Gorton (1993) or Allen, Morris and Postlewaite (1993) for different but related models.

(2010), who construct a two-period general equilibrium model with durable goods and collateral. Some of these models are designed to argue that the interaction between credit limits and asset prices is a powerful propagation mechanism (see Bernanke, Gertler and Gilchrisht 1999 for a survey of earlier models in a similar spirit). In contrast to those models, the balance sheets of firms play no role for the dynamics of asset prices in this paper: our firms face no borrowing constraints, and can finance investment at the rate of time preference. Similarly, our households can supply labor unconstrained to finance consumption and asset purchases. Instead, our results derive from the focus on liquidity – one could say from the *moneyness* of some assets – and instead of illustrating the propagation or amplification of exogenous shocks to fundamentals, our goal is to generate interesting dynamics entirely from the endogenous liquidity premium.

Other models of bubbles include Allen and Gale (2000) and Barlevy (2009), who emphasize agency problems. Farhi and Tirole (2010), in particular, consider an overlapping generations version of the corporate finance model in Holmstrom and Tirole (2008), where agency problems prevent firms from borrowing against future output. They show that bubbles are more likely when the supply of outside liquidity is scarce and corporate income is less pledgeable. Some of these results are consistent with what we find here, even thought the model is quite different, such as the fact that interesting dynamics are more likely to emerge when liquidity is scarce, and that there is an amount of outside liquidity that eliminates these equilibria; we think we deliver much richer asset price dynamics, and we like our emphasis on the relationship between asset prices, liquidity and trading frictions. This emphasis is common in New Monetarist Economics, as surveyed in Williamson and Wright (2010) and Nosal and Rocheteau (2010).[4]

The paper is organized as follows. Section 2 describes the environment. Section 3 analyzes the case where credit works perfectly. Sections 4, 5 and 6 have imperfect credit, so liquid assets are essential, and study stationary, non-stationary, and stochastic (sunspot) equilibria. Section 7 discusses endogenous private and public liquidity, and an application of the model to corporate finance. Section 8 concludes.

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[4]Papers in that literature concerned with liquidity and asset pricing includes Lagos (2007), Geromichalos, Licari and Suarez-Lledo (2007), Lagos and Rocheteau (2008), Lester, Postlewaite and Wright (2008), Rocheteau (2008), Li and Rocheteau (2009), Li and Li (2010). Those papers do not consider nonstationary equilibria, however, and do not endogenize market participation. Lagos and Wright (2003) do study nonstationary equilibria, but do not endogenize participation, and money is the only asset. Lagos and Rocheteau (2009) analyze (dealers') participation decisions but only look at steady states. Rocheteau and Wright (2005) model seller participation, but also consider only stationary equilibria, and money is the only asset. Guerrieri and Lorenzoni (2009) establish a link between liquidity and volatility, but their mechanism is different.

2 The Environment

Following Rocheteau and Wright (2005), the set of agents consists of a $[0,1]$ continuum of households and an $[0, S]$ continuum of firms. In each period of discrete time, they engage in two types of activity: in one subperiod they trade in a decentralized market, or DM, characterized by frictions detailed below; in another subperiod they trade in a frictionless centralized market, or CM. There are two nonstorable consumption goods: x_t is produced and consumed in the CM; y_t is is produced and consumed in the DM. Households can convert labor time into goods one-for-one in the CM (it is easy to use general technologies). The utility of a household is

$$\lim_{T \to \infty} \mathbb{E} \sum_{t=0}^{T} \beta^t \mathcal{U}(y_t, x_t, h_t), \tag{1}$$

where $\beta \in (0,1)$.[5] For simplicity, let $\mathcal{U}(y_t, x_t, h_t) = u(y_t) + U(x_t) - h_t$, where $u(y_t)$ and $U(x_t)$ are twice continuously differentiable, strictly increasing, and concave. Moreover, $u(0) = 0$, $u'(0) = \infty$. Let y^* solve $u'(y^*) = c'(y^*)$ and let x^* solve $U'(x^*) = 1$. To reduce notation, normalize $U(x^*) - x^* = 0$.

As is standard in search theory (e.g. Pissarides 2000), we model firm entry as follows. To participate in the DM at t, firms must invest $k^f > 0$ units of the CM good at $t - 1$. This allows it to generate at t any amount $y \in [0, 1]$ of the DM good, plus $x = f(1 - y)$ of the CM good, where f is twice continuously differentiable, strictly increasing and concave, $f'(0) = \infty$, and $f'(1) = 0$. This makes $c(y) \equiv f(1) - f(1 - y)$ the opportunity cost of selling y in the DM, where $c(0) = 0$, $c'(0) = 0$, and $c'(1) = \infty$. Assume $k^f > \beta f(1)$, so it is not profitable to produce only CM goods. One can think of firms as issuing equity to pay for the investment k^f: each share issued in the CM at $t-1$ at a normalized price of 1 entitles its holder to a fraction $1/k^f$ of profit at t. Assume portfolios of households are fully diversified, and denote by R_t the expected return. In addition to equity in these firms, there is another asset, in fixed supply $A > 0$, that one can think of as a standard Lucas tree, or maybe a house, yielding $\kappa > 0$ units of x (our numéraire) each period in the CM. Let q be the CM price of this asset in terms of x.

The DM involves bilateral random matching. The matching probabilities for households and firms are $\alpha(n)$ and $\alpha(n)/n$, respectively, where n is the measure of participating firms. As is standard, $\alpha'(n) > 0$,

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[5]We assume in the text the limit in (1) exists; in case it does not, Appendix B uses the catching-up criterion for optimization discussed in Brock (1970) and Seierstad and Sydsaeter (1987).

$\alpha''(n) < 0$, $\alpha(n) \leq \min\{1, n\}$, $\alpha(0) = 0$, $\alpha'(0) = 1$ and $\alpha(\infty) = 1$. One can make different assumptions concerning the possibility of credit in the DM. At one extreme is full commitment or enforcement. In this case, in any DM meeting, households can credibly promise a firm payment in the next CM. At the other extreme is no commitment. Together with the assumption that agents are anonymous – so that we cannot punish those who renege on debts, say because there is no monitoring or record keeping that permits identification after the fact – this rules out unsecured credit (the surveys mentioned in the Introduction provide discussion and references). In this case, DM trade is either quid pro quo, or equivalently, for our purposes, collateralized debt.[6]

To guarantee that equity shares in firms cannot be used for the same purpose as claims to trees, we assume they can be costlessly counterfeited and cannot be authenticated in the DM. By contrast, claims to trees can be authenticated or cannot be counterfeited, so they can be used as a means of payment or collateral. This distinction operationalizes the difference between liquid and illiquid assets, in the benchmark model, but we consider below the case where firms and trees are treated symmetrically. The idea of modeling differential liquidity in terms of information frictions, or recognizability, has been used recently in Lester, Postlewaite, and Wright (2008), Rocheteau (2008), Li and Rocheteau (2009), and much earlier work cited therein. In those models, the possibility of counterfeiting an asset at cost leads to equilibria with an endogenous upper bound $\bar{\tau}$ on the amount that can be used in DM trade. As the cost of counterfeiting vanishes, $\bar{\tau} \to 0$ and the asset is not used in the DM at all. We attribute this information problem to equity in firms but not trees because we want the stock of liquid assets to be fixed; later, we assume all assets are equally recognizable.

3   Perfect Credit

As a benchmark, suppose households can commit to repay DM debt. Let $W_t(a, s, d)$ be a household's value function in the CM at t with a units of the liquid asset, s illiquid shares in firms, and debt d from the

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[6]The above-mentioned equivalence is nicely described by David Andofatto in a recent blog: "On the surface, these two methods of payment [assets used for payment or used to collateralize debt] look rather different. The first entails immediate settlement, while the second entails delayed settlement. To the extent that the asset in question circulates widely as a device used for immediate settlement, it is called money (in this case, backed money). To the extent it is used in support of debt, it is called collateral. But while the monetary and credit transactions just described look different on the surface, they are equivalent in the sense that capital is used to facilitate transactions that might not otherwise have taken place."

previous DM, in units of x. Similarly, let $V_t(a, s)$ be their value function in the DM at t. Then we have

$$W_t(a_t, s_t, d_t) = \max_{x_t, h_t, a_{t+1}, s_{t+1}} \{U(x_t) - h_t + \beta V_{t+1}(a_{t+1}, s_{t+1})\} \tag{2}$$

$$\text{st} \quad x_t + d_t + q_t(a_{t+1} - a_t) + s_{t+1} = h_t + \kappa a_t + R_t s_t, \tag{3}$$

where q_t is the price of the liquid asset and R_t the gross return on shares. In Appendix B we show that a solution to the recursive problem solves the underlying sequence problem as long as the transversality condition $\lim_{t \to \infty} \beta^t q_t a_{t+1} = 0$ holds. Assuming x^* is large, $h_t \geq 0$ never binds, and we can substitute h_t from (3) into (2) to obtain

$$W_t(a_t, s_t, d_t) = (q_t + \kappa)a_t + R_t s_t - d_t + \max_{x_t \geq 0} [U(x_t) - x_t]$$

$$+ \max_{a_{t+1} \geq 0, s_{t+1} \geq 0} \{-q_t a_{t+1} - s_{t+1} + \beta V_{t+1}(a_{t+1}, s_{t+1})\}. \tag{4}$$

Notice (4) immediately implies (a_{t+1}, s_{t+1}) is independent of (a_t, s_t, d_t), which means we do not have to track the distribution of assets across agents in the DM as a state variable. This is the simplification implied by introducing some centralized trade into an otherwise standard search model.[7]

Lemma 1 $x_t = x^*$; W_t is linear in wealth $(q_t + \kappa)a_t + R_t s_t - d_t$; and (a_{t+1}, s_{t+1}) is independent of wealth.

Moving to the DM, in any meeting the firm gives the household output y, in exchange for τ_a liquid assets, τ_s stocks, and a promise (debt) of d payable in the next CM. To determine the terms of trade, for now, we use Kalai's (1977) proportional bargaining solution with $\theta \in [0, 1]$ denoting the household's share:

$$(y_t, \tau_{a,t}, \tau_{s,t}, d_t) = \arg \max_{y, \tau_a, \tau_s, d} [u(y) + W_t(a - \tau_a, s - \tau_s, d) - W_t(a, s, 0)] \tag{5}$$

$$\text{st} \quad u(y) + W_t(a - \tau_a, s - \tau_s, d) - W_t(a, s, 0) = \frac{\theta}{1 - \theta} [f(1 - y) + (q_t + \kappa)\tau_a + R_t \tau_s + d - f(1)] \tag{6}$$

(Note that we add some constraints on asset transfers in the next paragraph.) Intuitively, the proportional solution is pairwise efficient and gives the traders surpluses in fixed proportions, where the surplus is the difference between the payoff if they trade and if they do not trade. We use this to reduce the algebra, but the main results survive under generalized Nash bargaining or Walrasian pricing – indeed, with perfect

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[7]We also need quasi-linear utility in the CM for this to work. See the surveys mentioned in the Introduction for references to (much more complicated) search-based monetary models without a CM or without quasi-linearity.

credit the solutions are equivalent, but with imperfect credit the proportional solution has some advantages discussed in Footnote 8). Figure 2 provides a stylized representation of the solution, where U^H and U^F are the surpluses and the frontier solves $U^H + U^F = u(y^*) - c(y^*)$.

It is important to include the following constraints on asset transfers to the bargaining problem: $\tau_a \leq a$, and $\tau_s \leq \min(s, \bar{\tau}_s)$, where generally $\bar{\tau}_s$ is an upper bound that arises because of the recognizability problem discussed above – e.g. if stocks are freely counterfeitable and not verifiable then $\bar{\tau}_s = 0$. With these constraints, using the linearity of W_t and $c(y) = f(1) - f(1 - y)$, the bargaining solution becomes

$$(y_t, \tau_{a,t}, \tau_{s,t}, d_t) = \arg \max_{y, \tau_a, \tau_s, d} [u(y) - (q_t + \kappa)\tau_a - R_t\tau_s - d] \tag{7}$$

$$\text{st } (q_t + \kappa)\tau_a + R_t\tau_s + d = (1 - \theta) u(y) + \theta c(y) \tag{8}$$

$$\tau_a \leq a, \ \tau_s \leq \min(s, \bar{\tau}_s). \tag{9}$$

It is immediate that $y_t = y^*$ and $(q_t + \kappa)\tau_{a,t} + R_t\tau_{s,t} + d_t = (1 - \theta) u(y^*) + \theta c(y^*)$. Thus, we get efficient production $y_t = y^*$ in every DM meeting, just like we earlier found $x_t = x^*$ in the CM. Moreover, the exact form of payment – in terms of $\tau_{a,t}$, $\tau_{s,t}$ or d_t – is irrelevant as long as the total payment has CM value $(1 - \theta) u(y^*) + \theta c(y^*)$. Hence, we can impose without loss of generality that DM trade is conducted exclusively with debt, and the liquidity or illiquidity of assets is irrelevant for consumption or welfare.

Now consider asset pricing. Given the linearity of W_t and the bargaining solution,

$$\begin{aligned} V_t(a, s) &= \alpha(n_t) [u(y^*) + W_t(a, s, d)] + [1 - \alpha(n_t)] W_t(a, s, 0) \\ &= \alpha(n_t)\theta [u(y^*) - c(y^*)] + (q_t + \kappa)a + R_ts + W_t(0, 0, 0). \end{aligned} \tag{10}$$

Substituting (10) into (4), the choice of assets in period $t - 1$ solves

$$\max_{a \geq 0, s \geq 0} \left\{ - [q_{t-1} - \beta(q_t + \kappa)] a - [1 - \beta R_t] s \right\}.$$

If $\frac{q_t + \kappa}{q_{t-1}} > \beta^{-1}$ or $R_t > \beta^{-1}$, there is no solution to the problem. If $\frac{q_t + \kappa}{q_{t-1}} \leq \beta^{-1}$ or $R_t \leq \beta^{-1}$, the solution satisfies $[q_{t-1} - \beta(q_t + \kappa)] a = 0$ and $[1 - \beta R_t] s = 0$. Market clearing implies that for illiquid equity $R_t = \beta^{-1}$. For the liquid asset it implies
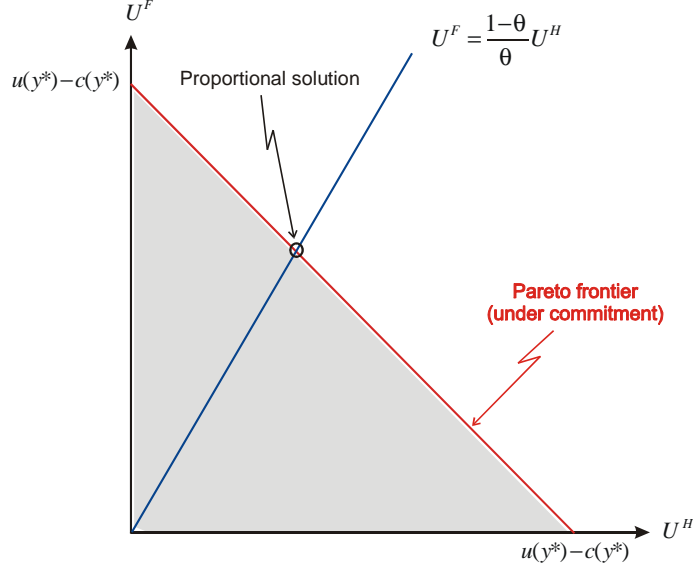
$$\frac{q_t + \kappa}{q_{t-1}} = \beta^{-1}. \tag{11}$$

Figure 2: Proportional bargaining: Perfect credit

Given nonnegativity $q_t \geq 0$, together with the transversality condition $\lim_{t \to \infty} \beta^t q_t = 0$, the only admissible solution to this difference equation is $q_t = q^*$, where by definition $q^* \equiv \frac{\kappa}{r}$ is the fundamental price.

It remains to determine entry. The expected value of a firm in the DM at t is

$$\Pi_t = \frac{\alpha(n_t)}{n_t}\left[f(1 - y^*) + d_t\right] + \left[1 - \frac{\alpha(n_t)}{n_t}\right]f(1). \tag{12}$$

In words, with probability $\alpha(n_t)/n_t$, the firm meets a household in the DM and sells y^* for a promise of $d_t = (1 - \theta)\, u(y^*) + \theta c(y^*)$ in the next CM, and then sells $f(1 - y_t)$ in the CM. And with complementary probability $1 - \frac{\alpha(n_t)}{n_t}$, the firm is unmatched in the DM and sells $f(1)$ in the CM. Simplifying, we get

$$\Pi_t = \frac{\alpha(n_t)}{n_t}(1 - \theta)\left[u(y^*) - c(y^*)\right] + f(1). \tag{13}$$

Firms participate in the DM as long as $\Pi_t \geq R_t k^f$. Using $\beta R_t = 1$, the measure of entrants n_t solves

$$\frac{\alpha(n_t)}{n_t}(1 - \theta)\left[u(y^*) - c(y^*)\right] \leq k, \quad = \text{ if } n_t > 0, \tag{14}$$

where $k \equiv \frac{k^f}{\beta} - f(1)$ is the effective entry cost.

This leads to the following definition and results.

Definition 1 *An equilibrium with perfect credit is a nonnegative sequence $\{(q_t, n_t)\}_{t=0}^{\infty}$ solving (11) and (14), with $\lim_{t \to \infty} \beta^t q_t = 0$. All other variables, including x_t, R_t etc. are determined as discussed above.*

Proposition 1 *Equilibrium with perfect credit exists, is unique and is stationary.*

These result follow from the observation that, for all t, $q_t = q^*$, and there is a unique $n_t = n$ solving (14). Also, $x_t = x^*$ and $y_t = y^*$. Hence, with perfect credit, asset prices are constant at their fundamental values, and consumption is constant at its efficient level. Overall efficiency, however, also requires $n = n^*$, which holds iff a standard Hosios (1990) condition holds, equating firms' share of the DM surplus to the elasticity of the matching function. For an arbitrary θ, even with perfect credit, equilibrium is not generally efficient, and entry could be too high or too low due to firms not internalizing their impact on the matching probabilities of other firms and households.

Proposition 2 *With perfect credit the equilibrium is efficient iff*

$$1 - \theta = \frac{n^* \alpha'(n^*)}{\alpha(n^*)},$$

where n^* solves $\alpha'(n) [u(y^*) - c(y^*)] = k$.

4   Imperfect Credit

We now describe the economy where households cannot commit to repay debt, which means $d_t = 0$. In this economy, the liquidity of assets can be critical. We continue to assume for now that shares in firms are not recognizable, so $\bar{\tau}_s = 0$. In fact, it is straightforward to allow credit in some meetings, but not others, and all results go through exactly as stated provided we reinterpret α as the probability of a no-credit meeting. Similarly, we can assume shares in firms are recognizable in some meetings, but not others, or make counterfeiting costly and use the endogenous upper bound $\bar{\tau}_s > 0$ derived in Rocheteau (2008) or Li and Rocheteau (2009) (recall that $\bar{\tau}_s = 0$ when counterfeiting is costless). Given this is understood, we proceed here with $d_t = \bar{\tau}_s = 0$ in all DM meetings. The CM value function of the household is still given by (2)-(3), but since credit is not feasible we omit the third argument of W_t.
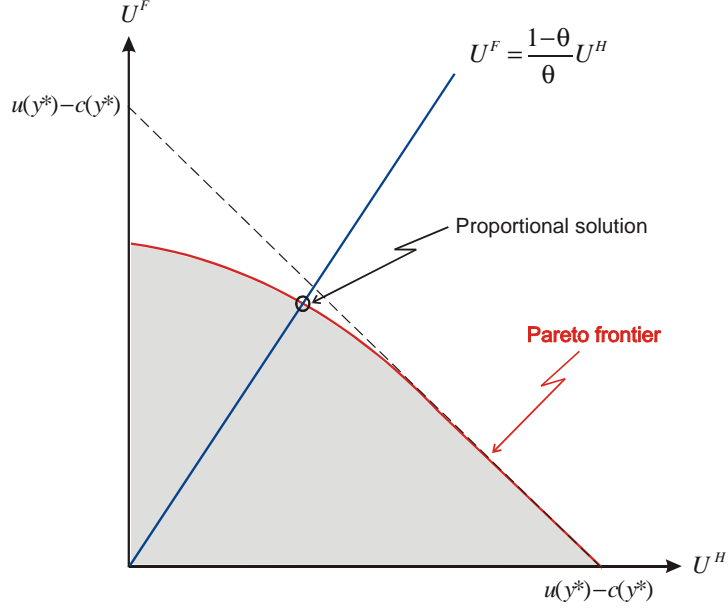
Figure 3: Proportional bargaining solution: Imperfect credit

Although we show in the Appendix the main results are preserved using Nash bargaining, we continue here to use proportional bargaining in the DM.[8] See Figure 3, where the frontier of the bargaining set is $U^H + U^F = u(y^*) - c(y^*)$ if $(q_t + \kappa)a \geq c(y^*) + U^F$, and $U^H = u \circ c^{-1} \left[(q_t + \kappa)a - U^F \right] - (q_t + \kappa)a$ otherwise. As compared to the previous section, the frontier is now linear when $y = y^*$ and strictly concave when $y < y^*$.[9] The constraints on the bargaining problem are $\tau_a \leq a$, and $\tau_s \leq \min(s, \bar{\tau}_s)$. The solution is given by (5)-(6) with $d_t = 0$ and $\tau_{s,t} = 0$. Substituting $(q_t + \kappa)\tau_a$ from (8) into (7), this simplifies to

$$y_t = \arg \max_y \theta \left[u(y) - c(y) \right] \tag{15}$$

$$\text{st} \quad \tau_{a,t} = \frac{(1 - \theta)u(y) + \theta c(y)}{q_t + \kappa} \leq a. \tag{16}$$

If $\tau_{a,t} \leq a$ does not bind then $y_t = y^*$ and $\tau_{a,t} = \frac{(1-\theta)u(y^*)+\theta c(y^*)}{q_t+\kappa}$. If it does bind, then y_t is the solution to

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[8]This has several desirable features when credit is infeasible. First, it guarantees the function V is concave. Second, the proportional solution is monotonic (each player's surplus increases with the total surplus), which means households have no incentive to hide some assets. These results cannot be guaranteed with Nash bargaining (Aruoba, Rocheteau and Waller 2007). Also, while Nash is attractive, due to its strategic foundations, in stationary models (see e.g. Osborne and Rubinstein 1990), this does not generally apply in nonstationary environments (Coles and Wright 1998; Ennis 2001).

[9]Readers may see an analogy between models with perfect or imperfect credit, on the one hand, and models with transferable or nontransferable utility, on the other. It is known models with nontransferable utility can generate more complicated equilibria than those with transferable utility (e.g. Burdett and Wright 1998).

$(q_t + \kappa)a = \omega(y_t)$, where

$$\omega(y_t) \equiv \theta c(y_t) + (1 - \theta)u(y_t). \tag{17}$$

The important point here is that y_t is a function of a household's liquid wealth. More generally, there is nothing particularly special about households in this regard, and there is no reason one cannot interpret our results as applying to trades between two firms, say, instead of a household and a firm. Payment by the transfer of assets by a consumer, or equivalently the use of assets to collateralize consumer debt, is not that different from the transfer of assets by a producer or the use of collateralized debt by a producer who wants to expand his capital stock or acquire any other input. While, of course, the economic interpretation may be different, the analytic results are identical. We think this is interesting because there is much emphasis in the related literature on collateral or corporate finance more generally (see e.g. Kiyotaki and More 2005, Holmstrom and Tirole 2008, or Farhi and Tirole 2010). We can capture not only what they have in mind, but we think more, with a simple reinterpretation of our model. In particular, we think it would be nice if that literature described in more detail the ways agents make transactions with each other – as opposed to transacting "with the market" – as we try to do here. In Section 7 we sketch the details of a corporate finance explicitly, but to ease the presentation, for now, we continue focusing on household liquidity.

In any case, following the reasoning in Section 3, we have

$$V_t(a, s) = \alpha(n_t)\theta \left\{ u\left[y((q_t + \kappa)a) \right] - c\left[y((q_t + \kappa)a) \right] \right\} + (q_t + \kappa)a + R_t s + W_t(0, 0). \tag{18}$$

By substituting $V_t(a, s)$ from (18) into (4), the household's choice of assets at $t - 1$ solves

$$\max_{a \geq 0} \left\{ -q_{t-1}a + \beta\alpha(n_t)\theta \left(u\left[y((q_t + \kappa)a) \right] - c\left[y((q_t + \kappa)a) \right] \right) + \beta(q_t + \kappa)a \right\} + \max_{s \geq 0} \left(-1 + \beta R_t \right) s.$$

As above, illiquid stocks are held only if $\beta R_t = 1$. The choice of liquid assets can be expressed

$$\max_{a \geq 0} \left\{ -\left(r\left(q_{t-1} - q^* \right) - \left(q_t - q_{t-1} \right) \right) a + \alpha(n_t)\theta \left\{ u\left[y((q_t + \kappa)a) \right] - c\left[y((q_t + \kappa)a) \right] \right\} \right\}, \tag{19}$$

where $q^* = \frac{\beta\kappa}{1-\beta} = \frac{\kappa}{r}$ is the fundamental price. The household chooses liquid assets to maximize the expected DM surplus minus the cost of holding the asset, where the cost is the capitalized difference between the price of the asset and its fundamental value, $r\left(q_{t-1} - q^* \right)$, net of the capital gain $q_t - q_{t-1}$.

It is immediate that the problem has a solution only if $r\left(q_{t-1} - q^*\right) \geq q_t - q_{t-1}$. We also have the following results. (All proofs are in the Appendix.)

Lemma 2 *Assume $r\left(q_{t-1} - q^*\right) \geq q_t - q_{t-1}$ for all $t \geq 1$. The household's choice of liquid assets, $\{a_t\}_{t=0}^{\infty}$, is optimal iff*

$$\lim_{t \to \infty} \beta^t q_t a_{t+1} = 0 \tag{20}$$

and, for all $t \geq 1$, the following is true:

(i) If $r\left(q_{t-1} - q^*\right) > q_t - q_{t-1}$, a_t is the unique solution to

$$- \left(r\left(q_{t-1} - q^*\right) - \left(q_t - q_{t-1}\right)\right) + \alpha(n_t)\theta \left\{ \frac{u'(y) - c'(y)}{\theta c'(y) + (1-\theta)u'(y)} \right\}(q_t + \kappa) \leq 0, \quad = \quad \text{if } a_t > 0, \tag{21}$$

where $y = y((q_t + \kappa)a_t)$.

(ii) If $r\left(q_{t-1} - q^*\right) = q_t - q_{t-1}$, then any $a_t \in \left[\frac{(1-\theta)u(y^*) + \theta c(y^*)}{q_t + \kappa}, +\infty\right)$ is optimal.

Summing the transversality condition (20) across agents, market clearing implies $\lim_{t \to \infty} \beta^t q_t A = 0$, where A is the fixed supply of the asset. Hence, $\lim_{t \to \infty} \beta^t q_t = 0$, which means that the asset price must asymptotically grow slower than the rate of time preference. From Lemma 2, if $\frac{q_t + \kappa}{q_{t-1}} < 1 + r$, the solution to the problem is unique and the distribution of liquid assets across households in the DM is degenerate. And if $\frac{q_t + \kappa}{q_{t-1}} = 1 + r$, the household's choice of a_t is not pinned down, but they all have at least enough liquid wealth to buy y^* in the DM. In this case, they are satiated in liquidity, and while the exact distribution of asset holdings is not determinate, it is not especially relevant since agents are indifferent to any asset position as described in part (ii) of the Lemma.

The expected value of a firm at t satisfies

$$\Pi_t = \frac{\alpha(n_t)}{n_t}\left[f(1 - y_t) + (q_t + \kappa)\tau_{a,t}\right] + \left[1 - \frac{\alpha(n_t)}{n_t}\right]f(1), \tag{22}$$

which as in the previous section implies

$$\frac{\alpha(n_t)}{n_t}(1-\theta)\left[u(y_t) - c(y_t)\right] \leq k, \quad = \text{if } n_t > 0. \tag{23}$$

This is similar to (14), except we replace y^* with y_t when credit is imperfect.

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In the following we will restrict our attention to equilibria where all households hold the same level of liquid assets, but from Lemma 2 this is without loss in generality.

Definition 2 *An equilibrium with imperfect credit is a nonnegative sequence $\{(q_t, n_t)\}_{t=0}^{\infty}$ solving (21) and (23) with $\lim_{t\to\infty} \beta^t q_t = 0$ and $a_t = A$ and some initial condition n_0. From these all other variables are determined in the obvious way.*

We start by characterizing stationary (steady-state) equilibria where (q_t, n_t) is constant, implying transversality is automatically satisfied. Consider the stationary version of the participation condition (23), and let $n(q)$ denote the solution for n given q.

Lemma 3 *Given q, $n(q) > 0$ iff*

$$(1 - \theta) \left\{ u\left[y((q + \kappa)A)\right] - c\left[y((q + \kappa)A)\right] \right\} > k. \tag{24}$$

If (24) holds, then $n'(q) > 0$ if $(q + \kappa)A < \omega(y^*)$ and $n'(q) = 0$ if $(q + \kappa)A > \omega(y^*)$, where $\omega(y)$ is defined in (17).

According to (24), firms participate in the DM iff their surplus, which is a fraction $1 - \theta$ of the total match surplus, is greater than the cost k. Moreover, if the value of liquid assets $(q + \kappa)A$ is too small to allow agents to trade y^*, an increase in the price q raises wealth and induces more firms to participate. This is the channel through which asset prices affect profits: higher prices increase the DM gains from trade, when credit is imperfect, because households have more liquid wealth with which to transact or, equivalently, use as collateral.

Consider next the stationary version of the asset-pricing condition (21), and let $q(n)$ denote the solution for q given n.

Lemma 4 *(i) If $(q^* + \kappa)A \geq \omega(y^*)$, then $q(n) = q^*$; (ii) If $(q^* + \kappa)A < \omega(y^*)$, then $q(n) > q^*$ for all $n > 0$, $q'(n) > 0$, $\lim_{n\to 0} q(n) = q^*$, and $\lim_{n\to\infty} q(n) = q_{\infty} > q^*$.*

If aggregate liquidity is large enough for households to buy y^* then q must be priced fundamentally; otherwise it exhibits a liquidity component $q(n) > q^*$, which in our terminology is a bubble. If n increases, it is easier

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for households to trade, so demand for liquid assets rises, as does the liquidity premium or bubble. This is the the channel through which firms' participation decisions affect asset prices.

To describe the set of stationary equilibria, define two thresholds for the participation cost:

$$k^* = (1-\theta)\left[u(y^*) - c(y^*)\right],$$

$$\tilde{k} = (1-\theta)\left\{u\left[y\left[(q^* + \kappa)A\right]\right] - c\left[y\left[(q^* + \kappa)A\right]\right]\right\}.$$

The threshold k^* corresponds to the maximum cost consistent with $n > 0$ when $y = y^*$, while $\tilde{k}$ is the maximum cost consistent with $n > 0$ when $q = q^*$. Notice that $\tilde{k} \le k^*$, with strict inequality if $(q^* + \kappa)A < \omega(y^*)$.

Proposition 3 *(i) If $(q^* + \kappa)A \ge \omega(y^*)$ then there is a unique stationary equilibrium and it has $q = q^*$, $y = y^*$, and*

$$\frac{\alpha(n)}{n} = \frac{k}{(1-\theta)\left[u(y^*) - c(y^*)\right]} \quad if \ k < k^*$$

$$n = 0 \ otherwise.$$

(ii) If $(q^* + \kappa)A < \omega(y^*)$, then there is a $\hat{k} \in \left[\tilde{k}, k^*\right)$ such that: if $k < \tilde{k}$, then all stationary equilibria have $n > 0$ and $q > q^*$; if $k \in \left(\tilde{k}, \hat{k}\right)$, then there is an inactive stationary equilibrium with $(n,q) = (0, q^*)$ and an even number of stationary equilibria with $n > 0$; if $k > \hat{k}$, then there is a unique stationary equilibrium and it has $n = 0$ and $q = q^*$. Moreover, for all $r < \frac{2\theta}{-\alpha''(0)(1-\theta)}$, there is a $\bar{A} < \frac{\omega(y^*)}{q^* + \kappa}$ such that for all $A < \bar{A}$, $\tilde{k} < \hat{k}$.

The outcome is depicted in Figure 4, in the case where liquidity is scarce in the sense that $(q^* + \kappa)A < \omega(y^*)$, for three values of k. Notice the origin in the figure is not $(0,0)$, but $(0, q^*)$, since the fundamental price q^* is the relevant lower bound on q. Summarizing the economic content of these results, if there is enough liquidity for households to buy y^* in the DM, assets are priced fundamentally, as in the model with perfect credit. But if liquid assets are in short supply, in any equilibrium with $n > 0$, the asset bears a premium, $q > q^*$, and households trade less than the efficient quantity, $y < y^*$. If k is too big, of course, the DM shuts down and there is no transactions role for assets, so they are priced fundamentally, $q = q^*$. There can be

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multiple stationary equilibria, including an inactive one with $n = 0$ and $q = q^*$, and an even number of active equilibria with $n > 0$ and $q > q^*$ (although we drew the two curves as convex and concave, all we really know is they are both upward sloping). This multiplicity arises from the complementarities between households' portfolio choices and firms' entry decisions discussed above. One way to verify this is to consider a version of the model with entry on the other side of the DM – a fixed number of firms and endogenous participation by households. In this version, the strategic complementarities between participation decisions and portfolio choices are removed, and there is a unique stationary equilibrium (details available upon request).
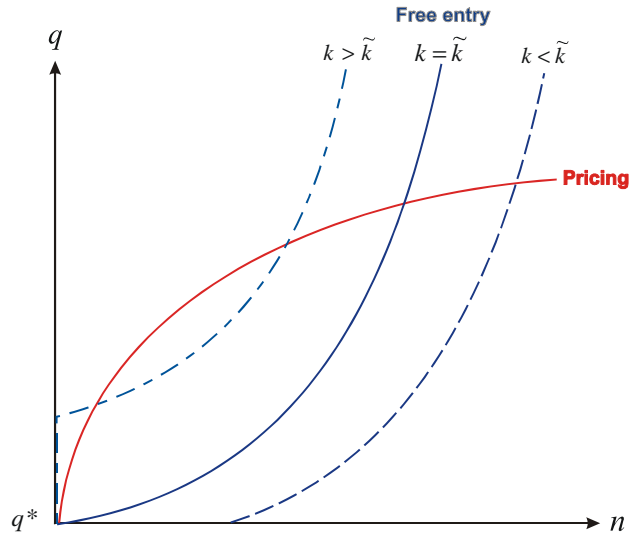

Figure 4: Equilibria when $A < \frac{\theta c(y^*) + (1-\theta)u(y^*)}{q^* + \kappa}$ and $\tilde{k} < \hat{k}$.

Notice that the asset can be valued above its fundamental price only when $n > 0$. In other words, bubbles are associated with increased economic activity. This may come as a surprise to some people, although it is similar to the message in Kocherlakota (2009), or to the familiar idea in monetary economics that monetary equilibria entail more trade than nonmonetary equilibria. The model also has predictions for the relationship between the price of the liquid asset and the capitalization of the stock market, or the total value of the firms, $n\Pi = \alpha(n)(1-\theta)[u(y) - c(y)] + nf(1)$. From the above results, there is a positive relation between q, y, and n across equilibria. So, the larger the price of the liquid asset, the larger is stock market capitalization. This is in contrast with some models, where bubbles tends to reduce the capital stock (see e.g. the overlapping

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generations model in Tirole 1985 or the search model in Lagos and Rocheteau 2008).

The model also generates predictions concerning asset prices and trading volume, where trade volume in the DM is the fraction of A used for transactions each period, $\mathcal{V} = \frac{\alpha(n)\tau_a}{A}$. This measure, which is obviously similar to the notion of velocity in monetary economics, has been used by Wallace (1996,2000) to analyze liquidity structure. Some observers, such as Cochrane (2002), claim that a positive relation between price and volume is a generic feature of bubbles. In case of multiple equilibria, $\mathcal{V}$ and q are positively related across equilibria. Of course, whenever there are multiple equilibria, comparative statics differ across them, so the following Proposition 4 focuses on the the best (see Proposition 5 below). In this equilibrium, if there is no shortage of liquidity, q is independent of A and k, and trade volume decreases with k simply because fewer firms enter. When liquidity is scarce, however, q decreases with A as well as k, and changes in k induce a positive relation between q and $\mathcal{V}$.

Proposition 4 *Consider the stationary equilibrium with the highest q. (i) If $A \geq \frac{\omega(y^*)}{q^*+\kappa}$ and $k < k^*$, then $\frac{\partial q}{\partial A} = \frac{\partial q}{\partial k} = 0$ and $\frac{\partial \mathcal{V}}{\partial k} < 0$. (ii) If $A < \frac{\omega(y^*)}{q^*+\kappa}$ and $k < \hat{k}$, then $\frac{\partial q}{\partial A} < 0$, $\frac{\partial q}{\partial k} < 0$ and $\frac{\partial \mathcal{V}}{\partial k} < 0$.*

The next results concern welfare, using the criterion

$$V = \sum_{t=0}^{\infty} \beta^t \left\{ \alpha(n) \left\{ u(y) - (q+\kappa)\tau_a \right\} + \kappa A + rnk^f \right\} = \frac{\alpha(n)\theta \left[u(y) - c(y) \right] + A\kappa + rnk^f}{1 - \beta}, \qquad (25)$$

which is the discounted sum of household's expected DM surplus, plus the utility generated by the returns on trees and stocks, the latter being equal (by free entry) to the number of firms times the entry cost.

Proposition 5 *Assume there exist multiple stationary equilibria, a sufficient condition for which is $A < \frac{\theta c(y^*)+(1-\theta)u(y^*)}{q^*+\kappa}$ and $k \in \left(\tilde{k}, \hat{k} \right)$. Then, V is increasing with q across equilibria.*

Equilibria where q is low are dominated by those with higher q. Indeed, welfare is minimized when assets are priced fundamentally. Again, this may be a surprise to some people, but not to monetary economists. Equilibria are not in general efficient. In the perfect credit economy, efficiency requires the Hosios (1990) condition; now it requires this plus an adequate supply of liquid assets.

Proposition 6 *Equilibrium is efficient iff*

$$A \geq \frac{\theta c(y^*) + (1 - \theta)u(y^*)}{q^* + \kappa} \ \text{and} \ \theta = 1 - \frac{n^* \alpha'(n^*)}{\alpha(n^*)},$$

where n^* solves $\alpha'(n)\left[u(y^*) - c(y^*)\right] = k$.

5 Dynamic Equilibria

We now consider non-stationary equilibria with imperfect credit (recall that perfect credit implies the only equilibrium is stationary). From (21), with imperfect credit, the price of the liquid asset satisfies the first-order difference equation

$$q_{t-1} = \Gamma(q_t) \equiv \frac{q_t + \kappa}{1 + r} \left\{ 1 + \alpha(n_t)\theta \left[\frac{u'(y_t) - c'(y_t)}{\theta c'(y_t) + (1 - \theta)u'(y_t)} \right] \right\}, \tag{26}$$

where

$$y_t = \min\left\{ y^*, \omega^{-1}\left[(q_t + \kappa)A\right] \right\}, \tag{27}$$

$$n_t = \psi^{-1}\left(\min\left\{ \frac{k}{(1 - \theta)\left[u(y_t) - c(y_t)\right]}, 1 \right\} \right), \tag{28}$$

and $\psi(n) \equiv \frac{\alpha(n)}{n}$, while $\omega(y) \equiv \theta c(y) + (1 - \theta)u(y)$. The price of the liquid asset at $t - 1$ equals the discounted sum of the price plus dividend at t, $\frac{q_t + \kappa}{1 + r}$, multiplied by a liquidity factor, the term in braces in (26). Any equilibrium $\{q_t\}_{t=0}^{+\infty}$ must also satisfy $q_t \geq q^*$, since the price cannot be less than the fundamental price. The next lemma shows that we can also restrict attention to bounded sequences.

Lemma 5 *A sequence $\{q_t\}_{t=0}^{\infty}$ that solves (26) satisfies $\lim_{t \to \infty} \beta^t q_t = 0$ iff q_t is bounded.*

To analyze $q_{t-1} = \Gamma(q_t)$, we define two thresholds. The first is the price above which households have enough liquidity to buy y^*, $\bar{q} = \frac{\omega(y^*)}{A} - \kappa$. The second is the price below which firms stop participating in the DM,

$$\underline{q} = \frac{\omega\left[\Delta^{-1}\left(\frac{k}{1-\theta}\right)\right]}{A} - \kappa \ \text{if} \ \frac{k}{1 - \theta} \leq u(y^*) - c(y^*)$$

$$= \infty \ \text{otherwise}$$

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where $\Delta(y) = u(y) - c(y)$ for $y \in [0, y^*]$. We claim $\Gamma(q_t)$ is linear whenever $q_t \notin \left(\underline{q}, \bar{q} \right)$, because $q_t < \underline{q}$ implies $n_t = 0$ and $q_t > \bar{q}$ implies $y_t = y^*$, and in either case the asset has no liquidity value at the margin. In contrast, $q_t \in \left(\underline{q}, \bar{q} \right)$ exceeds the fundamental price because the asset facilitates DM trade. So the nonlinear part of $\Gamma(q_t)$ reflects the existence of a liquidity premium or bubble.

Lemma 6 $\Gamma(q_t)$ *is continuous. If* $k \geq k^*$, *then* $\Gamma(q) = \frac{rq^* + q}{1+r}$ *for all* q. *If* $k < k^*$, *then* $\underline{q} < \bar{q}$ *and the following is true:* $\Gamma(q) = \frac{rq^* + q}{1+r}$ *for all* $q \notin \left(\underline{q}, \bar{q} \right)$ *and* $\Gamma(q) > \frac{rq^* + q}{1+r}$ *otherwise.*
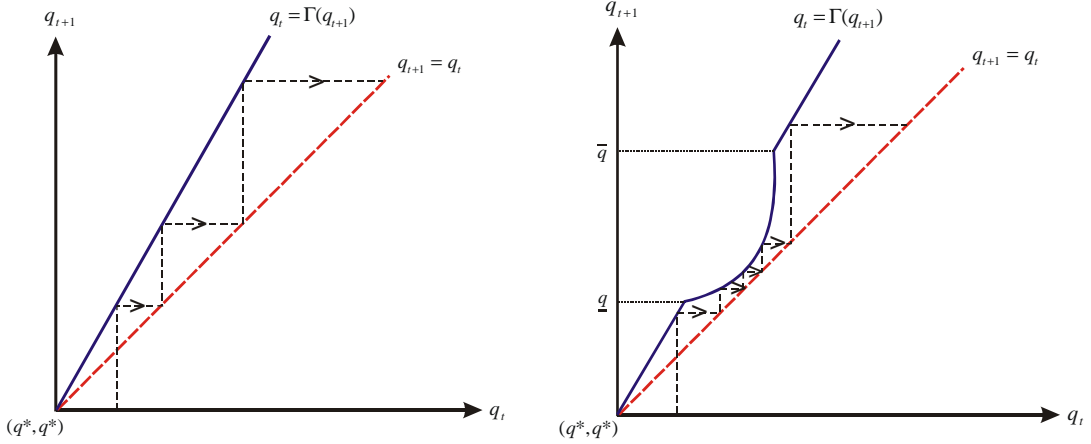

Figure 5: Left: $A \geq \frac{\omega(y^*)}{q^* + \kappa}$; Right: $A < \frac{\omega(y^*)}{q^* + \kappa}$ and $k > \hat{k}$.

We now dispense with some simple cases. Consider abundant liquidity, $A \geq \frac{\omega(y^*)}{q^* + \kappa}$. Then there are two subcases, $k < k^*$ and $k > k^*$. When $k < k^*$, the phase diagram is as shown in the left panel of Figure 5. In this case $q_{t-1} = \Gamma(q_t)$ is linear with slope $1 + r$, so any nonstationary solution to $q_{t-1} = \Gamma(q_t)$ grows asymptotically at rate r, which violates transversality. The unique equilibrium is $q_t = q^*$, the same as with perfect credit. Next, when $k > k^*$ firms do not enter irrespective of q, but the phase diagram still looks like the left panel of Figure 5. These cases with abundant liquid wealth are covered in part (i) of Proposition 7 below.

Now consider the case where liquidity is scarce, $A < \frac{\omega(y^*)}{q^* + \kappa}$, covered in Proposition 7 (ii). If $k \in \left(\hat{k}, k^* \right)$, there are values of q such that firms enter, shown as the nonlinear part of $q_{t-1} = \Gamma(q_t)$ in the right panel of
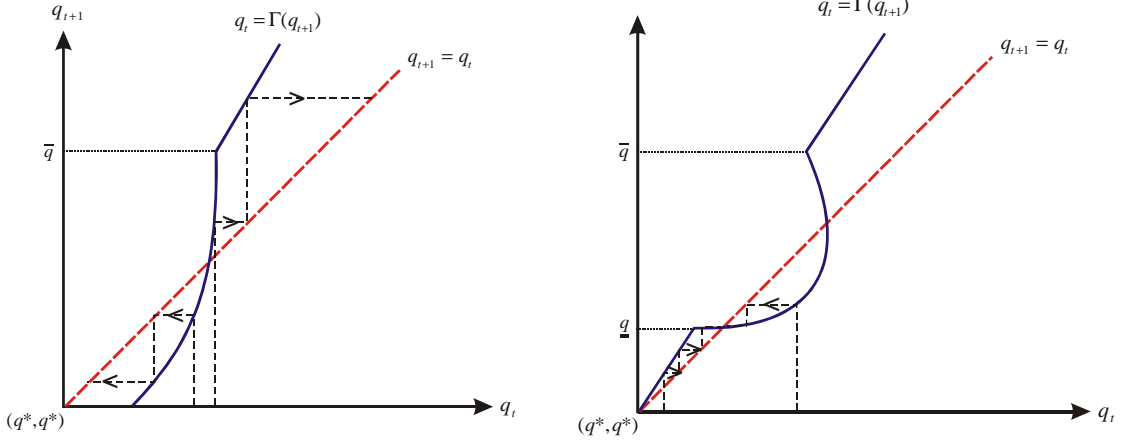
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Figure 6: Left: $A < \frac{\omega(y^*)}{q^*+\kappa}$ and $k < \tilde{k}$; Right: $k \in \left(\tilde{k}, \hat{k}\right)$.

Figure 5, but the unique equilibrium is still $q_t = q^*$ since anything else violates transversality. In contrast, if the entry cost is sufficiently low, $k < \tilde{k}$, in any equilibrium the DM is active and the price of the liquid asset is above its fundamental value. Moreover, if Γ looks like the left panel of Figure 6 – i.e. its slope at the stationary equilibrium is greater than one – the equilibrium is unique and it is stationary. When there are multiple stationary equilibria, $k \in \left(\tilde{k}, \hat{k}\right)$, there can also be multiple non-stationary equilibria, as in the right panel of Figure 6. In this case, there are a continuum of trajectories, starting from different q_0 between the fundamental price and the higher stationary price, that all converge to an intermediate stationary equilibrium.

Proposition 7   (i) If $A \geq \frac{\omega(y^*)}{q^*+\kappa}$, then $q_t = q^*$ and $n_t = n$ solves (14).

(ii) Assume $A < \frac{\omega(y^*)}{q^*+\kappa}$. If $k > \hat{k}$, then $q_t = q^*$ and $n_t = 0$. If $k < \tilde{k}$, then $q_t > q^*$ and $n_t > 0$ for all $t \geq 0$ in any equilibrium. If $k \in \left(\tilde{k}, \hat{k}\right)$, then there are multiple equilibria including $(q_t, n_t) = (q^*, 0)$ for all t, and a continuum of nonstationary equilibria.

In the equilibria described so far q_t varies monotonically over time. There can also be cycles, where q_t, n_t, and y_t fluctuate over time. We proceed by way of example, using $f(y) = y$, $\alpha(n) = 1 - e^{-n}$ and

$$u(y) = \frac{(y + 0.1)^{1-\eta} - 0.1^{1-\eta}}{1 - \eta}.$$

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For these examples, we fix $r = 0.1$, $\kappa = 0.1$ and $\theta = 0.4$, and vary the other parameters. In particular, if the utility parameter η is large, Γ bends backward as shown in Figure 7. In this case, an increase in q_{t+1} has two effects: it drives q_t up, as in any standard model; and it reduces the liquidity premium (when the asset price increases, total liquid wealth rises and it becomes worth less at the margin). If η is large the second effect dominates, and Γ slopes downward across the 45° line. If the slope of Γ is less than 1 in absolute value, there exists a continuum of q_0 in the neighborhood of steady state such that q_t and n_t converge nonmonotonically to steady state. Consequently, even when the stationary equilibrium is unique, as in the right panel of Figure 7, we can have indeterminacy of dynamic equilibria, and fluctuations in prices and quantities.
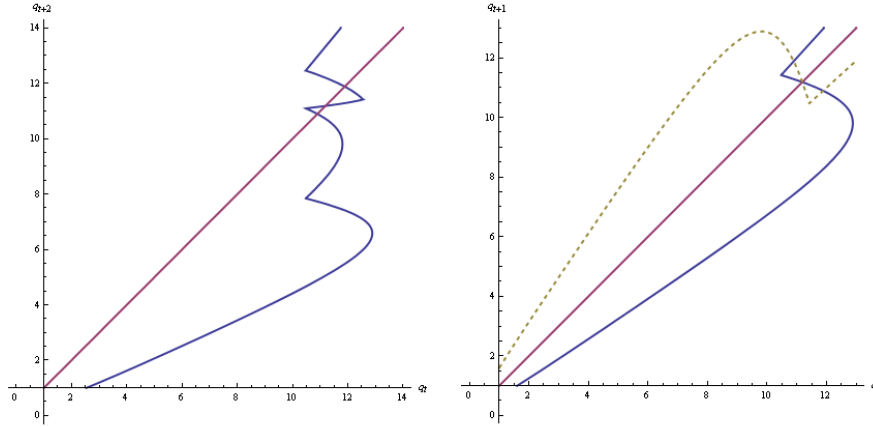

Figure 7: Two-period cycles: $\eta = 2$, $A = 0.5$, $k = 0.2$.

Moreover, when the slope of Γ on the 45° line passes -1, the system experiences a flip bifurcation, giving rise to 2-cycles.[10] In the left panel of Figure 7, 2-period cycles are fixed points of the second iterate of the system, $q_t = \Gamma^2(q_{t+2})$. Alternatively, as in the right panel, the cycles can be found at the intersection of $q_t = \Gamma(q_{t+1})$ and its inverse. The simple intuition for a 2-cycle is as follows. When q is low, agents anticipate it will increase and liquid wealth will rise – hence, a marginal unit of the asset will have a small liquidity premium. Conversely, if q is high, agents anticipate it will fall and liquidity will become scarce – hence, a

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[10] See Azariadis (1993, p.95-97) for a textbook treatment. Since the mathematics are well known, we do not dwell on technical details. The contribution here is intended to be our model of an economy where assets have a transactions role. Given the model, once we derive the relevant difference equation, the application dynamical system theory is standard. What we think is novel is the application of these tools in our model of the asset market.
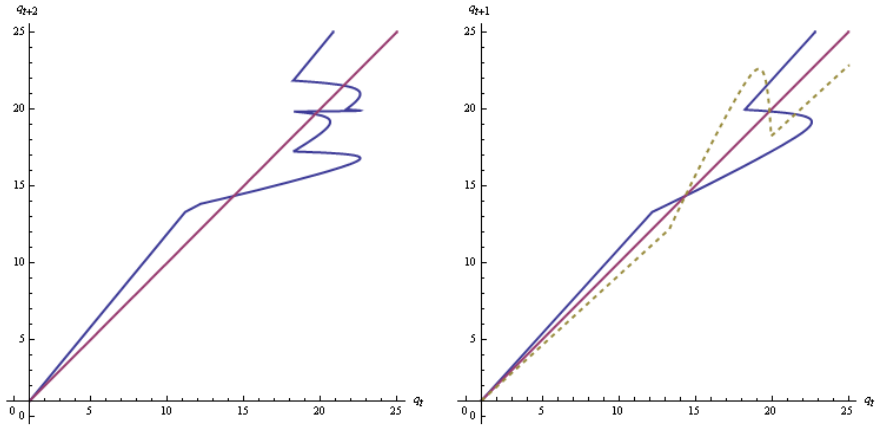
Figure 8: Multiple stationary equilibria and two-period cycles: $\eta = 3$, $A = 1.5$, $k = 20$

big liquidity premium. While Figure 7 has a unique stationary equilibrium with a 2-cycle around it, Figure 8 has multiple stationary equilibria with a 2-period cycle around the highest one. In both cases q alternates between a situation where households are liquidity constrained and one where they are not.
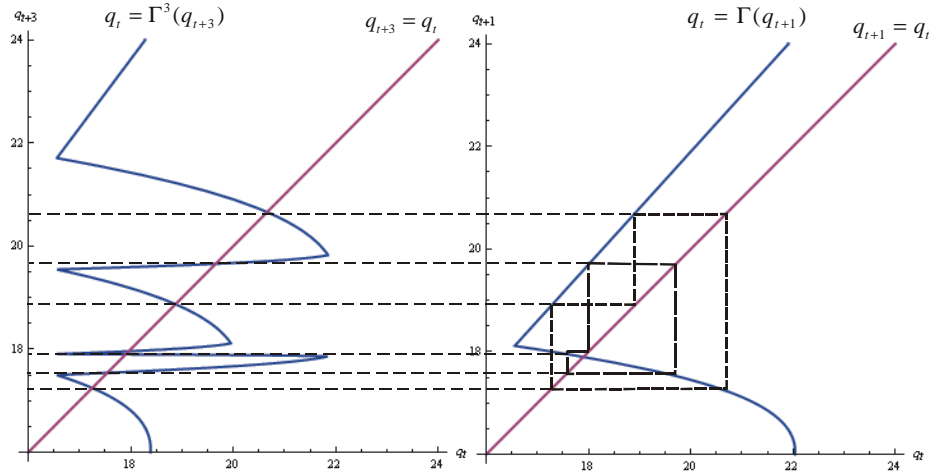

Figure 9: Three-period cycle: $\eta = 2.3$, $A = 0.5$, and $k = 0.2$.

The trajectory shown in Figure 1 in the Introduction, with fluctuating asset prices followed by a crash, corresponds to an example like the one in Figure 8 with parameter values $\eta = 3$, $A = 1.5$, and $k = 20$. During

the expansion phases, the return on the liquid asset is equal to the rate of time preference and households are not liquidity constrained, but the price cannot keep on increasing, or we would violate transversality. We again have fluctuations around a high-price stationary equilibrium, but now we crash at some point toward a lower-price equilibrium. The timing of the crash is indeterminate – we can make it happen whenever we like. All agents in the model know the bubble will burst, and they know exactly when, in this perfect foresight equilibrium, but there is nothing they can do to either avoid it or profit from it. Moreover, as η increases further, the system can generate periodic equilibria of higher order, including 3-cycles as shown in Figure 9. Once 3-cycles exist, then all periodic orbits exist, including ∞-cycles, or chaotic dynamics (Azariadis 1993, p.107). Hence, once the transactions role of assets and liquidity are modeled seriously, prices can display a wide range of dynamic behavior, even in perfect foresight equilibrium with rational agents.

6 Stochastic Equilibria

So far, we have described deterministic equilibria, where agents have perfect foresight. In this section we introduce extrinsic uncertainty – a sunspot – to construct equilibria where the economy fluctuates randomly between states with different asset prices, trade volume and output.[11] The sample space of the sunspot variable s is $S = \{\ell, h\}$, and s follows a Markov process with $\lambda_{ss'} = \Pr[s_{t+1} = s' s_t = s]$, with s observed by all agents at the start of each CM. We focus on equilibria where q_s and n_s are time-invariant functions of s (i.e., stationary sunspot equilibria). Of course, as usual, there are equilibria where agents ignore s. A proper sunspot equilibrium requires that prices or allocations are different in the two states $s = \ell$ and $s = h$.

Following standard reasoning, we can write the household problem as

$$\max_{a \geq 0} \left\{ -q_s a + \beta \alpha(n_s) \theta \left\{ u \left[y((\bar{q}_s + \kappa)a) \right] - c \left[y((\bar{q}_s + \kappa)a) \right] \right\} + \beta(\bar{q}_s + \kappa)a \right\}, \tag{29}$$

where $\bar{q}_s = \sum_{s' \in S} \lambda_{ss'} q_{s'}$, and the free entry condition as

$$\frac{\alpha(n_s)}{n_s}(1 - \theta) \left\{ u \left[y((\bar{q}_s + \kappa)A) \right] - c \left[y((\bar{q}_s + \kappa)A) \right] \right\} \leq k, \tag{30}$$

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[11]We argued earlier that it does not matter whether the asset is used in the DM as a means of payment or as collateral, since all parties are indifferent to whether the buyer repays his debt in the CM or defaults and leaves his collateral. This is clear with perfect foresight, but in sunspot equilibria the equivalence is more subtle, because the CM value of an asset can depend upon the realization of the state. Now repayment can be contingent on the state, of course, but suppose for the sake of argument it is not. This potentially gives a buyer strict incentive to default in some states. Given quasi-linear utility, however, this is actually immaterial, since default can be taken into account when the loan terms are negotiated.

with an equality if $n_s > 0$. The expected value of one unit of a before entering the next CM is $\bar{q}_s + \kappa$, where $\bar{q}_s$ is the expected price conditional on current s. The first-order condition of the household together with $a = A$ yield

$$q_s = \beta(\bar{q}_s + \kappa)\left\{1 + \alpha(n_s)\theta\left\{\frac{u'\left[y((\bar{q}_s + \kappa)A)\right] - c'\left[y((\bar{q}_s + \kappa)A)\right]}{(1 - \theta)u'\left[y((\bar{q}_s + \kappa)A)\right] + \theta c'\left[y((\bar{q}_s + \kappa)A)\right]}\right\}\right\}. \tag{31}$$

Definition 3 *A (proper, stationary, two-state) sunspot equilibrium has $(q_t, n_t) = (q_s, n_s)$ in state s, satisfying (30) and (31), with $(q_\ell, n_\ell) \neq (q_h, n_h)$.*

Although other outcomes are possible, for dramatic effect consider equilibria with $n_\ell = 0$ and $n_h > 0$, where the DM completely shuts down whenever $s = \ell$, and reopens whenever "animal spirits" stochastically switch back to $s = h$. Note that in any such an equilibrium, $q_\ell > q^*$, so there is a positive liquidity premium or bubble component to the asset price even when the DM is inactive, because of the expectation that it will reopen at some random date. We have the following result.[12]

Proposition 8 *For all $k \in \left(\tilde{k}, \hat{k}\right)$, if $\lambda_{\ell h}$ and $\lambda_{h\ell}$ are sufficiently small there exists a sunspot equilibrium where $0 = n_\ell < n_h$ and $q^* < q_\ell < q_h$.*

7   Applications

We consider three applications of our model. First we discuss the provision of liquid assets by the public sector, and ask whether government should try to eliminate liquidity premia or bubbles. We then study how the private sector might endogenously provide liquidity for itself. Then we use the model to think about firms undertaking collateralized loans in over-the-counter markets to finance the acquisition of capital or other intermediate goods.

7.1   Public Liquidity Provision

So far, we have seen that imperfect credit and scarce liquidity can generate various types of endogenous instability, including periodic, chaotic and stochastic equilibria. Here we investigate the effects of public

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[12] The methodology for constructing sunspot equilibria is similar Wright (1994) and Ennis (2001a). As in Azariadis and Guesnerie (1986) or Guesnerie (1986), one could also construct sunspot equilibria from the 2-period cycles in Section 5. One difference is that the method here generates sunspot equilibria that are very persistent, while the other method yields equilibria that are close to 2-cycles and hence not persistent.

liquidity provision. Suppose the government can issue one-period real bonds backed by its ability to tax: each bond issued in the CM at t is a claim to 1 unit of the numéraire x in the CM at $t + 1$. Assume bonds are recognizable (non-counterfeitable), which means that they can be traded, in the DM. The government budget constraint is $B_{t-1} = q_{b,t}B_t + T_t$, where B_t is the quantity of bonds issued at t, $q_{b,t}$ the price, and T_t is a lump-sum tax. The fundamental price of this bonds is $q_b^* = \beta$.

The household's problem can now be written

$$\max_{a \geq 0, b \geq 0} \left\{ -r\left[(q_{t-1} - q^*) - (q_t - q_{t-1}) \right] a - \left[(1+r)q_{b,t-1} - 1 \right] b + \alpha(n_t)\theta \left[u(y_t) - c(y_t) \right] \right\},$$

where $y_t = y^*$ if $\omega(y^*) \leq (q_t + \kappa)a + b$ and $\omega(y_t) = (q_t + \kappa)a + b$ otherwise. The FOC imply:

$$\frac{r(q_{t-1} - q^*) - (q_t - q_{t-1})}{q_t + \kappa} = (1+r)q_{b,t-1} - 1$$

$$= \alpha(n_t)\theta \left\{ \frac{u'(y_t) - c'(y_t)}{\theta c'(y_t) + (1-\theta)u'(y_t)} \right\}. \tag{32}$$

For the liquid asset and bonds to both be held, since they are equally liquid, they must have the same return, $\frac{q_t + \kappa}{q_{t-1}} = \frac{1}{q_{b,t-1}}$. Moreover, the real interest rate $\frac{1 - q_{b,t-1}}{q_{b,t-1}}$ is less than the discount rate whenever $y_t < y^*$. Using free-entry condition, welfare in equilibrium is

$$V = \alpha(n_0) \left[u(y_0) - c(y_0) \right] + n_0 f(1) + \sum_{t=1}^{\infty} \beta^t \alpha(n_t)\theta \left[u(y_t) - c(y_t) \right] + \frac{A\kappa}{1 - \beta}. \tag{33}$$

Thus, welfare at $t > 0$ is $\alpha(n_t)$ times the DM surplus $\theta\left[u(y_t) - c(y_t) \right]$, plus the CM output of the trees $A\kappa$. Also, at $t = 0$ households enjoy an expected surplus of $\alpha(n_0)\left[u(y_0) - (q_0 + \kappa)\tau_{a,0} \right]$ and firms earn revenue $\alpha(n_0)\left[-c(y_0) + (q_0 + \kappa)\tau_{a,0} \right] + n_0 f(1)$.

Proposition 9 *Assume that $(1 - \theta)\left[u(y^*) - c(y^*) \right] > k$. The optimal provision of bonds is such that $B \geq \omega(y^*) - (q^* + \kappa)A$. It achieves $q = q^*$, $q_b = q_b^*$, and $y = y^*$.*

This result says that when there is a shortage of private assets, $\omega(y^*) > (q^* + \kappa)A$, government should supplement the stock of liquidity with enough bonds so that agents can trade y^* in the DM, which means they are satiated in liquidity, and we deflate the liquidity premium to 0. Although this policy implies DM trade is efficient, as suggested by Proposition 6, the measure of firms is generically inefficient. Also note

that, if policy is not optimal, an increase B does not necessarily reduce q. Figure 10 shows an example where, in the absence of intervention, there exist one inactive and two active stationary equilibria. Suppose we introduce some bonds, but the total supply of liquid assets is not sufficient to allow agents to trade y^* in the DM. This eliminates the inactive and the low equilibria, but the high equilibrium remains. So if the economy is initially at the equilibrium with low q, an increase in liquidity can lead to a larger q.
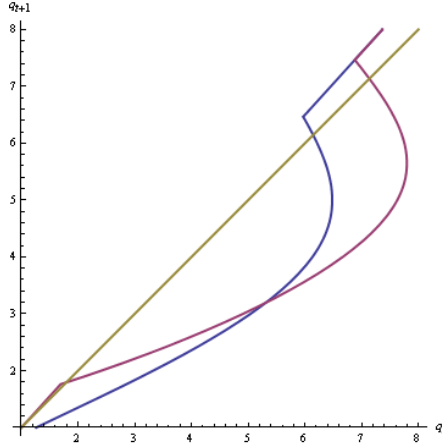

Figure 10: Phase lines: $\eta = 1.5$, $b = \kappa = r = 0.1$, $\theta = 0.6$, $A = 0.3$, and $B \in \{0, 0.3\}$

Also, the result that the provision of liquidity is optimal when agents are satiated, in the sense that they can trade y^*, is not robust to details of the specification. In particular, it is sensitive to our choice of DM pricing mechanism. In the Appendix we show that if we use Walrasian pricing instead of bargaining, it can be optimal for some parameter values to keep liquidity scarce and accept $y_t < y^*$. This can be the case because firms entering the DM do not internalize congestion on other firms, so entry can be too high. Policy can mitigate this by making liquid assets costly to hold, which requires the price to be above its fundamental value. This cannot happen under bargaining as in our benchmark model, but can under Walrasian pricing.[13] Additionally, we can obtain a similar result in a version of the model with bargaining but endogenous entry of households, instead of firms. (Details are available on request). In this version of the model, if households have too much bargaining power then it is optimal keep liquidity scarce. More work could be done on these

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[13] For related results showing how optimal policy can depend on the mechanism, see Rocheteau and Wright (2005,2009).

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policy problems, but this would take us away from our main objective here, which is to show how economies with scarce liquidity can display a wide range of interesting dynamic equilibria.

7.2 Private Liquidity Provision

There can be endogenous mechanisms that produce liquidity when assets are in short supply. To illustrate this idea, consider the case where stocks s_t (equity claims on firm revenue) can be authenticated at no cost in the DM, so that they can also be used to facilitate DM trade (for simplicity here we ignore government bonds). Then a household with a portfolio (a_t, s_t) obtains y_t units of DM output, where $y_t = y^*$ if $(q_t + \kappa)a_t + R_t s_t \geq \omega(y^*)$ and $(q_t + \kappa)a_t + R_t s_t = \omega(y_t)$ otherwise. The household problem is

$$\max_{a \geq 0, s \geq 0} \left\{ -\left(\beta^{-1} - \frac{q_t + \kappa}{q_{t-1}} \right) q_{t-1} a - \left(\beta^{-1} - R_t \right) s + \alpha(n_t)\theta \left[u\left(y_t\right) - c\left(y_t\right) \right] \right\}.$$

Again, all assets have the same rate of return,

$$R_t = \frac{q_t + \kappa}{q_{t-1}} = \beta^{-1} \left\{ 1 + \alpha(n_t)\theta \left[\frac{u'(y_t) - c'(y_t)}{\theta c'(y_t) + (1 - \theta)u'(y_t)} \right] \right\}^{-1}, \tag{34}$$

where

$$\omega(y_t) = \min \left[\omega(y^*), R_t \left(q_{t-1}A + n_t k^f \right) \right]. \tag{35}$$

The measure of firms solves

$$\beta \left\{ 1 + \alpha(n_t)\theta \left[\frac{u'(y_t) - c'(y_t)}{\theta c'(y_t) + (1 - \theta)u'(y_t)} \right] \right\} \left[\frac{\alpha(n_t)}{n_t}(1 - \theta) \left[u(y_t) - c(y_t) \right] + f(1) \right] = k^f. \tag{36}$$

If there is a shortage of liquid assets then $y_t < y^*$ and the real interest rate falls below the rate of time preference, which tends to increase the number of firms and hence inside liquidity.[14] This is a mechanism for the endogenous private provision of liquidity. An equilibrium is a bounded sequence $\{(n_t, q_t)\}_{t=0}^{\infty}$ that solves (34) and (36). This model is harder to solve, because we can not use (36) to determine n_t uniquely as a function of q_t. The measure of participating firms n_t affects profitability not only through trading probabilities, as in standard search models, but also through the liquidity that these firms provides by their very existence. Hence we analyze some examples.

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[14] The terminology of inside and outside liquidity is used in Holmstrom and Tirole (2008, p.57), presumably by analogy to the notions of inside and outside monies.

We first characterize equilibria where there is no shortage of liquidity so $y_t = y^*$. Then (34) implies $R_t = \beta^{-1}$, and the only bounded sequence of asset prices solving $\frac{q_t + \kappa}{q_{t-1}} = \beta^{-1}$ is the stationary solution $q_t = q^*$. From (36), $n_t = n^*$, so the outcome corresponds to the one with perfect credit. From (35) $y_t = y^*$ if and only if $q^* A + n^* k^f \geq \beta \omega(y^*)$, which generalizes the condition in the previous sections to the case where stocks are liquid.[15] Clearly, when stocks are liquid, a liquidity shortage is less likely.

Now consider the case where there is a shortage of liquid wealth and $\theta = 0$ (households have no bargaining power). From (34), assets have no liquidity value since households get no surplus from DM trade. Then $R_t = \beta^{-1}$ and $q_t = q^*$. If stocks were illiquid, as in the previous section, the number of firms would be uniquely determined. To see that this is no longer the case, define

$$\Gamma(n_t) = \beta \left[\frac{\alpha(n_t)}{n_t} \left[u(y_t) - c(y_t) \right] + f(1) \right] - k^f,$$

where, from (35), $\omega(y_t) = \min \left[\omega(y^*), \beta^{-1} \left(q^* A + n_t k^f \right) \right]$. An equilibrium is an n_t solving $\Gamma(n_t) = 0$. As shown in the Appendix and illustrated in Figure 11, if A is small there is an equilibrium with $n_t = 0$ and an even number with $n_t > 0$. Across equilibria DM output, the measure of active firms, and the total value of the stock market are positively related. This multiplicity arises because the creation of a firm has a positive externality on other firms, since it generates additional liquidity that facilitates DM trade.

Finally, we consider the case where firms have no bargaining power in the DM, $\theta = 1$. This illustrates another channel for private liquidity provision and multiple equilibria. From (34) and (36), $\frac{q}{q+\kappa} f(1) = k^f$, which gives

$$q = \frac{\kappa k^f}{f(1) - k^f}. \tag{37}$$

Under the assumption $-k^f + \beta f(1) < 0$, we have $q > \frac{\kappa}{r} = q^*$. For entry to occur the asset price must be greater than its value in the perfect credit economy, and the real interest must be lower than the rate of time preference. From (35),

$$y = \min \left[y^*, c^{-1} \left(\frac{\kappa f(1)}{f(1) - k^f} A + f(1) n \right) \right]. \tag{38}$$

Thus, y increases with the measure of firms since stocks are part of households' liquid wealth. From (36),

the measure of firms solves

$$\left\{1 + \alpha(n)\left[\frac{u'(y) - c'(y)}{c'(y)}\right]\right\}^{-1} - \beta\frac{f(1)}{k^f} = 0. \tag{39}$$

Provided that $\beta f(1)/k^f$ is close 1, there are multiple steady-state equilibria (see Appendix): an inactive equilibrium with $n = 0$ and $R = \beta^{-1}$; and an even number of active equilibria with $n > 0$, where $q > q^*$ and $R < \beta^{-1}$. Firms are willing to participate even though they receive no surplus in the DM because the real interest rate is low, and households accept a low rate of return on their wealth because of its liquidity value in the DM.
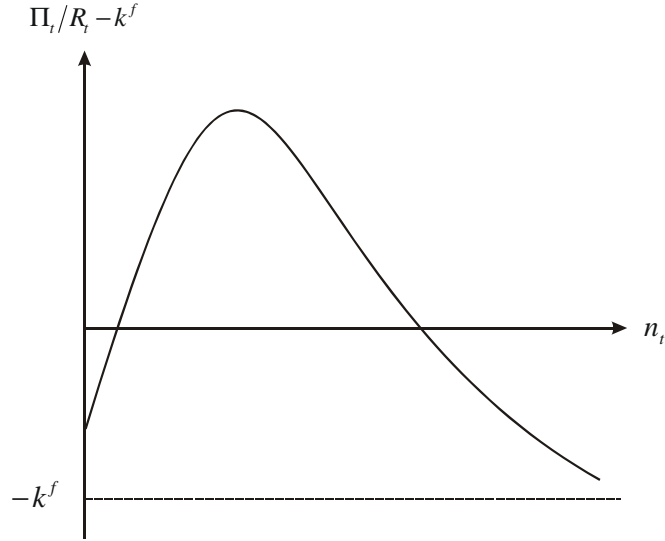
Figure 11: Multiple equilibria with liquid stocks

7.3 Corporate Finance

Consider a reinterpretation of the model where the liquid asset is used to reallocate capital among firms. Suppose that risk-neutral households only value CM goods, as in standard search models of the labor market. A unit measure of firms produce the CM good using intermediate goods according to the technology $zf(i)$, where $z \in \{z_L, z_H\}$ with $z_L < z_H$ is an idiosyncratic productivity shock. Each firm is endowed with one unit of the intermediate good in the CM, which fully depreciates in the next DM (one could also at the cost of simplicity endogeneize firms' CM investments in intermediate goods). At the beginning of each period,

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a firm draws a realization of z. With probability π_H, $z = z_H$ and with probability π_L, $z = z_L$. We now interpret the DM as an over-the-counter market where firms reallocate capital goods. Suppose a firm with high productivity meets (where we use this term loosely) a firm with low productivity with probability $\alpha\pi_L$, and a firm with low productivity meets a firm with high productivity with probability $\alpha\pi_H$.

Firms are not liquidity constrained in the CM – i.e., they can borrow at the interest rate r – but in the DM they can only issue loans secured with liquid assets. Defining

$$u(y) = z_H\left[f(1+y) - f(1)\right]$$

$$c(y) = z_L\left[f(1) - f(1-y)\right],$$

the efficient transfer of capital goods leads to $z_H f'(1+y^*) = z_L f'(1-y^*)$. Using the same reasoning as above, the price of the liquid asset solves

$$q_{t-1} = \frac{q_t + \kappa}{1+r}\left\{1 + \alpha\pi_H\pi_L\theta\left[\frac{z_H f'(1+y_t) - z_L f'(1-y_t)}{\theta z_L f'(1-y_t) + (1-\theta)z_H f'(1+y_t)}\right]\right\},$$

where

$$y_t = y^* \ \text{ if } (q_t+\kappa)A \geq \theta z_L\left[f(1) - f(1-y^*)\right] + (1-\theta)z_H\left[f(1+y^*) - f(1)\right]$$

$$(q_t+\kappa)A = \theta z_L\left[f(1) - f(1-y_t)\right] + (1-\theta)z_H\left[f(1+y_t) - f(1)\right] \ \text{ otherwise.}$$

Now all of the earlier results go through. Much more can be done with this model in terms of substantive applications, but we leave that to future work.

8   Conclusion

We studied the dynamics of asset markets in economies where frictions make credit difficult, which makes liquidity essential, and where market participation is costly. One message is that economies where liquidity plays a role are inherently unstable. There can be multiple steady-state equilibria that differ in terms of asset prices, participation, and stock market capitalization. Markets can break down and liquidity premia vanish because of self-fulfilling beliefs. The model can generate periodic equilibria, with fluctuating prices and allocations, even though fundamentals are constant. We can also generate stochastic crashes and recoveries

driven by extrinsic uncertainty (sunspots). Some of the equilibria are reminiscent of bubbles: a runup followed by a collapse in asset prices that cannot be explained by fundamentals. These results are consistent with rationality on the part of the agents in the model, who have perfect foresight or rational expectations. In terms of policy, the government can in principle supply enough liquidity to satiate households and thus eliminate instability; depending on the search frictions, this might or might not be optimal.

We close by saying that we do not want any particular example in this paper to be considered the definitive model of an asset market. The paper should be interpreted as an illustration of the power and flexibility of theories that take seriously the transactions role of assets – and by this we mean all assets, not merely fiat currency. Our conclusion is that much can be learned once it is understood that there are important complementarities between asset pricing, in general, and monetary theory.

Appendix A. Proofs

Proof of Lemma 2 The necessity and sufficiency of the transversality condition (20) is proved in Appendix B. The household's objective function in (19) is denoted

$$\Psi_t(a) \equiv -\left(r\left(q_{t-1} - q^*\right) - \left(q_t - q_{t-1}\right)\right)a + \alpha(n_t)\theta\left\{u\left[y((q_t + \kappa)a)\right] - c\left[y((q_t + \kappa)a)\right]\right\},$$

where $y((q_t + \kappa)a) = y^*$ if $(q_t + \kappa)a \geq (1-\theta)u(y^*) + \theta c(y^*)$ and $y((q_t + \kappa)a)$ is the solution to (17) otherwise. This objective function is concave. To see this, differentiate Ψ_t to obtain

$$\Psi_t'(a) = -\left(r\left(q_{t-1} - q^*\right) - \left(q_t - q_{t-1}\right)\right) + \alpha(n_t)\theta\left\{\frac{u'(y) - c'(y)}{\theta c'(y) + (1-\theta)u'(y)}\right\}(q_t + \kappa).$$

For all $a < \frac{(1-\theta)u(y^*) + \theta c(y^*)}{q_t + \kappa}$, $y'((q_t + \kappa)a) > 0$ and, since $\frac{u'(y) - c'(y)}{\theta c'(y) + (1-\theta)u'(y)}$ is decreasing in y, $\Psi''(a) < 0$. For all $a \geq \frac{(1-\theta)u(y^*) + \theta c(y^*)}{q_t + \kappa}$, $y((q_t + \kappa)a) = y^*$ and $\Psi'(a) = -\left(r\left(q_{t-1} - q^*\right) - \left(q_t - q_{t-1}\right)\right)$. We distinguish three cases:

(i) If $r\left(q_{t-1} - q^*\right) > q_t - q_{t-1}$, then $a_t \leq \frac{(1-\theta)u(y^*) + \theta c(y^*)}{q_t + \kappa}$, since otherwise $a_t\Psi_t'(a_t) < 0$ and the first-order condition is violated. Over the interval $[0, \frac{(1-\theta)u(y^*) + \theta c(y^*)}{q_t + \kappa}]$ $\Psi_t(a)$ is continuous and strictly concave, so (19) has a unique solution given by (21).

(ii) If $r\left(q_{t-1} - q^*\right) = q_t - q_{t-1}$, then any a_t such that $u'(y) = c'(y)$ is a solution. Hence, $a_t \in [\frac{(1-\theta)u(y^*) + \theta c(y^*)}{q_t + \kappa}, +\infty)$.

(iii) If $r\left(q_{t-1} - q^*\right) < q_t - q_{t-1}$, then the solution to (19) is unbounded.

Proof of Lemma 3 Take q and therefore $y\left[(q + \kappa)A\right]$ as given. From (23), any interior solution for n solves

$$\frac{\alpha(n)}{n} = \frac{k}{(1-\theta)\left[u(y) - c(y)\right]}. \tag{40}$$

Since $\frac{\alpha(n)}{n}$ is decreasing in n with $\frac{\alpha(0)}{0} = 1$ and $\lim_{n \to \infty} \frac{\alpha(n)}{n} = 0$, a solution $n > 0$ to (40) exists, and is unique, iff $\frac{k}{(1-\theta)[u(y) - c(y)]} < 1$, which gives (24). If $\frac{k}{(1-\theta)[u(y) - c(y)]} \geq 1$, then $\frac{\alpha(0)}{0}(1-\theta)\left[u(y) - c(y)\right] - k \leq 0$ and $n = 0$. Differentiating (40) we obtain

$$n'(q) = \frac{\alpha(n)(1 - \theta)\left[u'(y) - c'(y)\right]}{k\left(1 - \frac{\alpha'(n)n}{\alpha(n)}\right)}\frac{\partial y}{\partial q},$$

where $\frac{\alpha'(n)n}{\alpha(n)} < 1$ from the strict concavity of $\alpha(n)$ and the assumption $\alpha(0) = 0$. From (17), $y < y^*$ and $\frac{\partial y}{\partial q} = \frac{A}{\theta c'(y) + (1-\theta)u'(y)} > 0$ if $(q + \kappa)A < \omega(y^*)$ and $y = y^*$ and $\frac{\partial y}{\partial q} = 0$ if $(q + \kappa)A > \omega(y^*)$. Consequently, $n'(q) > 0$ if $(q + \kappa)A < \omega(y^*)$ and $n'(q) = 0$ if $(q + \kappa)A \geq \omega(y^*)$.

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Proof of Lemma 4   Define

$$\Gamma(q;n) \equiv \frac{r(q-q^*)}{q+\kappa} - \alpha(n)\theta \left\{ \frac{u'\left[y((q+\kappa)A)\right] - c'\left[y((q+\kappa)A)\right]}{\theta c'\left[y((q+\kappa)A)\right] + (1-\theta)u'\left[y((q+\kappa)A)\right]} \right\}.$$

From (21) q in stationary equilibrium solves $\Gamma(q;n) = 0$. Since $\frac{u'(y)-c'(y)}{\theta c'(y)+(1-\theta)u'(y)}$ is decreasing in y and $y(\cdot)$ is decreasing in $(q+\kappa)A$, $\Gamma'(q;n) > 0$, $\Gamma(q^*;n) \leq 0$ and $\Gamma(\infty) = r$. Hence, there is a unique $q \in [q^*, +\infty)$ such that $\Gamma(q;n) = 0$. We distinguish two cases:

(i) If $(q^* + \kappa)A \geq \omega(y^*)$, then (15)-(16) imply $y((q^* + \kappa)A) = y^*$ and $\Gamma(q^*;n) = 0$. So $q(n) = q^*$ for all n.

(ii) If $(q^* + \kappa)A < \omega(y^*)$, then $y((q^* + \kappa)A) < y^*$ and $\Gamma(q^*;n) < 0$ for all $n > 0$. Then the solution to $\Gamma(q;n) = 0$ is such that $q > q^*$. Differentiating $\Gamma(q;n) = 0$,

$$q'(n) = \frac{\alpha'(n)\theta}{\Gamma'(q;n)} \left\{ \frac{u'\left[y((q+\kappa)A)\right] - c'\left[y((q+\kappa)A)\right]}{\theta c'\left[y((q+\kappa)A)\right] + (1-\theta)u'\left[y((q+\kappa)A)\right]} \right\} > 0.$$

As $n \to 0$, $\alpha(n) \to 0$, and $\Gamma(q;n)$ converges uniformly to $\frac{r(q-q^*)}{q+\kappa}$. Hence $\lim_{n\to 0} q(n) = q^*$. As $n \to \infty$, $\alpha(n) \to 1$ and $q(n)$ approaches the solution $q_\infty > q^*$ to

$$\frac{r(q-q^*)}{q+\kappa} - \theta \left\{ \frac{u'\left[y((q+\kappa)A)\right] - c'\left[y((q+\kappa)A)\right]}{\theta c'\left[y((q+\kappa)A)\right] + (1-\theta)u'\left[y((q+\kappa)A)\right]} \right\} = 0.$$

Proof of Proposition 3   We distinguish two cases:

(i) $A \geq \frac{\theta c(y^*)+(1-\theta)u(y^*)}{q^*+\kappa}$. For all $q \geq q^*$, $A(q+\kappa) \geq A(q^*+\kappa) \geq \theta c(y^*) + (1-\theta)u(y^*)$. Then n solves (23) where $y((q+\kappa)A) = y^*$ and, from (21), $q = q^*$ since $u'\left[y((q+\kappa)A)\right] = c'\left[y((q+\kappa)A)\right]$.

(ii) $A < \frac{\theta c(y^*)+(1-\theta)u(y^*)}{q^*+\kappa}$. To show equilibrium exists, write (21) as $\Upsilon(q,k) = 0$ where

$$\Upsilon(q,k) \equiv \frac{r(q-q^*)}{q+\kappa} - \alpha\left[n(q,k)\right]\theta \left\{ \frac{u'\left[y((q+\kappa)A)\right] - c'\left[y((q+\kappa)A)\right]}{\theta c'\left[y((q+\kappa)A)\right] + (1-\theta)u'\left[y((q+\kappa)A)\right]} \right\},$$

and $n(q,k)$ solves (23). Then $\Upsilon(q,k)$ is continuous in q with $\Upsilon(q^*,k) \leq 0$ and $\Upsilon(q,k) = \frac{r(q-q^*)}{q+\kappa}$ for all q such that $(q+\kappa)A \geq \omega(y^*)$. Hence, there is a $q \in \left[q^*, \frac{\omega(y^*)-A\kappa}{A}\right]$ such that $\Upsilon(q,k) = 0$.

Next, we introduce the $\hat{k}$ that allows us to distinguish different configurations for the equilibrium set. Define $\mathcal{K} = \{k : \exists q > q^* \text{ st} \Upsilon(q,k) < 0\}$. From (23), $n(q,k)$ is decreasing in k. Hence, $\Upsilon(q,k)$ is increasing in k. So if $k \in \mathcal{K}$, then any $k' < k$ is in $\mathcal{K}$; if $k \notin \mathcal{K}$, then any $k' > k$ is not in $\mathcal{K}$. Moreover, $\mathcal{K}$ is open in $\mathbb{R}_+$ since $k \in \mathcal{K}$ implies $k' > k$ is in $\mathcal{K}$ provided that it is close to k. So $\mathcal{K} = [0, \hat{k})$ for some $\hat{k} \geq 0$. We now show $\hat{k} \in \left[\tilde{k}, k^*\right)$. If $k = k^*$, then $n(q,k) = 0$ and $\Upsilon(q,k) = \frac{r(q-q^*)}{q+\kappa}$ for all $q \geq q^*$. Hence, $k^* \notin \mathcal{K}$. For all $k < \tilde{k}$ and all $q \geq q^*$, $n(q,k) > 0$ and $\Upsilon(q^*,k) < 0$. Consequently, for all $k < \tilde{k}$, $k \in \mathcal{K}$, and by taking the limit when k approaches $\tilde{k}$, $\tilde{k} \leq \hat{k}$.

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We distinguish three subcases:

Consider the case $k < \tilde{k}$. Suppose there is an equilibrium with $n = 0$. Then, from (21), $q = q^*$. But $k < \tilde{k}$ implies $k < (1 - \theta) \{u [y [(q^* + \kappa)A]] - c [y [(q^* + \kappa)A]]\}$ and hence, from Lemma 3, $n > 0$, a contradiction. So any equilibrium is such that $n > 0$. Moreover, $\Upsilon(q^*, k) < 0$. Hence, the solution to $\Upsilon(q, k) = 0$ is such that $q > q^*$.

Consider the case $k \in \left(\tilde{k}, \hat{k}\right)$. First, we establish that $(n, q) = (0, q^*)$ is an equilibrium. If

$$k > \tilde{k} = (1 - \theta) \{u [y [(q^* + \kappa)A]] - c [y [(q^* + \kappa)A]]\},$$

then from (23) $n(q^*, k) = 0$ and $\Upsilon(q^*, k) = 0$. So $(n, q) = (0, q^*)$ is an equilibrium. Second, we establish that generically there exist an even number of steady-state equilibria with $n > 0$ and $q > q^*$. There is $\underline{q} > q^*$ such that $k = (1 - \theta) \{u [y [(\underline{q} + \kappa)A]] - c [y [(\underline{q} + \kappa)A]]\}$. For all $q \in [q^*, \underline{q}]$, $n(q, k) = 0$ and $\Upsilon(q, k) = \frac{r(q - q^*)}{q + \kappa} > 0$. If $k < \hat{k}$, then $k \in \mathcal{K}$ and there is a $q > q^*$ such that $\Upsilon(q, k) < 0$. For all $q \geq \bar{q} \equiv \frac{\omega(y^*)}{A} - \kappa > q^*$, $y [(q + \kappa)A] = y^*$ and $\Upsilon(q, k) \equiv \frac{r(q - q^*)}{q + \kappa} > 0$. Since Υ is continuous in q, there are an even number of values of $q \in (\underline{q}, \bar{q})$ such that $\Upsilon(q, k) = 0$. Moreover, since $q > \underline{q}$, $n(q, k) > 0$.

Consider the case $k > \hat{k}$. If $k > \hat{k}$, then $k \notin \mathcal{K}$ and there is no $q > q^*$ such that $\Upsilon(q, k) < 0$. Suppose there is a $q > q^*$ such that $\Upsilon(q, k) = 0$, which requires $n(q, k) > 0$. For all $k' \in (\hat{k}, k)$, $n(q, k') > n(q, k)$ and $\Upsilon(q, k') < \Upsilon(q, k) = 0$. So, $(\hat{k}, k) \subset \mathcal{K}$. A contradiction. So the only equilibrium is $(n, q) = (0, q^*)$.

Sufficient conditions for $\tilde{k} < \hat{k}$.

Let $\Upsilon(q, \tilde{k}) = C_1(q) - C_2(q)$ where

$$
\begin{aligned}
C_1(q) &= \frac{r(q - q^*)}{q + \kappa}, \\
C_2(q) &= \alpha \left[n(q, \tilde{k})\right] \theta \left\{ \frac{u' [y((q + \kappa)A)] - c' [y((q + \kappa)A)]}{\theta c' [y((q + \kappa)A)] + (1 - \theta)u' [y((q + \kappa)A)]} \right\}.
\end{aligned}
$$

Notice that $n(q^*, \tilde{k}) = 0$ implies $\Upsilon(q^*, \tilde{k}) = 0$. Consequently, a sufficient condition for $\tilde{k} \in \mathcal{K}$ is $C_1'(q^*) < C_2'(q^*)$. It can be checked that $C_1'(q^*) = \frac{r}{q^* + \kappa} < \infty$ and

$$C_2'(q^*) = \theta \left\{ \frac{u' [y((q^* + \kappa)A)] - c' [y((q^* + \kappa)A)]}{\theta c' [y((q^* + \kappa)A)] + (1 - \theta)u' [y((q^* + \kappa)A)]} \right\} \frac{dn}{dq} \bigg|_{q \downarrow q^*}$$

where we used that $\alpha' \left[n(q^*, \tilde{k})\right] = \alpha'(0) = 1$, and where, from (23),

$$\frac{dn}{dq} \bigg|_{q \downarrow q^*} = \frac{2A \frac{u' [y((q^* + \kappa)A)] - c' [y((q^* + \kappa)A)]}{\theta c' [y((q^* + \kappa)A)] + (1 - \theta)u' [y((q^* + \kappa)A)]}}{-\alpha''(0) \{u [y((q^* + \kappa)A)] - c [y((q^* + \kappa)A)]\}},$$

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where we have used that $\frac{\alpha(n)-\alpha'(n)n}{n^2} \approx \frac{-\alpha''(n)}{2}$ when n approaches 0 (since $n(q^*, \tilde{k}) = 0$). As A approaches 0,

$$\frac{u'\left[y((q^* + \kappa)A)\right] - c'\left[y((q^* + \kappa)A)\right]}{\theta c'\left[y((q^* + \kappa)A)\right] + (1 - \theta)u'\left[y((q^* + \kappa)A)\right]} \rightarrow \frac{1}{1 - \theta},$$

where we used that $c'(0) = 0$, and

$$\frac{u\left[y((q^* + \kappa)A)\right] - c\left[y((q^* + \kappa)A)\right]}{A} \rightarrow \frac{q^* + \kappa}{1 - \theta},$$

from L'Hôpital's rule. Consequently, $\lim_{A \to 0} C_2'(q^*) = \frac{\theta}{1-\theta} \frac{2}{-\alpha''(0)(q^*+\kappa)}$. So, as A approaches 0, $C_1'(q^*) < C_2'(q^*)$ is equivalent to $r < \frac{\theta}{1-\theta} \frac{2}{-\alpha''(0)}$. By the continuity of $C_2'(q^*)$ with respect to A, for all $r < \frac{\theta}{1-\theta} \frac{2}{-\alpha''(0)}$, there is a $\bar{A} < \frac{\theta c(y^*)+(1-\theta)u(y^*)}{q^*+\kappa}$ such that for all $A < \bar{A}$, $\tilde{k} < \hat{k}$.

Proof of Proposition 4

(i) From Proposition 3, if $A \geq \frac{\omega(y^*)}{q^*+\kappa}$, then $q = q^*$. Hence, $\frac{\partial q}{\partial A} = \frac{\partial q}{\partial k} = 0$. Moreover, $\mathcal{V} = \frac{\alpha(n)\omega(y^*)}{(q^*+\kappa)A}$ where $n > 0$ (since $k < k^*$) solves $\frac{\alpha(n)}{n} = \frac{k}{(1-\theta)[u(y^*)-c(y^*)]}$. Hence, $\frac{\partial \mathcal{V}}{\partial k} = \frac{\alpha'(n)\omega(y^*)}{(q^*+\kappa)A} \frac{\partial n}{\partial k} < 0$ since $\frac{\partial n}{\partial k} = \frac{n^2}{\left[\frac{\alpha'(n)n}{\alpha(n)}-1\right]\alpha(n)(1-\theta)[u(y^*)-c(y^*)]} < 0$.

(ii) From the proof of Proposition 3, q is the solution to $\Upsilon(q, k, A) = 0$. If $k < \hat{k}$, an active equilibrium exists and it is such that $q > q^*$, since $A < \frac{\omega(y^*)}{q^*+\kappa}$. It can be checked that Υ is increasing in A and k. Moreover, at the equilibrium with the highest q, the curve representing $\Upsilon(q, k, A)$ as a function of q intersects the horizontal axis from below, i.e., $\Upsilon_q = \frac{\partial \Upsilon}{\partial q} > 0$. So $\frac{\partial q}{\partial k} = \frac{-\Upsilon_k}{\Upsilon_q} < 0$ and $\frac{\partial q}{\partial A} = \frac{-\Upsilon_A}{\Upsilon_q} < 0$. Since $\tau_a = A$, $\mathcal{V} = \alpha(n)$. From (23), it is immediate that n increases with q and decreases with k taking q as given. Hence, $\frac{\partial \mathcal{V}}{\partial k} = \alpha'(n)\left[\frac{\partial n}{\partial k} + \frac{\partial n}{\partial q}\frac{\partial q}{\partial k}\right] < 0$.

Proof of Proposition 5 According to Proposition 3, if $A < \frac{\theta c(y^*)+(1-\theta)u(y^*)}{q^*+\kappa}$ and $k \in \left(\tilde{k}, \hat{k}\right)$, there is an equilibrium with $n = 0$ and $q = q^*$. Household welfare is at a minimum and equal to $V = \frac{A\kappa}{1-\beta}$. There are also an even number of active equilibria with $n > 0$ and $q > q^*$. From (17) and (23),

$$\begin{aligned}
\omega(y) &= (q + \kappa)A, \\
\frac{\alpha(n)}{n} &= \frac{k}{(1 - \theta)\left[u(y) - c(y)\right]}.
\end{aligned}$$

The stationary equilibrium with higher q is associated with higher y, higher n, and, from (25), higher welfare.

Proof of Proposition 6.   Household welfare is

$$
\begin{aligned}
V & = \sum_{t=0}^{\infty} \beta^t \left\{ \alpha(n_t) \left[u\left(y_t\right) - (q_t + \kappa)\tau_{a,t} \right] + n_t \Pi_t + A\kappa - n_{t+1}k^f \right\} \\
& = \sum_{t=0}^{\infty} \beta^t \left\{ \alpha(n_t) \left[u\left(y_t\right) - (q_t + \kappa)\tau_{a,t} \right] + \alpha(n_t) \left[-c\left(y_t\right) + (q_t + \kappa)\tau_{a,t} \right] + n_t f(1) + A\kappa - n_{t+1}k^f \right\} \\
& = \sum_{t=0}^{\infty} \beta^t \left\{ \alpha(n_t) \left[u\left(y_t\right) - c\left(y_t\right) \right] + n_t f(1) + A\kappa - n_{t+1}k^f \right\},
\end{aligned}
$$

where from the first line to the second line we have used the expression for the firm's profits in (22). The first-order condition with respect to y_t gives $u'(y_t) = c'(y_t)$, i.e., $y_t = y^*$. The first-order condition with respect to n_t gives $\alpha'(n_t)\left[u(y^*) - c(y^*)\right] \le k^f/\beta - f(1) = k$, or $n_t = n^*$. From Proposition 3, $y_t = y^*$ if and only if $A \ge \frac{\omega(y^*)}{q^* + \kappa}$. From (23), $n_t = n^*$ if and only if $\frac{\alpha(n^*)}{n^*}(1 - \theta) = \alpha'(n^*)$.

Proof of Lemma 5   If $\{q_t\}_{t=0}^{\infty}$ is bounded, it is immediate that the transversality condition holds. Next, we show that any unbounded sequence $\{q_t\}_{t=0}^{\infty}$ violates $\lim_{t \to \infty} \beta^t q_t = 0$. The function $\Gamma(q)$ is continuous and over the interval $[q^*, \bar{q}]$ it reaches a maximum $q_{\max}$, where $(\bar{q} + \kappa)A = \omega(y^*)$. For all $q_{t+1} \ge \bar{q}$, $\Gamma(q_{t+1}) = \beta(q_{t+1} + \kappa)$. If $\{q_t\}_{t=0}^{\infty}$ is unbounded, there is a T such that for all $t \ge T$, $q_t > \max(q^*, q_{\max})$ and $q_t = \beta(q_{t+1} + \kappa)$. The solution to this difference equation is $q_{T+t} = (q_T - q^*)(1 + r)^t + q^*$. Hence, $\lim_{t \to \infty} \beta^t q_{T+t} = q_T - q^* > 0$.

Proof of Lemma 6.   The continuity of the function Γ comes from the continuity of $u(y)$, $c(y)$, and $\alpha(n)$, and hence the continuity of $\omega(y)$ and $\psi(n)$.

(i) If $k \ge k^* = (1 - \theta)\left[u(y^*) - c(y^*)\right]$, then $\min\left[\frac{k}{(1-\theta)[u(y_t)-c(y_t)]}, 1\right] = 1$ for all $y_t \le y^*$ and, from (28), $n_t = 0$ since $\frac{\alpha(0)}{0} = \alpha'(0) = 1$. From (26), $\Gamma(q_t) = \frac{rq^* + q_t}{1+r}$.

(ii) If $k < k^*$, then $\underline{q} = \frac{\omega\left[\Delta^{-1}\left(\frac{k}{1-\theta}\right)\right]}{A} - \kappa < \bar{q} = \frac{\omega(y^*)}{A} - \kappa$ since $\Delta^{-1}\left(\frac{k}{1-\theta}\right) < y^*$ and $\omega' > 0$. For all $q \le \underline{q}$, $n_t = 0$ and, from (26), $\Gamma(q_t) = \frac{rq^* + q_t}{1+r}$. For all $q \ge \bar{q}$, $y_t = y^*$ and, from (26), $\Gamma(q_t) = \frac{rq^* + q_t}{1+r}$. For all $q \in \left(\underline{q}, \bar{q}\right)$, $y_t < y^*$ and $n_t > 0$, and then

$$
\alpha(n_t)\theta \left\{ \frac{u'(y_t) - c'(y_t)}{\theta c'(y_t) + (1 - \theta)u'(y_t)} \right\} > 0.
$$

From (26), $\Gamma(q_t) > \frac{rq^* + q_t}{1+r}$.

Proof of Proposition 7.

(i) The condition $A \geq \frac{\omega(y^*)}{q^*+\kappa}$ is equivalent to $q^* \geq \bar{q} = \frac{\omega(y^*)}{A} - \kappa$. From Lemma 6, $\Gamma(q) = \frac{rq^*+q}{1+r}$ for all $q \geq q^*$. Any solution to $q_{t-1} = \Gamma(q_t) = \frac{rq^*+q_t}{1+r}$ is of the form

$$q_t = C(1+r)^t + q^*,$$

where $C \geq 0$. The transversality condition, $\lim_{t\to\infty} \frac{q_t}{(1+r)^t} = 0$, implies $C = 0$ and hence $q_t = q^*$. From (15)-(16), $(q^* + \kappa) A \geq \omega(y^*)$, $y_t = y^*$. Given $q_t = q^*$ and $y_t = y^*$, n_t is determined by (28).

(ii) Suppose first that $k \geq k^* = (1-\theta)[u(y^*) - c(y^*)]$. From (28), $n_t = 0$. Consequently, $q_{t-1} = \frac{q_t+\kappa}{1+r}$ for all $q_t \geq q^*$, and the only solution that satisfies the transversality condition, $\lim_{t\to\infty} \frac{q_t}{(1+r)^t} = 0$, is $q_t = q^*$. Assume next that $A < \frac{\omega(y^*)}{q^*+\kappa}$ and $k > \hat{k}$. From the proof of Proposition 3, $\Upsilon(q, k) = \frac{(1+r)[q - \Gamma(q)]}{q+\kappa}$. We established that if $A < \frac{\omega(y^*)}{q^*+\kappa}$ and $k > \hat{k}$, then $\Upsilon(q^*, k) = 0$ and $\Upsilon(q, k) > 0$ for all $q > q^*$. Hence, $q^* = \Gamma(q^*)$ and $q_{t-1} = \Gamma(q_t) < q_t$ for all $q_t > q^*$. If $q_0 \neq q^*$, then $\{q_t\}_{t=0}^\infty$ is monotone increasing and unbounded, which from Lemma 5 cannot be an equilibrium. (If the sequence has a finite limit, $q_\infty > q^*$, then by the continuity of Γ this limit satisfies $q_\infty = \Gamma(q_\infty)$, which is a contradiction.) Hence, the only admissible trajectory is $q_t = q^*$. Since $\hat{k} \geq \tilde{k} = (1-\theta)\{u[y[(q^*+\kappa)A]] - c[y[(q^*+\kappa)A]]\}$, then $n_t = 0$.

Consider next the case $k < \tilde{k}$. Let $\check{n}$ denote the solution to

$$\frac{\alpha(\check{n})}{\check{n}}(1-\theta)[u(\check{y}) - c(\check{y})] = k,$$

where $\omega(\check{y}) = A(q^*+\kappa) < \omega(y^*)$. Since $k < \tilde{k} = (1-\theta)\Delta(\check{y})$, $\check{n} > 0$. Moreover,

$$\Gamma(q^*) = \frac{q^*+\kappa}{1+r}\left\{1 + \alpha(\check{n})\theta\left\{\frac{u'(\check{y}) - c'(\check{y})}{\theta c'(\check{y}) + (1-\theta)u'(\check{y})}\right\}\right\} > q^*,$$

and, for all $q_t > q^*$, $\Gamma(q_t) \geq \frac{q_t+\kappa}{1+r} > \frac{q^*+\kappa}{1+r} = q^*$. Consequently, for all $q_t \geq q^*$, $q_{t-1} > q^*$, which implies that $q_t = q^*$ is not part of an equilibrium. Any trajectory is such that $q_t > q^*$ for all t.

Consider the case $k \in (\tilde{k}, \hat{k})$. The condition $k > \tilde{k}$ is equivalent to $(1-\theta)\Delta[y(A(q^*+\kappa))] < k$, and hence $q^* < \underline{q}$. Hence, from Lemma 6, $\Gamma(q^*) = q^*$ and $(q_t, n_t) = (q^*, 0)$ is an equilibrium. From Proposition 3, if $k \in (\tilde{k}, \hat{k})$, then there are an even number of active steady-state equilibria. We denote $q_L > \underline{q} > q^*$ the asset price at the lowest active steady-state equilibrium. From Lemma 6, for all $q_{t+1} \in (q^*, \underline{q})$, $q_t = \Gamma(q_{t+1}) = \frac{rq^*+q_{t+1}}{1+r} < q_{t+1}$. Consequently, Γ is located above the 45° line in the space (q_t, q_{t+1}) for all $q_t \in (q^*, q_L)$, it intersects the 45° line by above, $\left.\frac{\partial q_{t+1}}{\partial q_t}\right|_{q_t=q_L} \leq 1$, and with a positive slope, $\left.\frac{\partial q_{t+1}}{\partial q_t}\right|_{q_t=q_L} > 0$ (since $\Gamma(q)$ is single valued). Hence, $\left.\frac{\partial q_{t+1}}{\partial q_t}\right|_{q_t=q_L} = \frac{1}{\Gamma'(q_L)} \in (0, 1]$.

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If $\frac{1}{\Gamma'(q_L)} \in (0,1)$, then the linearized system $q_{t+1} = q_L + \frac{q_t - q_L}{\Gamma'(q_L)}$ admits a continuum of solutions corresponding to different initial conditions that converge to q_L. If $\Gamma'(q_L) = 1$, then the phase line is tangent to the 45° line. It can be checked on a phase diagram that all trajectories such that $q_0 \in (q^*, q_L)$ converge to q_L.

Proof of Proposition 8 From (31), and using that $n_\ell = 0$, $q_\ell = \beta(\lambda_{\ell\ell}q_\ell + \lambda_{\ell h}q_h + \kappa)$, i.e.,

$$q_\ell = \frac{\lambda_{\ell h}q_h + \kappa}{r + \lambda_{\ell h}}. \tag{41}$$

From (31), the price in the high state satisfies

$$q_h = \beta(\bar{q}_h + \kappa)\left\{1 + \alpha(n_h)\theta\left\{\frac{u'\left[y((\bar{q}_h + \kappa)A)\right] - c'\left[y((\bar{q}_h + \kappa)A)\right]}{(1 - \theta)u'\left[y((\bar{q}_h + \kappa)A)\right] + \theta c'\left[y((\bar{q}_h + \kappa)A)\right]}\right\}\right\}, \tag{42}$$

where $\bar{q}_h = \frac{(r + \lambda_{\ell h} - r\lambda_{h\ell})q_h + \lambda_{h\ell}\kappa}{r + \lambda_{\ell h}}$. Define

$$\Upsilon(q_h, k, \lambda_{h\ell}, \lambda_{\ell h}) \equiv q_h - \beta(\bar{q}_h + \kappa)\left\{1 + \alpha(n_h)\theta\left\{\frac{u'\left[y((\bar{q}_h + \kappa)A)\right] - c'\left[y((\bar{q}_h + \kappa)A)\right]}{(1 - \theta)u'\left[y((\bar{q}_h + \kappa)A)\right] + \theta c'\left[y((\bar{q}_h + \kappa)A)\right]}\right\}\right\},$$

where n_h is an implicit function of q_h and k defined by (30). The price q_h is part of an equilibrium if $\Upsilon(q_h, k, \lambda_{h\ell}, \lambda_{\ell h}) = 0$. From Proposition 3, if $k \in \left(\tilde{k}, \hat{k}\right)$ then there is a $q_h > q^*$ such that $\Upsilon(q_h, k, 0, \lambda_{\ell h}) = 0$. By the continuity of Υ and the Implicit Function Theorem, for $\lambda_{h\ell}$ sufficiently small, there is a $q_h > q^*$ such that $\Upsilon(q_h, k, \lambda_{h\ell}, \lambda_{\ell h}) = 0$. For $n_\ell = 0$, we need to check that

$$(1 - \theta)\left\{u\left[y((\bar{q}_\ell + \kappa)A)\right] - c\left[y((\bar{q}_\ell + \kappa)A)\right]\right\} \leq k.$$

From (41), as $\lambda_{\ell h}$ tends to 0, $\bar{q}_\ell$ approaches q_ℓ and q_ℓ approaches q^*. Since $k > \tilde{k}$,

$$(1 - \theta)\left\{u\left[y((q^* + \kappa)A)\right] - c\left[y((q^* + \kappa)A)\right]\right\} < k.$$

Hence, for $\lambda_{\ell h}$ sufficiently close to 0, $n_\ell = 0$.

Proof of Proposition 9 From (33), the household's welfare in equilibrium is

$$
\begin{aligned}
V &= \sum_{t=0}^{\infty}\beta^t\left\{\alpha(n_t)\left[u(y_t) - (q_t + \kappa)\tau_{a,t}\right] + n_t\Pi_t - n_{t+1}k^f + A\kappa\right\} \\
&= \alpha(n_0)\left[u(y_0) - (q_0 + \kappa)\tau_{a,0}\right] + n_0\Pi_0 + \sum_{t=0}^{\infty}\beta^t\alpha(n_t)\left[u(y_t) - (q_t + \kappa)\tau_{a,t}\right] + \frac{A\kappa}{1 - \beta} \\
&= \alpha(n_0)\left[u(y_0) - c(y_0)\right] + n_0 f(1) + \sum_{t=1}^{\infty}\beta^t\alpha(n_t)\theta\left[u(y_t) - c(y_t)\right] + \frac{A\kappa}{1 - \beta}, \tag{43}
\end{aligned}
$$

where from the to the second line we used $-k^f + \beta\Pi_t = 0$ and the transversality condition, and from the second to the third we use $u(y_t) - (q_t + \kappa)\tau_{a,t} = \theta[u(y_t) - c(y_t)]$ and $n_0\Pi_0 = \alpha(n_0)[-c(y_0) + (q_0 + \kappa)\tau_{a,0}] + n_0 f(1)$. Assuming $n_t < 0$, it solves

$$\frac{\alpha(n_t)}{n_t}(1-\theta)[u(y_t) - c(y_t)] = k.$$

Then n_t is maximum when $u(y_t) - c(y_t)$ is maximized, at $y_t = y^*$. Consequently, the first and third terms on the right side of (43) are maximum when $y_t = y^*$ for all $t \geq 0$. From (32), $y_t = y^*$ implies

$$q_t - (1+r)q_{t-1} + rq^* = 0,$$
$$(1+r)q_{b,t-1} - 1 = 0.$$

The solution to the second equation is $q_{b,t} = q_b^* = \beta$ for all $t \geq 0$. The solution to the first equation that satisfies transversality is $q_t = q^*$. The condition $y_t = y^*$ requires $(q^* + \kappa)A + B \geq \omega(y^*)$.

Appendix B. Transversality

Consider the sequence problem of the household. At time 0 we take as given $\{q_t\}_{t=0}^\infty$, $\{R_t\}_{t=0}^\infty$ and $\{n_t\}_{t=0}^\infty$. Clearly we can set $x_t = x^*$ and focus on other choices. The household chooses an asset plan to maximize lifetime expected utility. An asset plan specifies a portfolio of assets for all (stochastic trading) histories in the DM. Thus, a trading shock is represented by a binary variable $\chi_t \in \{0,1\}$ where $\chi_t = 1$ if the household is matched in the DM at t and $\chi_t = 0$ otherwise. A partial history is $\chi^t = (\chi_0, \chi_1, ..., \chi_t) \in \{0,1\}^{t+1}$. The distribution of probabilities of the partial histories is

$$\Pr\left[(\chi_0, \chi_1, ..., \chi_t)\right] = \prod_{j=0}^{t}\left\{\chi_j\alpha(n_j) + \left(1 - \chi_j\right)[1 - \alpha(n_j)]\right\}.$$

An asset plan is an initial $(a_0, s_0) \in \mathbb{R}_{2+}$ and a sequence of functions $\{a_t, s_t\}_{t=1}^\infty$ mapping partial history χ^{t-1} into a portfolio, $a_t : \{0,1\}^t \to \mathbb{R}_+$ and $s_t : \{0,1\}^t \to \mathbb{R}_+$, for all $t \geq 1$.

The expected discounted utility up to period T is

$$U_T(\{a_t, s_t\}) = \mathbb{E}\sum_{t=0}^{T}\beta^t\left\{\chi_t\left\{u\left[y((q_t + \kappa)a_t)\right] - (q_t + \kappa)\tau_a((q_t + \kappa)a_t)\right\}\right. \tag{44}$$
$$\left. + (q_t + \kappa)a_t - q_t a_{t+1} + R_t s_t - s_{t+1}\right\}.$$

Thus, if the household is matched in period t, y_t is consumed, in the DM in exchange for τ_a units of the asset, where y and τ_a are functions of liquid wealth. In the CM the household enjoys $\kappa a_t + R_t s_t$ and supplies

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$q_t(a_{t+1} - a_t) + s_{t+1}$ units of labor. The expectation in (44) is taken with respect to the random trading histories, χ^t.

From the proportional bargaining solution,

$$u\left[y((q_t + \kappa)a_t)\right] - (q_t + \kappa)\tau_a((q_t + \kappa)a_t) = \theta\Delta\left[y((q_t + \kappa)a_t)\right],$$

where $\Delta(y) = u(y) - c(y)$. Hence, utility from $t = 0$ to $t = T$ can be written

$$U_T\left(\{a_t, s_t\}\right) = \mathbb{E}\sum_{t=0}^{T}\beta^t\left\{\chi_t\theta\Delta\left[y((q_t + \kappa)a_t)\right] + (q_t + \kappa)a_t - q_t a_{t+1} + R_t s_t - s_{t+1}\right\}. \tag{45}$$

Rearrange terms in (45) as:

$$U_T\left(\{a_t, s_t\}\right) = \alpha(n_0)\theta\Delta\left[y((q_0 + \kappa)a_0)\right] + (q_0 + \kappa)a_0 + R_0 s_0$$

$$+\mathbb{E}\sum_{t=1}^{T}\beta^t\left\{\chi_t\theta\Delta\left[y((q_t + \kappa)a_t)\right] - \left[(1 + r)q_{t-1} - (q_t + \kappa)\right]a_t - (1 + r - R_t)s_t\right\}$$

$$-\mathbb{E}\left[\beta^T\left(s_{T+1} + q_T a_{T+1}\right)\right].$$

By the Law of Iterated Expectations,

$$\mathbb{E}\mathbb{E}\left[\chi_t\theta\Delta\left[y((q_t + \kappa)a_t)\right]\big|\,\chi^{t-1}\right] = \mathbb{E}\alpha(n_t)\theta\Delta\left[y((q_t + \kappa)a_t)\right],$$

where we have used that

$$\mathbb{E}\left[\chi_t\theta\Delta\left[y((q_t + \kappa)a_t(\chi^{t-1}))\right]\big|\,\chi^{t-1}\right] = \theta\Delta\left[y((q_t + \kappa)a_t(\chi^{t-1}))\right]\mathbb{E}\left[\chi_t|\,\chi^{t-1}\right]$$

$$= \alpha(n_t)\theta\Delta\left[y((q_t + \kappa)a_t(\chi^{t-1}))\right].$$

Hence,

$$U_T\left(\{a_t, s_t\}\right) = \alpha(n_0)\theta\Delta\left[y((q_0 + \kappa)a_0)\right] + (q_0 + \kappa)a_0 + R_0 s_0$$

$$+\mathbb{E}\sum_{t=1}^{T}\beta^t\left[\mathcal{U}(a_t; q_t, q_{t-1}, n_t) - (1 + r - R_t)s_t\right] - \mathbb{E}\left[\beta^T\left(s_{T+1} + q_T a_{T+1}\right)\right],$$

where

$$\mathcal{U}(a_t; q_t, q_{t-1}, n_t) = \alpha(n_t)\theta\Delta\left[y((q_t + \kappa)a_t)\right] - \left[(1 + r)q_{t-1} - (q_t + \kappa)\right]a_t.$$

The difficulty when evaluating asset plans is that $\lim_{T\to\infty} U_T$ might not exist for all feasible plans. To circumvent this problem, we use the catching-up criterion of Brock (1970) and Seierstad and Sydsaeter (1987, p.232), according to which a plan (a_t, s_t) catches up to a plan $(\tilde{a}_t, \tilde{s}_t)$ if

$$\underline{\lim}_{T\to\infty}\left[U_T(a_t, s_t) - U_T(\tilde{a}_t, \tilde{s}_t)\right] \geq 0.$$

A plan is optimal if it catches up to any other plan with the same initial conditions, (a_0, s_0).

First, any candidate for an optimal plan is such that $a_t(\chi^{t-1}) = a_t^*$ for all $\chi^{t-1} \in \{0, 1\}^t$ where

$$a_t^* \in \arg\max_{a \geq 0} \mathcal{U}(a; q_t, q_{t-1}, n_t) \text{ for all } t \geq 1,$$

where a_t^* is independent of the history χ^{t-1}. The proof is by contradiction. Suppose that there is some $j \geq 1$ such that $a_j \neq a_j^*$. Then, one can construct an alternative plan, $(\hat{a}_t, \hat{s}_t)$, such that $(\hat{a}_t, \hat{s}_t) = (a_t, s_t)$ for all $t \neq j$ and $(\hat{a}_\tau, \hat{s}_\tau) = (a_j^*, s_j)$. Then,

$$U_T(\{a_t, s_t\}) - U_T(\{\hat{a}_t, \hat{s}_t\}) = \mathbb{E}\beta^\tau \left[\mathcal{U}(a_t; q_t, q_{t-1}, n_t) - \mathcal{U}(a_t^*; q_t, q_{t-1}, n_t)\right] < 0, \quad \text{for all } T \geq j,$$

and hence (a_t, s_t) is not catching up to $(\hat{a}_t, \hat{s}_t)$. This establishes that a solution to the sequence problem is a solution to (19). By a similar reasoning, in any equilibrium, $\beta R_t \leq 1$ and $s_t = s_t^*$ where $(1 + r - R_t) s_t^* = 0$.

Second, scale down the candidate plan $\{a_t^*, s_t^*\}_{t=0}^\infty$ by a factor $1 - \varepsilon \in (0, 1)$ (except the initial asset holdings a_0 and s_0) and define

$$D_T(\varepsilon) = \frac{U_T(\{a_t^*, s_t^*\}) - U_T(\{(1-\varepsilon)a_t^*, (1-\varepsilon)s_t^*\})}{\varepsilon}.$$

Then $\lim_{\varepsilon \to 0} D_T(\varepsilon) = -\beta^T \left(s_{T+1}^* + q_T a_{T+1}^*\right)$, using

$$\lim_{\varepsilon \to 0} \frac{\mathcal{U}(a_t^*; q_t, q_{t-1}, n_t) - \mathcal{U}((1-\varepsilon)a_t^*; q_t, q_{t-1}, n_t)}{\varepsilon} = 0,$$

since, from the first-order condition, $\frac{\partial \mathcal{U}(a_t^*; q_t, q_{t-1}, n_t)}{\partial a_t} = 0$ if $a_t^* \neq 0$. Consequently, a necessary condition for the plan to be optimal is $\varliminf_{T \to \infty} D_T(0) \geq 0$, or

$$\varliminf_{T \to \infty} \left[-\beta^T \left(s_{T+1}^* + q_T a_{T+1}^*\right)\right] \geq 0.$$

(Otherwise, for all $T > 0$, there is a $\hat{T} \geq T$ and a $\varepsilon > 0$ such that $D_{\hat{T}}(\varepsilon) < 0$, i.e., $U_{\hat{T}}(\{a_t^*, s_t^*\}) - U_{\hat{T}}(\{(1-\varepsilon)a_t^*, (1-\varepsilon)s_t^*\}) < 0$.) Equivalently,

$$\varlimsup_{T \to \infty} \left[\beta^T (s_{T+1}^* + q_T a_{T+1}^*)\right] = 0,$$

or

$$\lim_{T \to \infty} \beta^T (s_{T+1}^* + q_T a_{T+1}^*) = 0. \tag{46}$$

This leads to $\lim_{T \to \infty} \beta^T s_{T+1}^* = 0$ and $\lim_{T \to \infty} \beta^T q_T a_{T+1}^* = 0$.

To show sufficiency, notice that for any plan, $\{a_t, s_t\}$ with initial conditions (a_0, s_0)

$$U_T(\{a_t, s_t\}) \leq \alpha(n_0)\theta\Delta \left[y((q_0 + \kappa)a_0)\right] + (q_0 + \kappa)a_0 + R_0 s_0 + \mathbb{E}_0 \sum_{t=1}^T \beta^t \mathcal{U}(a_t^*; q_t, q_{t-1}, n_t).$$

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Consequently,

$$\overline{\lim}_{T \to \infty} U_T(\{a_t, s_t\}) \leq \alpha(n_0)\theta\Delta\left[y((q_0 + \kappa)a_0)\right] + (q_0 + \kappa)a_0 + R_0 s_0 + \mathbb{E}_0 \sum_{t=1}^{\infty} \beta^t \mathcal{U}(a_t^*; q_t, q_{t-1}, n_t),$$

where the right side is the household's expected lifetime utility under the plan (a_t^*, s_t^*) if (46) holds.

Appendices Not Necessarily Intended for Publication

Appendix C. Generalized Nash Bargaining

In the text we used proportional bargaining. Here we show the main insights are preserved using the generalized Nash solution. The terms of trade in a match are now

$$(y, \tau) = \arg \max_{\tau \leq \omega} \left[u(y) - \tau \right]^\theta \left[\tau - c(y) \right]^{1-\theta},$$

where ω represents the household's liquid wealth. If $\tau \leq \omega$ does not bind, then $y = y^*$ and $\tau = \theta c(y^*) + (1 - \theta) u(y^*)$. If $\tau \leq \omega$ binds, the households spends all its wealth and y solves

$$(q_t + \kappa) a_t = \omega(y_t) = \frac{\theta u'(y_t) c(y_t) + (1 - \theta) c'(y_t) u(y_t)}{\theta u'(y_t) + (1 - \theta) c'(y_t)}. \tag{47}$$

The household choice of DM consumption is then

$$\max_{y_t \in [0, y^*]} \left\{ - \left(r \left(q_{t-1} - q^* \right) - (q_t - q_{t-1}) \right) \frac{\omega(y_t)}{q_t + \kappa} + \alpha(n_t) \left[u\left(y_t \right) - \omega(y_t) \right] \right\}.$$

The objective function is continuous in y_t and it is maximized over a compact set. Hence, it has a solution. If the cost of holding assets is zero $r \left(q_{t-1} - q^* \right) = q_t - q_{t-1}$, and the DM is active $n_t > 0$, the household purchases $\tilde{y}$ so that $u'(\tilde{y}) = \omega'(\tilde{y})$. If $\theta < 1$ then $\tilde{y} < y^*$, and the household has no incentive to accumulate more wealth than $\omega(\tilde{y})$. To see this, notice that $u'(y^*) - \omega'(y^*) < 0$. In that case, as in Lagos and Rocheteau (2008) we let households bring only a fraction of their assets to the DM. They choose to do so if $(q_t + \kappa) a_t > \omega(\tilde{y})$. If it is costly to hold assets, $r \left(q_{t-1} - q^* \right) > q_t - q_{t-1}$, households do not hold more assets than they intend to spend in the DM. The solution, assuming interiority, is given by the first-order condition

$$\frac{r \left(q_{t-1} - q^* \right) - (q_t - q_{t-1})}{q_t + \kappa} = \alpha(n_t) \left[\frac{u'(y_t)}{\omega'(y_t)} - 1 \right]. \tag{48}$$

Assuming that $\frac{u'(y_t)}{\omega'(y_t)}$ is decreasing, the solution to the household's problem is obviously unique. Even if $\frac{u'(y_t)}{\omega'(y_t)}$ is not decreasing, a version of the argument in Wright (2010) can be used to establish uniqueness.

If $n_t > 0$, market clearing implies

$$a_t = A \quad \text{if} \quad r \left(q_{t-1} - q^* \right) > q_t - q_{t-1}, \tag{49}$$

$$a_t = \frac{\omega(\tilde{y})}{q_t + \kappa} \leq A \quad \text{if} \quad r \left(q_{t-1} - q^* \right) = q_t - q_{t-1}, \tag{50}$$

where a_t represents the household's asset holdings brought into the DM. Finally,

$$a_t \geq 0 \quad \text{if} \quad n_t = 0 \quad \text{and} \quad r \left(q_{t-1} - q^* \right) = q_t - q_{t-1}, \tag{51}$$

Then n_t solves

$$\frac{\alpha(n_t)}{n_t}\left[\omega(y_t) - c(y_t)\right] \le k, \quad = \text{if } n_t > 0. \tag{52}$$

An equilibrium is list $\{y_t\}$, $\{a_t\}$, $\{n_t\}$ and $\{q_t\}$ solving (47)-(52).

Consider active stationary equilibria. Consider first equilibria where $q_t = q_{t-1} = q = q^*$. From market clearing, $a = \frac{\omega(\tilde{y})}{q^*+\kappa} \le A$ and $y = \tilde{y}$. The measure of firms solves

$$\frac{\alpha(n)}{n}\left[\omega(\tilde{y}) - c(\tilde{y})\right] = k.$$

The measure of participating firms is positive if $\omega(\tilde{y}) - c(\tilde{y}) > k$.

Consider next equilibria with $q > q^*$. The asset pricing condition is

$$\frac{r\,(q - q^*)}{q + \kappa} = \alpha(n)\left[\frac{u' \circ \omega^{-1}\left[(q+\kappa)A\right]}{\omega' \circ \omega^{-1}\left[(q+\kappa)A\right]} - 1\right], \quad \text{if } (q^* + \kappa)A < \omega(\tilde{y}).$$

The free-entry condition is

$$\frac{\alpha(n)}{n}\left[(q+\kappa)A - c \circ \omega^{-1}\left[(q+\kappa)A\right]\right] = k \quad \text{if } (q+\kappa)A \le \theta c(y^*) + (1-\theta)u(y^*),$$

$$\frac{\alpha(n)}{n}(1-\theta)\left[u(y^*) - c(y^*)\right] = k \quad \text{if } (q+\kappa)A \ge \theta c(y^*) + (1-\theta)u(y^*).$$

Define

$$\tilde{k} = (q^* + \kappa)A - c \circ \omega^{-1}\left[(q^* + \kappa)A\right],$$

$$k^* = \omega(\tilde{y}) - c(\tilde{y}).$$

Following the proof of Proposition 3, we have:

Proposition 10 *(i) If $A \ge \frac{\omega(\tilde{y})}{q^*+\kappa}$ and $k < k^*$, then there is a unique active stationary equilibrium and it is such that $q = q^*$, $y = \tilde{y}$, and*

$$\frac{\alpha(n)}{n} = \frac{k}{\omega(\tilde{y}) - c(\tilde{y})}.$$

(ii) If $A < \frac{\omega(\tilde{y})}{q^*+\kappa}$, then there is a $\hat{k} \in \left[\tilde{k}, k^*\right)$ such that: if $k < \tilde{k}$, then all active stationary equilibria are such that $q > q^*$; if $k \in \left(\tilde{k}, \hat{k}\right)$, then there is an even number of active stationary equilibria; if $k > \hat{k}$, then there is no active stationary equilibrium.

Consider non-stationary equilibria. From (48), q follows

$$q_{t-1} = \Gamma(q_t) \equiv \frac{q_t + \kappa}{1 + r}\left\{1 + \alpha(n_t)\left\{\frac{u'(y_t) - \omega'(y_t)}{\omega'(y_t)}\right\}\right\},$$

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where $y_t = \tilde{y}$ if $(q_t + \kappa)A \geq \omega(\tilde{y})$ and $y_t = \omega^{-1}\left((q_t + \kappa)A\right)$ otherwise. Then n_t solves

$$n_t = \psi^{-1}\left(\min\left[\frac{k}{\omega(y_t) - c(y_t)}, 1\right]\right),$$

with $\psi(n) = \frac{\alpha(n)}{n}$. This system is qualitatively similar to the one in the paper.

Appendix D. Public Liquidity Provision under Walrasian Pricing

Suppose that the DM is a Walrasian market where the price of the liquid asset in terms of DM output is p_t and the price of bonds is $p_{b,t}$. Assume that agents' ability to trade in this market is limited due to congestion effects: a household is able to trade with probability $\alpha(n_t)$ while a participating firm is able trade with probability $\frac{\alpha(n_t)}{n_t}$.[16] Note that entry by firms in this setup means entry into the group trying to get into the Walrasian market, but of these only a fraction $\frac{\alpha(n_t)}{n_t}$ succeed.

Free entry now implies

$$\frac{\alpha(n_t)}{n_t} \max_{\tau_{a,t}, \tau_{b,t}} \left[(q_t + \kappa)\tau_{a,t} + \tau_{b,t} - c(p_t\tau_{a,t} + p_{b,t}\tau_{b,t})\right] \leq k, \quad = \text{ if } n_t > 0, \tag{53}$$

where $\tau_{a,t}$ and $\tau_{b,t}$ are the quantities of assets and bonds obtained by firms trading in the DM. The price of DM output in terms of CM output is determined by the marginal cost condition $c'(y_t) = \frac{q_t + \kappa}{p_t} = \frac{1}{p_{b,t}}$. Similarly, a representative household who trades in the DM solves

$$\max_{\tau_{a,t} \leq A, \tau_{b,t} \leq B} \left[u(p_t\tau_{a,t} + p_{b,t}\tau_{b,t}) - (q_t + \kappa)\tau_{a,t} - \tau_{b,t}\right].$$

It can easily be checked that if $(q_t + \kappa)A + B_t \geq y^*c'(y^*)$ then $y_t = y^*$, and $(q_t + \kappa)A + B_t = y_tc'(y_t)$ otherwise. The asset price satisfies (32) with $\theta = 1$, since under Walrasian pricing households extract the whole DM marginal surplus. Household welfare is then

$$V = \alpha(n_0)\left[u(y_0) - c(y_0)\right] + n_0 f(1) + \sum_{t=1}^{\infty} \beta^t \alpha(n_t)\left[u(y_t) - c'(y_t)y_t\right] + \frac{A\kappa}{1-\beta}.$$

Consider a planner choosing $\{n_t, y_t\}_{t=1}^{\infty}$ subject to (53) (we ignore the initial choice of y_0, which should be set to y^* if n_0 is taken as given). From the allocation $\{n_t, y_t\}_{t=1}^{\infty}$ one can back out asset prices $\{q_t, q_{b,t}\}_{t=0}^{\infty}$ from (32) with $\theta = 1$. If $y_t = y^*$, the supply of liquidity must be such that $(q_t + \kappa)A + B_t \geq y^*c'(y^*)$. The planner problem is essentially static. Assuming an interior solution, it maximizes $\hat{W}_t = \alpha(n_t)\left[u(y_t) - c(y_t)\right] - kn_t$ subject to (53). In the neighborhood of $y_t = y^*$, we have

$$\frac{d\hat{W}_t}{dy_t} < 0 \text{ if } \frac{c'(y^*)y^* - c(y^*)}{u(y^*) - c(y^*)} > \frac{\alpha'(n^*)n^*}{\alpha(n^*)}.$$

So if the firm's share is less than its contribution to the matching process, it is optimal to keep liquidity scarce, $(q_t + \kappa)A + B_t < c'(y^*)y^*$, which implies $y_t < y^*$. In this case, firms do not internalize the congestion they impose on other firms in the market, so the number of entrants is too high. Policy can mitigate this effect by making liquid assets costly to hold, which requires the asset price to be above its fundamental value.

Appendix E. Private Provision of Liquidity

Case where $\theta = 0$. An equilibrium is an n solving $\Gamma(n) = 0$ where

$$\Gamma(n) = \beta \left[\frac{\alpha(n)}{n} \left[u(y) - c(y) \right] + f(1) \right] - k^f,$$

where, from (35), $\omega(y) = \min \left[\omega(y^*), \beta^{-1} \left(q^* A + n k^f \right) \right]$. As $n \to 0$, $\Gamma(n) \to \beta \left[u(\underline{y}) - c(\underline{y}) + f(1) \right] - k^f$, where $\omega(\underline{y}) = \beta^{-1} q^* A$. As $n \to \infty$, $\Gamma(n) \to \beta f(1) - k^f < 0$. Consequently, if $\beta \left[u(\underline{y}) - c(\underline{y}) + f(1) \right] - k^f < 0$, which is true if A is small, whenever there is a positive solution to $\Gamma(n) = 0$ there are multiple solutions: an equilibrium with $n = 0$ and an even number with $n > 0$.

Case where $\theta = 1$. An equilibrium is an n solving $\Gamma(n) = 0$ where

$$\Gamma(n) = \left\{ 1 + \alpha(n) \left[\frac{u'(y) - c'(y)}{c'(y)} \right] \right\}^{-1} - \beta \frac{f(1)}{k^f},$$

and

$$y = \min \left[y^*, c^{-1} \left(\frac{\kappa f(1)}{f(1) - k^f} A + f(1)n \right) \right].$$

If $n = 0$ then $\Gamma(n) = 1 - \beta \frac{f(1)}{k^f} > 0$ and if $n = \infty$ then $y = y^*$ and $\Gamma(n) = 1 - \beta \frac{f(1)}{k^f} > 0$. If $\frac{\kappa f(1)}{f(1) - k^f} A < c(y^*)$ and $n \in (0, \infty)$ then $\Gamma(n) < 1 - \beta \frac{f(1)}{k^f}$. So if $\beta f(1)/k^f$ is sufficiently close 1, there are multiple steady-state equilibria. There is an inactive equilibrium with $n = 0$ and $R = \beta^{-1}$. There are also an even number of active equilibria where $q > q^*$ and $R < \beta^{-1}$.

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References

[1] Abreu, Dilip and Markus Brunnermeier (2003). "Bubbles and crashes," *Econometrica* 71, 173–204.

[2] Acharya, Viral and Lasse Pedersen (2005). "Asset pricing with liquidity risk," *Journal of Financial Economics* 77, 375–410.

[3] Allen, Franklin and Douglas Gale (2000). "Bubbles and crises," *Economic Journal* 110, 236–255.

[4] Allen, Franklin and Gary Gorton (1993). "Churning bubbles," *Review of Economic Studies* 60, 813–836.

[5] Allen, Franklin, Stephen Morris and Andrew Postlewaite (1993). "Finite bubbles with short sale constraints and asymmetric information," *Journal of Economic Theory* 61, 206-230.

[6] Aruoba, Boragan, Guillaume Rocheteau and Christopher Waller (2007). "Bargaining and the value of money," Journal of Monetary Economics 54, 2636–2655.

[7] Azariadis, Costas (1993). *Intertemporal Macroeconomics.* Blackwell: Malden, Massachusetts.

[8] Azariadis, Costas and Roger Guesnerie (1986). "Sunspots and cycles," *Review of Economic Studies* 53, 725–737.

[9] Barlevy, Gadi (2009). "A leverage-based model of speculative bubbles," *Working Paper*, Federal Reserve Bank of Chicago.

[10] Bernanke, Ben, Mark Getler and Simon Gilchrist (1999). "The financial accelerator in a quantitative business cycle framework," in *Handbook of Macroeconomics*, Volume 1C, Chapter 21, Elsevier: Amsterdam.

[11] Brock, William (1970). "On the existence of weakly maximal programmes in a multi-sector economy," *Review of Economic Studies* 37, 275–280.

[12] Burdett, Kenneth and Randall Wright (1998). "Two-sided search with nontransferable utility," *Review of Economic Dynamics* 1, 220–45

[13] Caballero, Ricardo (2006). "On the macroeconomics of asset shortages," *NBER Working Paper* 12753.

[14] Christian, Gilles and Stephen LeRoy (1992). "Bubbles and charges," International Economic Review 33, 323–339.

[15] Cochrane, John (2002). "Stocks as money: Convenience yield and the tech-stock bubble," *Working Paper*.

[16] Coles, Melvyn and Randall Wright. (1998). "A dynamic equilibrium model of search, bargaining and money," *Journal of Economic Theory* 78, 32–54.

[17] Duffie, Darrell, Nicolae Garleanu and Lasse Pedersen (2005). "Over-the-counter markets," *Econometrica* 73, 1815–1847.

[18] Duffie, Darrell, Nicolae Garleanu and Lasse Pedersen (2008). "Valuation in over-the-counter markets," *Review of Financial Studies* 20, 1865–1900.

[19] Ennis, Huberto (2001). "On random matching, monetary equilibria, and sunspots," *Macroeconomic Dynamics* 5, 132–141.

[20] Farhi, Emmanuel and Jean Tirole (2010). "Bubbly liquidity," *Working Paper*.

[21] Ferraris, Leo and Makoto Watanabe (2008). "Collateral secured loans in a monetary economy," *Journal of Economic Theory* 143, 405–424.

[22] Ferguson, Niall (2008). *The Ascent of Money: A Financial History of the World.* New York: Penguin Press.

[23] Geanakoplos, John and William Zame (2010). "Collateral equilibrium," *Working Paper*.

[24] Geromichalos, Athanasios, Juan Manuel Licari and Jose Suarez-Lledo (2007). "Monetary policy and asset prices," *Review of Economic Dynamics* 10, 761–779.

[25] Gilles, Christian and Stephen LeRoy (1992). "Bubbles and charges," *International Economic Review* 33, 323–339.

[26] Grandmont, Jean-Michel (1985). "On endogenous competitive business cycles," *Econometrica* 53, 995–1045.

[27] Greenspan, Alan and James Kennedy (2007). "Sources and uses of equity extracted from homes," *Staff Working Papers in the Finance and Economics Discussion Series of the Federal Reserve Board* 2007-20.

[28] Guerrieri, Veronica and Guido Lorenzoni (2009). "Liquidity and trading dynamics," *Econometrica* 77, 1751–1790.

[29] Guesnerie, Roger (1986). "Stationary sunspot equilibria in an N-commodity world," *Journal of Economic Theory* 40, 103–28.

[30] Holmstrom, Bengt and Jean Tirole (2008). *Wicksell lectures: Inside and outside liquidity.* Forthcoming.

[31] Hosios, Arthur (1990). "On the efficiency of matching and related models of search and unemployment" *Review of Economic Studies* 57, 279–298.

[32] Kalai, Ehud (1977). "Proportional solutions to bargaining situations: Interpersonal utility comparisons," Econometrica 45, 1623–1630.

[33] Kiyotaki, Nobuhiro (1998). "Credit and business cycles," *The Japanese Economic Review* 49, 18–35.

[34] Kiyotaki, Nobuhiro and John Moore (1997). "Credit cycles," *Journal of Political Economy* 105, 211–248.

[35] Kiyotaki, Nobuhiro and John Moore (2005). "Liquidity and asset prices," *International Economic Review* 46, 317–349.

[36] Kiyotaki, Nobuhiro and Randall Wright (1989). "On money as a medium of exchange," *Journal of Political Economy* 97, 927–954.

[37] Kiyotaki, Nobuhiro and Randall Wright (1993). "A search-theoretic approach to monetary economics," *American Economic Review* 83, 63–77.

[38] Kocherlakota, Naryana (2000). "Creating business cycles through credit constraints," *Federal Reserve Bank of Minneapolis Quarterly Review* 24, 2–10.

[39] Kocherlakota, Naryana (2009). "Bursting bubbles: Consequences and cures," *Working Paper*.

[40] Krishnamurthy, Arvind and Annette Vissing-Jorgensen (2008). "The aggregate demand for treasury debt," *Working Paper*.

[41] Lagos, Ricardo (2007). "Asset prices and liquidity in an exchange economy," *Working Paper*.

[42] Lagos, Ricardo and Guillaume Rocheteau (2008). "Money and capital as competing media of exchange," *Journal of Economic Theory* 142, 247–258.

[43] Lagos, Richard and Guillaume Rocheteau (2009). "Liquidity in asset markets with search frictions," *Econometrica* 77, 403–426.

[44] Lagos, Ricardo and Randall Wright (2003). "Dynamics, cycles, and sunspot equilibria in 'genuinely dynamic, fundamentally disaggregative' models of money," *Journal of Economic Theory* 109, 156–171.

[45] Lagos, Ricardo and Randall Wright (2005). "A unified framework for monetary theory and policy analysis," *Journal of Political Economy* 113, 463–484.

[46] LeRoy, Stephen (2004). "Rational exuberance," *Journal of Economic Literature* 42, 783–804.

[47] Lester, Benjamin, Andrew Postlewaite and Randall Wright (2008). "Information, liquidity, asset prices and monetary policy," *Working Paper*.

[48] Li, Yiting and Ying-Syuan Li (2010). "Liquidity, asset prices and credit constraints," *Working Paper*.

[49] Li, Yiting and Guillaume Rocheteau (2009). "Liquidity constraints," *Working Paper*.

[50] Lucas, Robert E. Jr. (1978). "Asset prices in an exchange economy," *Econometrica* 46, 1429–1445.

[51] Lucas, Robert E. Jr., and Edward Prescott (1974): "Equilibrium search and unemployment," *Journal of Economic Theory* 7, 188–209.

[52] Mills, David C. Jr. and Robert R. Reed (2009). "Default risk and collateral in the absence of commitment," *Working Paper.*

[53] Nosal, Ed and Guillaume Rocheteau (2010). *Money, Payments, and Liquidity.* MIT Press, Forthcoming.

[54] Osborne, Martin J. and Ariel Rubinstein (1990). *Bargaining and markets.* Academic Press.

[55] Pissarides, Christopher (2000). *Equilibrium Unemployment.* Second edition, MIT Press: Cambridge.

[56] Reinhart, Carmen and Kenneth Rogoff (2009). *This Time Is Different: Eight Centuries of Financial Folly.* Princeton University Press: Princeton.

[57] Rocheteau, Guillaume (2008). "A monetary approach to asset liquidity," *Working Paper of the Federal Reserve Bank of Cleveland* 0901.

[58] Rocheteau, Guillaume and Randall Wright (2005). "Money in search equilibrium, in competitive equilibrium, and in competitive search equilibrium," *Econometrica* 73, 175–202.

[59] Rocheteau, Guillaume and Randall Wright (2009). "Inflation and welfare with trading frictions," in *Monetary Policy in Low Inflation Economics.* Eds. Ed Nosal and David Altig, Cambridge University Press: Cambridge.

[60] Santos, Manuel and Michael Woodford (1997). "Rational asset pricing bubbles," *Econometrica* 65, 19–57.

[61] Seierstad, Atle and Knut Sydsaeter (1987). *Optimal control theory with economic applications.* North-Holland: Amsterdam.

[62] Tirole, Jean (1985). "Asset bubbles and overlapping generations," *Econometrica* 53, 1499–1528.

[63] Wallace, Neil (1980). "The overlapping generations model of fiat money," *Models of Monetary Economies.* Eds. John Kareken and Neil Wallace. Federal Reserve Bank of Minneapolis.

[64] Wallace, Neil (1996). "A dictum for monetary theory," *Federal Reserve Bank of Minneapolis Quarterly Review* 22, 20–26.

[65] Wallace, Neil (2000). "A model of the liquidity structure based on asset indivisibility," *Journal of Monetary Economics* 45, 55–68.

[66] Williamson, Stephen and Randall Wright (2010). "New monetarist economics: models," Forthcoming in *Handbook of Monetary Economics*, Eds. Benjamin Friedman and Michael Woodford. North Holland.

[67] Wright, Randall (1994). "A note on sunspot equilibria in search models of fiat money," *Journal of Economic Theory* 64, 234–241.

[68] Wright, Randall (2010). "A uniqueness proof for monetary steady state," *Journal of Economic Theory* 145, 382–91.