

A Production-Based Model for the Term Structure

Urban J. Jermann*

Wharton School of the University of Pennsylvania and NBER

January 29, 2013

Abstract

This paper considers the term structure of interest rates implied by a production-based asset pricing model where the fundamental drivers are investment in equipment and structures, and inflation. The model matches the average yield curve up to five year maturity almost perfectly. Longer term yields are roughly as volatile as in the data. The model also generates time-varying bond risk premiums. In particular, when running Fama-Bliss regressions of excess returns on forward premiums, the model produces slope coefficients of roughly half the size of the empirical counterparts. Closed-form expressions highlight the importance of the capital depreciation rates for interest rate dynamics.

*I gratefully acknowledge comments from Andy Abel, Yasser Boualam, David Chapman, Nick Rousanov, and a referee, as well as from seminar and conference participants at various venues. Email: jermann@wharton.upenn.edu.

1 Introduction

There are many models of the term structure of interest rates, only a few tie interest rates to macroeconomic fundamentals. Among fundamentals-based models, most are driven by consumption. Given the relative success of production-based models in matching features of stock returns at the aggregate level and in the cross-section, it seems promising to further extend the production-based approach to the term structure of interest rates.

Consumption-based models of the term structure face a number of difficulties. Many of these are related to the equity premium puzzle (Mehra and Prescott, 1985), according to which empirically reasonable consumption volatility and risk aversion are too small to match the sizable historical equity premium. Backus, Gregory, and Zin (1989) find that complete markets models cannot explain the sign, the magnitude nor the variability of the term premium. In this class of models, expected consumption growth and real yields are positively correlated, Chapman (1997) reports some supportive evidence for this property, as do Berardi and Torous (2005). Considering richer model specifications, several more recent studies report more positive results for explaining term premiums; for instance, Wachter (2006), Bansal and Shaliastovich (2010), Piazzesi and Schneider (2007), Ehling, Gallmeyer, Heyerdahl-Larsen, and Illeditsch (2011), and Rudebusch and Swansson (2008). General equilibrium models that start from a consumption-based model and add elements of endogenous production still face difficulties with jointly explaining the term structure and macroeconomic aggregates, as shown in Binsbergen, Fernandez-Villaverde, Koijen, and Rubio-Ramirez (2010).

Production-based asset pricing models have linked stock returns to fundamentals such as investment and productivity. Cochrane (1991) establishes the link between a firm's return to investment and its market return, he also shows a tight empirical relation between aggregate investment and stock returns. Production-based models have been used to explain the value premium (Zhang (2005)), and properties of external financing behavior (Li, Livdan and Zhang (2009)). Production-based models have also shown to be useful for understanding the cross-section of stock returns more generally, see for instance Berk, Green, and Naik (1999), Liu, Whited, and Zhang (2009), Belo (2010), Tuzel (2010), and Eisfeldt and Papanikolaou (2012).¹

The objective of this paper is to extend the production-based approach to price nominal bonds

¹For additional examples of the production-based approach applied to stocks, see, Carlson, Fisher, and Giammarino (2004), Li, Vassalou, and Xing (2003), Warusawitharana (2010), Kogan (2004), Cooper (2006), Pastor and Veronesi (2009), Kuehn (2009), and Eberly and Wang (2010).

of different maturities. Specifically, this paper builds on Jermann (2010) that has analyzed the determinants of the equity premium and presented a model that can quantitatively match first and second moments of the real returns on stocks and short term real bonds. I start from the same two-sector investment model based on equipment and structures. The paper here extends the analysis to the term structure of nominal bonds and explicitly introduces inflation. The paper is also related to Cochrane's 1988 working paper that presents a two-sector investment model and shows that the real forward premium from the model can track quite well its empirical counterpart over 1952-1986. In my paper, I present a more detailed analysis of the term structure, explicitly introduce inflation, and consider nominal bonds. The real side of the model is also more general, importantly, I allow for general curvature in the capital adjustment cost functions as opposed to Cochrane's quadratic specification

The main quantitative findings are that the model, calibrated to match the equity premium and the volatility of stock returns as well as the mean and volatility of short term yields, matches the average yield curve up to five year maturity almost perfectly. Longer term yields are roughly as volatile as in the data. The model also generates time-varying bond risk premiums. In particular, when running Fama-Bliss regressions of excess returns on forward premiums, the model produces slope coefficients of roughly half the size of the empirical counterparts.

My model is a two-sector version of a q-theory investment model. Firms' optimal investment choices generate the well known equivalence between market returns and investment returns. The short term real risk free rate can be seen as a long-short portfolio of the two risky investment returns. With the help of a continuous-time version of the model, the economic forces that drive the quantitative results are revealed explicitly. In particular, the short rate is shown to be a weighted average of the two expected investment returns, where the weights are constant and simple functions of the adjustment cost curvature parameters. Expected returns and the market price of risk are driven by the two investment to capital ratios that display important low frequency components. The volatility of the short rate is also a function of the investment to capital ratios. Thus, even with homoscedastic shocks, the model endogenously produces time-varying bond risk premiums. A key new finding is that the difference in depreciation rates between structures and equipment plays a crucial role for whether interest rates commove positively or negatively with investment, and for whether the implied term premium for bonds with a short maturity is positive or negative.

Section 2 presents the model, and section 3 the quantitative analysis. Section 4 analyzes a continuous-time version of the model. Section 5 concludes.

2 Model

This section starts by presenting the real side of the model which was first used in Jermann (2010). Inflation is then introduced.

2.1 Real model

Assume an environment where uncertainty is modelled as the realization of s , one out of a set of two (s_1, s_2) , with s_t the current period realization and $s^t \equiv (s_0, s_1, \dots, s_t)$ the history up to and including t . Assume a revenue function with two capital stocks $K_j(s^{t-1})$ for $j = 1, 2$,

$$F\left(\{K_j(s^{t-1})\}_{j \in (1,2)}, s^t\right) = \sum_{j=1}^2 A_j(s^t) K_j(s^{t-1}). \quad (1)$$

As is standard, $K_j(s^{t-1})$ is chosen one period before it becomes productive. $F(\cdot)$ represents the resources available after the firm has optimally chosen and paid factors of production that are selected within the period, for instance labor.² $A_j(s^t)$ is driven by productivity shocks and other factors affecting the marginal product of capital. It is key that there are as many capital stocks as there are states of nature next period. Without this property, it would not be possible to recover state prices from the firm's production choices.

Capital of type j accumulates through

$$K_j(s^t) = K_j(s^{t-1})(1 - \delta_j) + I_j(s^t), \quad (2)$$

where δ_j is the depreciation rate, and $I_j(s^t)$ is investment. The total cost of investing in capital of type j includes convex adjustment costs and is given by

$$H_j(K_j(s^{t-1}), I_j(s^t)) = \left\{ \frac{b_j}{\nu_j} (I_j(s^t) / K_j(s^{t-1}))^{\nu_j} + c_j \right\} K_j(s^{t-1}), \quad (3)$$

with $b, c > 0$, $\nu > 1$. For each capital stock, different values for b , c , and ν will be allowed. The most important parameter is the curvature ν , as it determines the volatility of percentage changes in the marginal adjustment cost, and thus Tobin's q , relative to investment volatility. The other parameters play a minor role for the main asset pricing properties this paper focuses on.

Taking as given state prices $P(s^t)$, a representative firm solves the following problem,

$$\max_{\{I, K'\}} \sum_{t=0}^{\infty} \sum_{s^t} P(s^t) \left[F\left(\{K_j(s^{t-1})\}_{j \in (1,2)}, s^t\right) - \sum_{j=1}^2 H_j(K_j(s^{t-1}), I_j(s^t)) \right] \quad (4)$$

²This revenue function could, for instance, be derived from a production function $(\sum_j a_{j,t} K_{j,t})^\alpha N_t^{1-\alpha}$, where $a_{j,t}$ are shocks, $0 < \alpha < 1$, and where labor N is paid its marginal product.

$$\text{s.t. } K_j (s^t) = K_j (s^{t-1}) (1 - \delta_j) + I_j (s^t), \quad \forall s^t, j, \quad (5)$$

with s^0 and $K_j (s^{-1})$ given, and $P (s^0) = 1$ without loss of generality.

First-order conditions are

$$q_j (s^t) = H_{j,2} (K_j (s^{t-1}), I_j (s^t)), \quad (6)$$

and

$$1 = \sum_{s_{t+1}} P (s_{t+1} | s^t) \left(\frac{F_{K_j} (s^t, s_{t+1}) - H_{j,1} (s^t, s_{t+1}) + (1 - \delta_j) q_j (s^t, s_{t+1})}{q_j (s^t)} \right), \quad (7)$$

for each j , where the notation $P (s_{t+1} | s^t)$ shows the value of a unit of the numeraire in s_{t+1} conditional on s^t and in units of the numeraire at s^t . $H_{j,i} (K_j (s^{t-1}), I_j (s^t))$ represents the derivative with respect to the i^{th} element of the function. Eq. (7) determines the marginal q, $q_j (s^t)$, the marginal investment cost, and given that the production function and adjustment cost functions are homogenous of degree one, it also equals the average q (Tobin's q).

It is our objective to recover the state prices that drive firms' decisions from investment and capital choices. To do this, define the investment return as

$$R_j^I (s^t, s_{t+1}) \equiv \left(\frac{F_{K_j} (s^t, s_{t+1}) - H_{j,1} (s^t, s_{t+1}) + (1 - \delta_j) q_j (s^t, s_{t+1})}{q_j (s^t)} \right). \quad (8)$$

Knowing investment returns for the two types of capital, we can combine the first-order conditions,

$$\begin{bmatrix} R_1^I (s^t, \mathfrak{s}_1) & R_1^I (s^t, \mathfrak{s}_2) \\ R_2^I (s^t, \mathfrak{s}_1) & R_2^I (s^t, \mathfrak{s}_2) \end{bmatrix} \begin{bmatrix} P (\mathfrak{s}_1 | s^t) \\ P (\mathfrak{s}_2 | s^t) \end{bmatrix} = \mathbf{1}, \quad (9)$$

and recover the state prices with the matrix inversion

$$\begin{bmatrix} P (\mathfrak{s}_1 | s^t) \\ P (\mathfrak{s}_2 | s^t) \end{bmatrix} = \left(\begin{bmatrix} R_1^I (s^t, \mathfrak{s}_1) & R_1^I (s^t, \mathfrak{s}_2) \\ R_2^I (s^t, \mathfrak{s}_1) & R_2^I (s^t, \mathfrak{s}_2) \end{bmatrix} \right)^{-1} \mathbf{1}. \quad (10)$$

With these state prices, any claim can be priced; multi-period claims are priced by iterating forward. To efficiently compute stock returns, I can use the equivalence between investment returns and market returns that comes from the homogeneity assumptions. In particular, consider the market returns to the aggregate capital stock

$$R^M (s^t, s_{t+1}) \equiv \frac{D (s^t, s_{t+1}) + V (s^t, s_{t+1})}{V (s^t)}, \quad (11)$$

where $D (s^t, s_{t+1}) = F (\{K_j (s^{t-1})\}, s^t) - \sum_j H_j (K_j (s^{t-1}), I_j (s^t))$ represents the dividends paid by the firm and $V (s^t, s_{t+1})$ the ex-dividend value of the firm. Assuming constant returns to scale

in $F(\cdot)$ and $H_j(\cdot)$, it can easily be shown that this return will be equal to a weighted average of the investment returns:

$$R^M(s^t, s_{t+1}) = \sum_j \frac{q_j(s^t) K_j(s^t)}{\sum_i q_i(s^t) K_i(s^t)} \cdot R_j^I(s^t, s_{t+1}). \quad (12)$$

To evaluate the asset pricing implications of this model, we can postulate a stochastic process for the firms' investment growth rates

$$\lambda_j^I(s^t, s_{t+1}) \equiv I_j(s^t, s_{t+1}) / I_j(s^t), \quad j = 1, 2, \quad (13)$$

and for the marginal product terms $A_j(s^t)$. Assuming initial conditions for the two capital stocks and investment, $K_j(s^{-1})$ and $I_j(s^{-1})$, the investment growth process can be used to generate histories for investment and capital stocks. Based on these, one can generate investment returns, $R_j^I(s^t, s_{t+1})$, state prices $P(s_{t+1}|s^t)$ and the prices and returns of any security of interest. Overall, this simple model allows one to produce asset prices from firms' investment behavior and technology shocks that can then be compared to data on asset prices. As described in detail in Jermann (2010), to insure stationary returns for the aggregate capital stock, we require investment growth rates across the two types of capital not only to be perfectly (conditionally) correlated (as is implicit in the two-state assumption), but also to have the same realizations. That is,

$$\lambda_j^I(s^t, s_{t+1}) = \lambda^I(s^t, s_{t+1}) \quad \text{for } j = 1, 2. \quad (14)$$

Given our empirical implementation with investment in equipment and in structures, assuming equal volatility of investment growth rates is consistent with the data.³

2.2 Pricing nominal bonds

To price nominal bonds, I need to explicitly introduce inflation. It is assumed that inflation is given exogenously. In the model, the real value of firms' cash flows is not affected by inflation. I assume that inflation is not a priced factor, in the sense that inflation risk will only require a risk premium to the extent that inflation is correlated with the real side of the model. This approach seems essentially comparable to the way nominal bonds are priced in basic consumption-based asset pricing models.

³See Jermann (2010) for more details. It is also shown how to introduce time-varying investment specific technologies to accommodate possible differences in growth rates across capital stocks. Given the limited effect on the main asset pricing properties of this extension, the more simple specification is used here.

Assume inflation can have two possible realizations, $\lambda^P(z_t)$, with $z_t \in (\mathfrak{z}_1, \mathfrak{z}_2)$. For transparency, I present here the most basic case. It is straightforward to extend the approach to handle an inflation process with more than two states and a richer dependence on history. Combined with the real side presented before, we now have an extended state space that has a total of four states, the product of $(\mathfrak{s}_1, \mathfrak{s}_2)$ and $(\mathfrak{z}_1, \mathfrak{z}_2)$. In this extended state space, we have

$$P(s_{t+1}|s^t, z^t) = P(s_{t+1}, \mathfrak{z}_1|s^t, z^t) + P(s_{t+1}, \mathfrak{z}_2|s^t, z^t), \quad (15)$$

that is, for two states that differ only by next period's inflation realization, the sum of the state prices equals the price for delivery conditional on a real state realization but independent of the inflation realization. To determine the state prices in the extended state space, we make two assumptions. First, we assume that the firms' technology and investment decisions are not contingent on inflation, so that

$$P(s_{t+1}|s^t, z^t) = P(s_{t+1}|s^t). \quad (16)$$

With this assumption, the state prices derived from the real model (without considering inflation) correspond to the sums of two state prices in the extended setup here. To impose that inflation is not a priced factor, the second assumption is: If two states of nature only differ by their inflation realization, then the ratio of their state prices is assumed to equal the ratio of their (physical) probabilities. Thus, for instance, we have

$$\begin{aligned} P(s_{t+1}, \mathfrak{z}_1|s^t, z^t) &= \left(\frac{\Pr(s_{t+1}, \mathfrak{z}_1|s^t, z^t)}{\Pr(s_{t+1}, \mathfrak{z}_1|s^t, z^t) + \Pr(s_{t+1}, \mathfrak{z}_2|s^t, z^t)} \right) P(s_{t+1}|s^t), \text{ and} \quad (17) \\ P(s_{t+1}, \mathfrak{z}_2|s^t, z^t) &= \left(1 - \frac{\Pr(s_{t+1}, \mathfrak{z}_1|s^t, z^t)}{\Pr(s_{t+1}, \mathfrak{z}_1|s^t, z^t) + \Pr(s_{t+1}, \mathfrak{z}_2|s^t, z^t)} \right) P(s_{t+1}|s^t), \end{aligned}$$

where $\Pr(s_{t+1}, z_{t+1}|s^t, z^t)$ denotes a physical probability.

Having specified an inflation process, we can derive the price of a one-period nominal bond that pays one dollar at time $t + 1$ as

$$\begin{aligned} V_t^{\mathfrak{s}(1)}(s^t, z^t) &= \left(\frac{\Pr(\mathfrak{s}_1, \mathfrak{z}_1|s^t, z^t)}{\Pr(\mathfrak{s}_1, \mathfrak{z}_1|s^t, z^t) + \Pr(\mathfrak{s}_1, \mathfrak{z}_2|s^t, z^t)} \right) P(\mathfrak{s}_1|s^t) \frac{1}{\lambda^P(\mathfrak{z}_1)} \quad (18) \\ &+ \left(1 - \frac{\Pr(\mathfrak{s}_1, \mathfrak{z}_1|s^t, z^t)}{\Pr(\mathfrak{s}_1, \mathfrak{z}_1|s^t, z^t) + \Pr(\mathfrak{s}_1, \mathfrak{z}_2|s^t, z^t)} \right) P(\mathfrak{s}_1|s^t) \frac{1}{\lambda^P(\mathfrak{z}_2)} \\ &+ \left(\frac{\Pr(\mathfrak{s}_2, \mathfrak{z}_1|s^t, z^t)}{\Pr(\mathfrak{s}_2, \mathfrak{z}_1|s^t, z^t) + \Pr(\mathfrak{s}_2, \mathfrak{z}_2|s^t, z^t)} \right) P(\mathfrak{s}_2|s^t) \frac{1}{\lambda^P(\mathfrak{z}_1)} \\ &+ \left(1 - \frac{\Pr(\mathfrak{s}_2, \mathfrak{z}_1|s^t, z^t)}{\Pr(\mathfrak{s}_2, \mathfrak{z}_1|s^t, z^t) + \Pr(\mathfrak{s}_2, \mathfrak{z}_2|s^t, z^t)} \right) P(\mathfrak{s}_2|s^t) \frac{1}{\lambda^P(\mathfrak{z}_2)}. \end{aligned}$$

Multi-period bonds are priced recursively. For instance,

$$V_t^{\mathfrak{s}(2)}(s^t, z^t) = \sum_{s_{t+1}, z_{t+1}} P(s_{t+1}, z_{t+1} | s^t, z^t) V_{t+1}^{\mathfrak{s}(1)}(s^{t+1}, z^{t+1}) / \lambda^P(z_{t+1}). \quad (19)$$

As a special case, consider inflation to be independent of investment. In particular, assume

$$\frac{\Pr(\mathfrak{s}_1, \mathfrak{z}_1 | s^t, z^t)}{\Pr(\mathfrak{s}_1, \mathfrak{z}_2 | s^t, z^t)} = \frac{\Pr(\mathfrak{s}_2, \mathfrak{z}_1 | s^t, z^t)}{\Pr(\mathfrak{s}_2, \mathfrak{z}_2 | s^t, z^t)}, \text{ for a given } (s^t, z^t), \quad (20)$$

then, after some algebra,

$$V_t^{\mathfrak{s}(1)}(s^t, z^t) = \{P(\mathfrak{s}'_1 | s^t) + P(\mathfrak{s}'_2 | s^t)\} E\left(\frac{1}{\lambda^P} | s^t, z^t\right). \quad (21)$$

That is, with inflation independent of investment, the price of a nominal bond is simply the price of a real bond times the expected loss due to inflation, $E\left(\frac{1}{\lambda^P} | s^t, z^t\right)$, without any compensation for inflation risk. More generally, with $\frac{\Pr(\mathfrak{s}_1, \mathfrak{z}_1 | s^t, z^t)}{\Pr(\mathfrak{s}_1, \mathfrak{z}_2 | s^t, z^t)} \neq \frac{\Pr(\mathfrak{s}_2, \mathfrak{z}_1 | s^t, z^t)}{\Pr(\mathfrak{s}_2, \mathfrak{z}_2 | s^t, z^t)}$, this will no longer be the case, and a risk premium is needed in equilibrium.

3 Quantitative analysis

This section describes the calibration procedure and then presents quantitative results. The calibration fixes a first set of parameter values to match some direct empirical counterparts; a second set of parameter values is determined to match the first and second moments of US stock and short term bond returns. The test of the model consists in examining the implications for the term premium and the predictability of bond excess returns.

3.1 Calibration

The joint dynamics of investment growth and inflation are summarized by a first-order VAR. Table 1 shows four versions of the VAR estimated for investment in equipment and in structures, and for inflation measured by the CPI and the GDP deflator, for the period 1952-2010. There is moderately positive correlation between the innovations of investment growth and inflation. Inflation forecasts investment typically with a negative coefficient, while investment forecasts inflation with a positive coefficient. The calibration of the model targets the values of these moments averaged across the four empirical counterparts. As shown in Table 1, consistent with our modelling assumption, the historical volatility of equipment and structures are very similar.

In the model, the joint process for investment growth and inflation is represented by a four-state Markov chain. The model implied VAR reported in Table 1 is based on a very long simulated

sample. The model implied VAR cannot perfectly match the calibration target, mainly because with the limited number of grid points, the Markov chain cannot perfectly match the high persistence for inflation. The model implied VAR also does not fit perfectly, because of the bounds imposed when the I/K ratios of the two types of capital move far away from their steady state values. This is needed to insure a finite firm value. In particular, a lower and an upper limit for I/K_E at 0.0983 and 0.1745 are set, respectively. When the simulated process with the original investment growth rate realizations would go outside a bound, the realized growth rates are limited to reach exactly the lower or the upper bound, respectively. Asset pricing implications are not significantly affected by the exact values of these bounds. More details, and the values for the parameters of the Markov chain, are given in Appendix D. The appendix also shows that none of the conclusions of our analysis depends on the exact specification of this process, because the model implied term structure behavior is not very sensitive to small changes in the investment/inflation process.

Table 1: VAR(1) for Investment Growth and Inflation Dynamics

	CPI				GDP Deflator				Model	
	Equip.		Struct.		Equip.		Struct.		λ_t^I	λ_t^P
	λ_t^I	λ_t^P	λ_t^I	λ_t^P	λ_t^I	λ_t^P	λ_t^I	λ_t^P	λ_t^I	λ_t^P
λ_{t-1}^I	.12	.09	.36	.03	.12	.06	.35	.03	.20	.03
(t-stat)	(.9)	(3.9)	(2.7)	(1.0)	(.9)	(3.7)	(2.6)	(1.5)		
λ_{t-1}^P	-.81	.84	-.35	.81	-.78	.92	-.16	.89	-.57	.77
(t-stat)	(-2.2)	(12)	(-.9)	(10)	(-1.7)	(16)	(-.4)	(15)		
Sd(ε_t)	.0731	.0137	.0738	.0154	.0744	.0090	.0743	.0098	.070	.0121
Corr	.22		.34		.14		.24		.23	

$Sd(\varepsilon_t)$ and $Corr$ refer to moments of the innovations, λ^I is investment growth, λ^P is inflation; growth rates are first differenced logarithms.

Drift terms are included in the realizations of the Markov chain. They correspond to the historical averages of investment growth rates and inflation of 0.0355 and 0.0363, respectively. Several parameters describing firm technology are set to match long term averages of plausible empirical counterparts. These are the depreciation rates, δ_E and δ_S , the average relative value of the capital stocks, $\overline{K_E/K_S}$, and the adjustment cost parameters b_E, b_S, c_E, c_S . Except for the role of the depreciation rates for interest rate dynamics, these parameters impact our main

quantitative results only moderately. The sources for the values are described in Jermann (2010).

Parameter	Symbol	Value
Depreciation rates	δ_E, δ_S	0.11245, 0.031383
Relative value of capital stocks	$\overline{K_E/K_S}$	0.6
Adjustment cost parameters	b_E, b_S, c_E, c_S so that \bar{q}	1.5
Adjustment cost curvatures	ν_E, ν_S	2.4945, 3.9815
Marginal products of capital	A_E, A_S so that \bar{R}_E, \bar{R}_S	1.04819, 1.05706

There are 4 remaining parameters: the two adjustment cost curvature parameters, ν_E and ν_S , and the marginal product coefficients A_E and A_S . These parameters are important for asset pricing implications, and they do not have obvious direct empirical counterparts. Following Jermann (2010), these four parameters are set to match four moments from the data, under the assumption that $\nu_S > \nu_E$. The target moments are the unconditional mean and standard deviations of the aggregate US stock market and the one-year yields for the period 1952-2010. Practically, for a fixed sample of 100,000 periods, I search over the four-dimensional parameter space.

With the assumption that $\nu_S > \nu_E$, expected returns in the calibrated model are larger for structures than for equipment, which is consistent with the empirical analysis in Tuzel (2010). More direct evidence presented in Jermann (2010) also suggests that the adjustment cost curvature should be larger for structures than for equipment. For instance, the fact that the first-order serial correlation of the growth rates is somewhat higher for structures than for equipment can be interpreted as an expression of the desire to smooth investment over time due to the relatively higher adjustment cost. Some of the parameter estimates in Israelsen (2010), which contains a structural estimation of a two-sector adjustment cost model with structures and equipment, are not consistent with this assumption. There are several differences in the models and the data that can potentially account for the different estimates.

As shown in Table 3, the model can perfectly match the four targeted moments. Each of the four parameters affect all four moments, but with different degrees of sensitivities. In particular, for the short term yield, the levels of the marginal product coefficients A_E and A_S are important. To see this, consider the deterministic model version with constant investment growth rates,

where the constant return and interest rate, R , is given by

$$R = \frac{A - c}{b(\lambda^I - (1 - \delta))^{\nu-1}} + \left(1 - \frac{1}{\nu}\right) \lambda^I + \frac{1}{\nu}(1 - \delta), \quad (22)$$

which is increasing in the marginal product term A .⁴ The implied magnitudes of the adjustment costs are low, amounting to 5% and 6% of investment on average, for equipment and structures, respectively. Marginal costs are 1.4 and 1.2 on average.

Table 3: Equity returns and short term yields

	Model	Data
$E(r_M - y^{(1)})$ %	4.64	4.64
$\sigma(r_{M,r})$ %	17.13	17.13
$E(y^{(1)})$ %	5.29	5.29
$\sigma(y^{(1)})$ %	2.98	2.98

Yields, y , are from Fama and Bliss, defined as $-\ln(\text{price})/\text{maturity}$,

stock returns are the logs of value-weighted returns from CRSP,

$r_{M,r}$ is the stock return deflated by the CPI-U. Data covers 1952-2010.

To insure that state prices are always nonnegative, we make the marginal product term for structures stochastic, such that $A_S(1 \pm x(s^t))$. The size of the shock $x(\cdot)$ is determined so that the state prices that would go negative without the shock, are equal to 0. In the benchmark simulation, x is nonzero less than 7% of the time. This is further discussed below.

3.2 Results

I consider the model's implication for nominal bonds of different maturities and compare these to the Fama and Bliss data series of US Treasury bonds' implied discount bonds. Table 4 shows the means and standard deviations for yields of different maturities. The model matches the slope of the empirical term structure very well. The term premium (yield differential) for a five-year bond over a one-year bond is about half a percent as in the data. At least as far back as Shiller (1979), it has been recognized that models that satisfy the expectations hypothesis have

⁴Returns in the model could be compared to an unlevered return to capital. For simplicity and comparability with the literature, this is not done here.

difficulties matching the volatility of yields with longer maturities. The model here does a good job with yield volatilities; the standard deviations for longer term yields are only moderately lower than their empirical counterparts.

Table 4: Term structure					
	Maturity (years)				
	1	2	3	4	5
Nominal yields					
Mean - Model %	5.29	5.45	5.60	5.74	5.88
Mean - Data %	5.29	5.49	5.67	5.81	5.90
Std - Model %	2.98	2.73	2.52	2.34	2.17
Std - Data %	2.98	2.93	2.85	2.80	2.75
Real yields					
Mean - Model %	1.75	1.98	2.20	2.41	2.60
Std - Model %	2.81	2.57	2.37	2.21	2.06

Yields are from Fama and Bliss 1952-2010, defined as $-\ln(\text{price})/\text{maturity}$.

What determines the term premium in this model? As can be seen in Table 4, the model produces a positive real term premium and a smaller negative risk premium for inflation risk. In particular, the five-year term premium equals $5.88 - 5.29 = 0.59$, while the term premium for real yields equals $2.60 - 1.75 = 0.85$. This implies a negative premium for inflation risk of $85 - 59 = -26$ basis points.

Bond excess returns are in general negatively correlated with the stochastic discount factor, so that high returns are realized in lowly valued states (relative to their probabilities). High investment implies lower real interest rates (and thus higher real bond returns) at the same time as lower state prices (and stochastic discount factors). Section 4 will illustrate the mechanisms that drive these results with a continuous-time version of the model.

The negative inflation risk premium is due to the implied negative correlation between inflation and the stochastic discount factor. This property is brought about by the positive correlation between innovations in investment and inflation, as well as the positive coefficient on investment in

the VAR’s equation for inflation. When investment is high, inflation and expected inflation have a tendency to be high too. Therefore, inflation makes a negative contribution to the returns of nominal bonds at times when investment, and thus stock returns, are high. The fact that inflation enters the VAR’s equation for investment with a negative coefficient has a minor quantitative effect.⁵

Having shown that the model’s term premium matches its empirical counterpart reasonably well in some dimensions, we now consider return predictability. In particular, we consider the popular Fama-Bliss excess return regressions that seek to forecast excess return for n -period bonds with the forward premium of n -period bonds. These regressions provide a direct test of the expectations hypothesis, according to which excess returns should not be forecastable. As shown in Table 5, empirically, as is well known, excess returns are in fact forecastable with the forward premium. For every one point increase in the forward premium, excess returns increase by roughly the same amount. Typically, most fundamentals-based asset pricing models have a hard time producing the extent of return predictability observed in the data. As shown in Table 5, the model doesn’t perfectly match the data, but it can produce coefficients that are roughly half the size of the empirical counterparts.

Table 5: Fama-Bliss excess return regressions				
$rx_{t+1}^{(n)} = \alpha + \beta \left(f_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+1}^{(n)}$				
	Maturity (years)			
	2	3	4	5
Model - β	.3007	.3501	.4564	.5499
Data - β	.7606	1.0007	1.2723	.9952

Yields are from Fama and Bliss 1952-2010, $rx_{t+1}^{(n)}$ is the excess return of an n -period discount bond, $f_t^{(n)}$ is the forward rate, $(p_t^{(n-1)} - p_t^{(n)})$, with $p_t^{(n)}$ the log of the price discount bond, and $y_t^{(1)}$ is the one-period yield.

⁵The direct historical evidence on the real term structure from inflation-indexed bonds is short and inconclusive. As documented in Piazzesi and Schneider (2007), the term structure of inflation-indexed bonds has been on average upward sloping for the US and downward sloping for the UK.

The model’s ability to produce strong return forecastability stems from the implied time-variation in the market price of risk and in risk premiums. In the model, expected returns and the market price of risk are a function of the state variables, in particular, the investment to capital ratios of the two types of capital.

As for average yields, inflation risk plays a relatively minor role for the Fama-Bliss regressions. To illustrate the effect of inflation, we run two alternative versions of the Fama-Bliss regression. First, inflation risk is eliminated from the model by setting the standard deviation of inflation to zero. Second, the nominal forward premium is replaced as a regressor by the forward premium for real bonds. In both cases, as seen in Table 6, the slope coefficients increase moderately.

Table 6: Fama-Bliss excess return regressions					
Role of inflation risk					
$rx_{t+1}^{(n)} = \alpha + \beta \left(f_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+1}^{(n)}$					
		Maturity (years)			
		2	3	4	5
Model - β	no inflation risk	0.3596	0.4307	0.5685	0.6898
Model - β	real forward premium	0.3082	0.3634	0.4770	0.5773
Model - β	benchmark	.3007	.3501	.4564	.5499
Data - β		.7606	1.0007	1.2723	.9952

Yields are from Fama and Bliss 1952-2010, $rx_{t+1}^{(n)}$ is the excess return of an n -period discount bond, $f_t^{(n)}$ is the forward rate, $(p_t^{(n-1)} - p_t^{(n)})$, with

Yields are from Fama and Bliss 1952-2010, defined as $-\ln(\text{price})/\text{maturity}$.

To conclude this section, I consider here the importance of realizations at the borders of the state space. In particular, in order to guarantee positive state prices throughout, shocks to the marginal product of structures are introduced such that the marginal product term becomes $A_S(1 \pm x(s^t))$, with the value for $x(s^t)$ set so that a particular state price is equal to zero, if it were to drop below zero without the shock. For the benchmark calibration, this shock kicks in somewhat less than 7% of the time, when I/K for structures becomes very small. It turns out that the model implied regression coefficients are partially driven by realizations with low I/K

ratios for structures.

Fig. 1 shows expected excess returns and the forward premium as a function of the I/K ratio for structures. Both display the negative (conditional) relation which contributes to the positive regression coefficient in the Fama-Bliss regression. The figure also shows plots for the market price of risk and the conditional equity premium. In line with the expected excess returns for the two-period bond, both are downward sloping. However, the two top panels display some nonlinearity for lower values of the I/K ratio, which suggests that regression coefficients are partially determined by this region. For instance, we can run the Fama-Bliss regression with only those observations for which none of the next periods' state prices is zero, which roughly eliminates the 7% of observations with the lowest realization for the I/K ratio. In this case, for the two-period bond, the slope coefficient drops to .19 (from .30 reported in Table 5) and for the five-year bond it drops to .33 (from .55 in Table 5). One could consider alternative ways to insure positive state prices. However, given the lack of theoretical guidance and the fact that empirically this essentially concerns relatively rare outcomes, some arbitrariness is unavoidable. Overall, while the regression coefficients appear somewhat sensitive to auxiliary assumptions, endogenously time-varying risk premiums are a robust feature of this model. See Appendix A for additional evidence about the robustness of our results with respect to these assumptions.

3.3 *Additional results*

Given the extensive empirical literature on term structure behavior, this subsection examines additional model implications and compares these to their empirical counterparts. In particular, the autocorrelation of yields and the forecasting ability of term spreads for inflation and excess stock returns are considered. Of course, one would not expect this parsimoniously parameterized small scale model to perfectly match all moments in the data. Nevertheless, the model does a very good job with the autocorrelations, and it can reproduce at least qualitatively the considered evidence on the forecasting ability of term spreads.

As is well known, yields are highly autocorrelated. Table 7, panel A, displays autocorrelations for the various maturities over a one-year period. These are between .85 and .9, with higher autocorrelations for the longer maturities. The yields in the model almost match these values.

There is a long tradition in examining the information content of the term structure for future economic variables. For instance, Mishkin (1990), or Fama (1990), study the information in the term structure for future inflation. Following Mishkin, I regress the change in the n-year inflation

rate from the one-year inflation rate, $\ln \lambda_{t,t+n}^P - \ln \lambda_{t,t+1}^P$, on the term spread between the n-year and the one-year yield,

$$\ln \lambda_{t,t+n}^P - \ln \lambda_{t,t+1}^P = \alpha_n + \beta_n \left(n \cdot y_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+n}. \quad (23)$$

As shown in Table 7, panel B, the empirical slope coefficients β_n are between .28 and .37, with R2s between .11 and .14, depending on the maturities. In the model, the slope coefficients are somewhat lower, between .10 and .14, and the term spread's forecasting ability is also lower, with R2s between .01 and .03. The model can generate declining slope coefficients and declining R2s as a function of maturity, as observed in the data.

Empirical studies have also examined the term spread's ability to predict stock returns. Examples are Fama and French (1989), and more recently, Cochrane and Piazzesi (2005). I consider here a simple univariate regression that can capture some of the term spread's forecasting ability for stocks:

$$rxe_{t,t+m} = \alpha_m + \beta_m \left(y_t^{(5)} - y_t^{(1)} \right) + \varepsilon_{t+m}. \quad (24)$$

Stock market returns (value-weighted returns from CRSP) in excess of the one-year yield for different horizons, $rxe_{t,t+m}$, are regressed on the five-year term spread. As shown in Table 7, panel C, empirical slope coefficients are between 4 and 11 with R2s between .04 and .10. That is, a high term spread forecasts high excess returns for stocks (as seen earlier, a high term spread also forecasts high excess returns for bonds). The model can produce positive slope coefficients, however, these are somewhat smaller than in the data; the R2s are also smaller in the model. Overall, the model reproduces qualitatively how in the data the term spread is linked to future inflation and future stock returns, but, the relations in the model are somewhat weaker than in the data.

Table 7: Autocorrelations and Forecasting with Term Spreads

	Maturity (years)				
	1	2	3	4	5
A. Autocorrelations					
Data	.848	.871	.886	.894	.903
Model	.868	.872	.872	.877	.879
B. $\ln \lambda_{t,t+n}^P - \ln \lambda_{t,t+1}^P = \alpha_n + \beta_n \left(n \cdot y_t^{(n)} - y_t^{(1)} \right) + \varepsilon_{t+n}$					
Data β_n		.37	.32	.29	.28
(t-stat)		(2.4)	(1.9)	(1.6)	(1.5)
R^2		.14	.12	.11	.11
Model β_n		.14	.12	.11	.10
R^2		.03	.02	.02	.01
C. $rxet_{t,t+m} = \alpha_m + \beta_m \left(y_t^{(5)} - y_t^{(1)} \right) + \varepsilon_{t+m}$					
Data β_m	4.3	6.7	9.7	10.9	11.2
(t-stat)	(1.9)	(3.2)	(3.7)	(3.5)	(2.0)
R^2	.04	.05	.09	.10	.09
Model β_m	2.6	3.7	4.3	4.7	4.9
R^2	.03	.02	.02	.02	.01

$\lambda_{t,t+n}^P$ is the rate of inflation, $y_t^{(n)}$ is the yield of an n -period discount bond, $rxet_{t,t+m}$ is the market return in excess of the one-period yield, t-stats use the Hansen-Hodrick correction.

4 Continuous-time analysis

The continuous-time version of the model without inflation provides additional insights into the behavior of the short rate, the market price of risk, and the term premium. In particular, we will see that the capital depreciation rates play a key role for interest dynamics, while not impacting much other asset pricing implications.

Assume a discount factor process

$$\frac{d\Lambda}{\Lambda} = -r(\cdot) dt - \sigma(\cdot) dz \quad (25)$$

where dz is a univariate Brownian motion, and $r(\cdot)$ and $\sigma(\cdot)$ are functions of state variables driven by dz . This environment can be taken as the continuous-time counterpart of our two-state discrete-time setup. Solving for the firm's optimal investment policies yields the returns to the two types of capital

$$\frac{dR_j}{R_j} = \mu_j(\cdot) dt + \sigma_j(\cdot) dz, \text{ for } j = 1, 2. \quad (26)$$

The absence of arbitrage implies that

$$0 = E_t \left(\frac{d\Lambda_t}{\Lambda_t} \right) + E_t \left(\frac{dR_{jt}}{R_{jt}} \right) + E_t \left(\frac{d\Lambda_t}{\Lambda_t} \frac{dR_{jt}}{R_{jt}} \right), \quad (27)$$

so that (where we now longer explicitly acknowledge the state dependence of drift and diffusion terms)

$$0 = -r dt + \mu_j dt - \sigma_j \sigma dt, \text{ for } j = 1, 2. \quad (28)$$

Finally, the two coefficients of the discount factor process can be recovered by solving the system of these two equations as

$$r = \frac{\sigma_2}{\sigma_2 - \sigma_1} \mu_1 - \frac{\sigma_1}{\sigma_2 - \sigma_1} \mu_2 \quad (29)$$

$$\sigma = \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1}. \quad (30)$$

Clearly, In order to be able to recover the discount factor process from the two returns, the two diffusion terms need to be different, that is, $\sigma_2 - \sigma_1 \neq 0$. As in the discrete time analysis, once we specify processes for investment in the two types of capital, we have a fully specified discount factor process.

4.1 Market price of risk

As shown in the Appendix B, the realized return to a given capital stock equals

$$\left\{ \begin{array}{l} \frac{A_j - c_j}{b_j \left(\frac{I_{j,t}}{K_{j,t}} \right)^{\nu_j - 1}} - (\nu_j - 1) \left(1 - \frac{1}{\nu_j} \right) I_{j,t} / K_{j,t} - \delta_j \\ + (\nu_j - 1) [(\lambda^{I,j} - 1) + \delta_j + \frac{1}{2} (\nu_j - 2) \sigma_{I,j}^2] \end{array} \right\} dt + (\nu_j - 1) \sigma_{I,j} dz, \quad (31)$$

where $(\lambda^{I,j} - 1)$ and $\sigma_{I,j}$ are drift and diffusion terms of investment. Consider first the steady state level, as this is informative about average (unconditional) moments. In this case,

$I/K = \lambda^I - 1 + \delta$.⁶ To further simplify the expressions, assume equal investment volatility across the two capital stocks, $\sigma_{I,j} = \sigma_I$, as in the benchmark calibration. The market price of risk at the steady state can then be written as

$$\sigma|_{ss} = \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} = \frac{\bar{R}_2 - \bar{R}_1}{(\nu_2 - \nu_1)\sigma_I} + \frac{\nu_1 + \nu_2 - 3}{2}\sigma_I. \quad (32)$$

with

$$\bar{R} = \frac{A - c}{b(\lambda^I - (1 - \delta))^{\nu-1}} + \left(1 - \frac{1}{\nu}\right)\lambda^I + \frac{1}{\nu}(1 - \delta), \quad (33)$$

the latter representing the return of a given capital stock at steady state in a deterministic environment with a constant growth rate λ^I . The deterministic return \bar{R} is a useful quantity to summarize the effects of the technology parameters on the implied discount rate process. If we were to consider a deterministic environment, \bar{R} would have to be the same for the two capital stocks. But, as seen in Eq. (32), in this stochastic setup, the difference between \bar{R} across types of capital is an important contributor to the market price of risk. Starting from $\nu_2 > \nu_1$ and $\sigma_I > 0$, our calibration selects $\bar{R}_2 - \bar{R}_1 > 0$ and $\nu_1 + \nu_2 - 3 > 0$ to produce a large enough (positive) equity premium and to generate enough volatility for the market return. That is, $\sigma|_{ss} > 0$, and in this case the discount factor is lower when investment and the market return are high.

Away from steady-state, the market price of risk will change as a function of the state variables. Movements in the model's endogenous state variables, the investment to capital ratios I/K in the two sectors, are the main drivers. In particular, as is clear from Eq. (31), with $A - c > 0$, an increase in I/K reduces the expected return for this type of capital. Intuitively, with high I/K the value of capital, q , is high, and thus expected returns are low. If I/K gets very small (close to 0) the expected return can become very large. For our quantitative experiments, this effect is particularly strong when I_2/K_2 is low, so that μ_2 increases, driving up the market price of risk. As shown in Jermann (2010), the time-variation in σ , and hence the variation in the equity premium, is substantial enough to have implied at some occasions a negative conditional equity premium for the model calibrated to the US economy.

4.2 Dynamics of the short rate

Reproducing the solution for the short rate from above

$$r = \frac{\sigma_2}{\sigma_2 - \sigma_1}\mu_1 - \frac{\sigma_1}{\sigma_2 - \sigma_1}\mu_2,$$

⁶In particular, consider a path where $dz = 0$ for a very long time, such that λ^I and σ_I have converged, and I_t/K_t has converged to $\lambda^I - 1 + \delta$.

the short rate can be seen as a weighted average of the two expected returns μ_1 and μ_2 . With positively correlated returns, and assuming (without loss of generality) $\sigma_2 > \sigma_1 > 0$, the weight on the less volatile return, $\frac{\sigma_2}{\sigma_2 - \sigma_1}$, is then larger than one, and the other weight is negative. The economic intuition is the following. There are two risky investment returns that are perfectly (conditionally) correlated, so that the risk free rate is created with a portfolio of the two risky returns for which the two return realizations exactly offset each other. This portfolio puts a positive weight on the less volatile return and a negative weight (that is smaller in absolute value) on the more volatile return.

Specializing to the case $\sigma_{I_j} = \sigma_I$ as in the benchmark calibration, the equation for the short rate becomes

$$r = \frac{\nu_2 - 1}{\nu_2 - \nu_1} \mu_1 - \frac{\nu_1 - 1}{\nu_2 - \nu_1} \mu_2, \quad (34)$$

which implies that the "weights" attached to the expected returns are simple functions of the constant adjustment cost curvature parameters ν_j . Thus, the short rate is driven only by movements in expected returns.

In our simulations, expected returns are positively correlated, but not perfectly so. The expectation of the less volatile return, μ_1 , has a relatively bigger weight, but it is also less volatile than μ_2 . In our benchmark calibration, the impact of μ_1 dominates. To see why this is the case, and to illustrate this mechanism further, assume that drift and diffusion of investment, λ^I and σ_I , are constant, so that the only state variables are $I_{1,t}/K_{1,t}$ and $I_{2,t}/K_{2,t}$, and consider the differential equation for the short rate

$$dr = \mu_r(\cdot) dt + \sigma_r(\cdot) dz.$$

Under the made assumptions, the diffusion term equals

$$\sigma_r = \left[\frac{\nu_2 - 1}{\nu_2 - \nu_1} \frac{d\mu_1}{d\frac{I_{1,t}}{K_{1,t}}} \frac{I_{1,t}}{K_{1,t}} - \frac{\nu_1 - 1}{\nu_2 - \nu_1} \frac{d\mu_2}{d\frac{I_{2,t}}{K_{2,t}}} \frac{I_{2,t}}{K_{2,t}} \right] \sigma_I, \quad (35)$$

with

$$\frac{d\mu}{d\frac{I}{K}} = -(\nu - 1) \left[\frac{A - c}{b \left(\frac{I}{K} \right)^\nu} + 1 - \frac{1}{\nu} \right]. \quad (36)$$

After rearranging terms

$$\sigma_r = \frac{(\nu_2 - 1)(\nu_1 - 1)}{\nu_2 - \nu_1} \times \left[\left(\frac{A_2 - c_2}{b_2 \left(\frac{I_{2,t}}{K_{2,t}} \right)^{\nu_2 - 1}} + \left(1 - \frac{1}{\nu_2} \right) \frac{I_{2,t}}{K_{2,t}} \right) - \left(\frac{A_1 - c_1}{b_1 \left(\frac{I_{1,t}}{K_{1,t}} \right)^{\nu_1 - 1}} + \left(1 - \frac{1}{\nu_1} \right) \frac{I_{1,t}}{K_{1,t}} \right) \right] \sigma_I. \quad (37)$$

Some properties of the interest rate are now obvious. If any of the curvature parameters ν is close to 1, then the short rate can be arbitrarily smooth. In addition, interest volatility moves with the state variables, that is, interest rates are heteroscedastic even with homoscedastic investment.

At the steady state,

$$\sigma_r|_{ss} = \frac{(\nu_2 - 1)(\nu_1 - 1)}{\nu_2 - \nu_1} \sigma_I [\bar{R}_2 + \delta_2 - \bar{R}_1 - \delta_1]. \quad (38)$$

As shown in Eq. (38), in addition to the terms that are also included in the market price of risk, ν_j , \bar{R}_j and σ_I , the depreciation rates appear explicitly. The equation implies that for the case with $\bar{R}_1 = \bar{R}_2$, if the sector with the higher curvature, ν , has the higher depreciation rate, δ , then the short rate commoves positively with investment shocks ($\sigma_r|_{ss} > 0$), and if the sector with the higher curvature has a lower depreciation rate, the short rate commoves negatively with investment ($\sigma_r|_{ss} < 0$). Essentially, with a higher depreciation rate, the investment to capital ratio is higher, and thus a given percentage change in investment produces a larger change in the investment to capital ratio and, everything else equal, in the expected return. Our benchmark calibration implies that structures are harder to adjust than equipment, $\nu_2 - \nu_1 > 0$, also structures depreciate less than equipment, $\delta_2 - \delta_1 < 0$.⁷ With this particular parameter configuration, at steady state, innovations in the short rate are negatively related to innovations in investment ($\sigma_r|_{ss} < 0$). Appendix C considers the more general case when $\sigma_{I_1} \neq \sigma_{I_2}$, as well as the case for quadratic adjustment costs; the difference in capital depreciation rates remain a key determinants of interest rate dynamics.

This mechanism also plays a crucial role for the term premium. While the setup does not allow for closed form solutions for the term premium, the solution for the short rate offers some insights. To illustrate this, consider the holding period return of a discount bond with a maturity of N ,

$$\frac{dP(N, t)}{P} - \frac{1}{P} \frac{\partial dP(N, t)}{\partial N} dt, \quad (39)$$

and a corresponding yield to maturity for which

$$dy^{(N)} = \mu_{Y^N} dt + \sigma_{Y^N} dz. \quad (40)$$

Then, the holding period return satisfies

$$\frac{dP(N, t)}{P} - \frac{1}{P} \frac{\partial dP(N, t)}{\partial N} dt = [\cdot] dt - N \sigma_{Y^N} dz. \quad (41)$$

⁷The benchmark calibration also implies $\bar{R}_2 - \bar{R}_1 > 0$, but this difference is substantially smaller than $\delta_1 - \delta_2$.

That is, stochastic changes in the yield drive the stochastic return movements. As is clear from Eq. (28), the instantaneous risk premium for this bond equals

$$-N\sigma_{Y^N}\sigma. \quad (42)$$

Returns in my model are driven by a single Brownian motion, so that, conditionally, all returns are perfectly correlated (positively or negatively). However, from the perspective of asset prices, the model has two endogenous state variables, the investment to capital ratios in the two sectors. For the reported simulations, the yields of different maturities move positively together and their volatilities are not too different. Thus, the expression for σ_r can be a useful approximation for σ_{Y^N} for shorter maturities, and $-N\sigma_r\sigma$ can be a useful approximation for the term premium.

To get a sense of how informative such an approximation is, compare $-\sigma_r\sigma$ to the conditional log excess return of a discount bond that has a one period maturity when it is sold, that is, the conditional log excess return of a two-period bond, $E_t\left(r_{t+1}^{(2)} - y_t^{(1)}\right)$, in the discrete time model at steady state.

Table 8: Term premium: continuous-time versus discrete-time model

	Cont.-time	Discrete-time
	$-\sigma_r\sigma$	$E_t\left(r_{t+1}^{(2)} - y_t^{(1)}\right)$
Benchmark	.0036	.0033
$\delta_1 = \delta_2, \bar{R}_1 = \bar{R}_2,$	0	0.000016
$\delta_1 = \delta_2$	-0.00044	-.00036
$\bar{R}_1 = \bar{R}_2, \delta_1 = .112 > \delta_2 = .0313$.0026	.0023
$\bar{R}_1 = \bar{R}_2, \delta_1 = .0313 < \delta_2 = .112$	-.0026	-.0027

For the discrete time model the term premium is computed at the steady-state levels of I/K_j . In the benchmark case $[\bar{R}_1, \bar{R}_2, \delta_1, \delta_2]$ are equal to $[1.048, 1.057, .112, .0313]$. When \bar{R}_j , or δ_j , are equalized across sectors, they take the average value.

As shown in Table 8 $-\sigma_r\sigma$ is clearly very informative about the size of the term premium in the

discrete-time model.⁸ Substantively, this confirms the importance of different capital depreciation rates for the term premium at short maturities, as shown in Eq. (38).

5 Conclusion

This paper extends the q theory of investment to price the term structure of nominal bonds. The quantitative model does a reasonable job matching averages and volatilities of the nominal term structure for the US. The model also displays time-varying risk premiums for bonds, providing a possible explanation for observed departures from the expectations hypothesis. In the model, real and nominal risks affect returns on nominal bonds. Quantitatively, real interest risk plays a more important role for risk premiums than inflation risk. In the paper, the modeling of the production technology and the role of inflation for firms' profits is kept simple for transparency and tractability. Further enriching the model with features used in corporate finance or in macroeconomics would appear to offer fruitful avenues for future research.

⁸For comparability, the discrete-time model here has IID investment growth and no inflation.

References

- Backus, D., Gregory, A., Zin, S., 1989. Risk premiums in the term structure: evidence from artificial economies. *Journal of Monetary Economics* 24, 371–399.
- Bansal R., Shaliastovich I., 2010. A long-run risks explanation of predictability puzzles in bond and currency markets. Unpublished working paper. University of Pennsylvania.
- Belo, F., 2010. Production-based measures of risk for asset pricing. *Journal of Monetary Economics* 57, 146–163.
- Berardi, A., Torous W., 2005. Term structure forecasts of long-term consumption growth, *Journal of Financial and Quantitative Analysis*, 241-258.
- Berk, J., Green, R., Naik, V., 1999. Optimal investment, growth options and security returns. *Journal of Finance* 54, 1153–1607.
- Binsbergen, J. van, Fernandez-Villaverde, J., Koijen, R., Rubio-Ramirez, J., 2012. The term structure of interest rates in a DSGE model with recursive preferences, *Journal of Monetary Economics* 59, 634-48.
- Carlson, M., Fisher, A., Giammarino, R., 2004. Corporate investment and asset price dynamics: implications for the cross-section of returns, *Journal of Finance* 59, 2577–2603.
- Chapman, D., 1997. The cyclical properties of consumption growth and the real term structure, *Journal of Monetary Economics* 39, 145-172.
- Cochrane, J., 1988. Production-based asset pricing. Unpublished working paper 2776. National Bureau of Economic Research, Cambridge, MA.
- Cochrane, J., 1991. Production-based asset pricing and the link between stock returns and economic fluctuations. *Journal of Finance* 46, 207–234.
- Cochrane, J., Piazzesi, M., 2005. Bond risk premia, *American Economic Review* 95, 138-160.
- Cooper, I., 2006, Asset pricing implications of nonconvex adjustment costs and irreversibility of investment. *Journal of Finance* 61, 139-170.
- Eberly, J., Wang, N., 2010. Reallocating and pricing illiquid capital: Two productive trees. Unpublished working paper.

- Ehling, P., Gallmeyer, M., Heyerdahl-Larsen, C., Illeditsch, P., 2011. Beliefs About Inflation and the Term Structure of Interest Rates. Unpublished working paper University of Pennsylvania.
- Eisfeldt, A., Papanikolaou, D., 2012. Organization capital and the cross-section of expected returns. Forthcoming Journal of Finance.
- Fama, E., 1990. Term-structure forecasts of interest rates, inflation and real returns. Journal of Monetary Economics 25, 59-76.
- Fama, E., French, K., 1989. Business conditions and expected returns on stocks and bonds. Journal of Financial Economics 25, 23-49.
- Israelsen, R., 2010. Investment based valuation and managerial expectations. Unpublished working paper. Indiana University.
- Jermann, U. J., 2010. The equity premium implied by production. Journal of Financial Economics 98, 279-296.
- Kogan, L., 2004. Asset prices and real investment. Journal of Financial Economics 73, 411-431.
- Kuehn, L., 2009. Disentangling investment returns and stock returns: The importance of time-to-built, unpublished working paper. Carnegie Mellon University.
- Li, E., Livdan, D., Zhang, L., 2009. Anomalies, Review of Financial Studies 22, 4301-4334.
- Li, Q., Vassalou, M., Xing, Y., 2006. Sector investment growth rates and the cross-section of equity returns. The Journal of Business, 79 (3), 1637-1665.
- Liu L. X., Whited, T. M., Zhang L, 2009. Investment-based expected stock returns. Journal of Political Economy 117 (6), 1105-1139.
- Mehra, R., Prescott, E. 1985. The equity premium puzzle. Journal of Monetary Economics 15, 145-161.
- Miskhin, F., 1990. The information in the longer maturity term structure about future inflation. The Quarterly Journal of Economics 105, 815-828.
- Pastor, L., Veronesi, P., 2009. Technological revolutions and stock prices. American Economic Review 99, 1452-1483.

- Piazzesi, M., Schneider M., 2007. NBER Macroeconomics Annual 2006, Cambridge MA: MIT Press p. 389-442.
- Rudebusch, G., Swansson E., 2008. Examining the Bond Premium Puzzle with a DSGE Model. *Journal of Monetary Economics* 55, 111-126.
- Shiller, R., 1979. The Volatility of Long-Term Interest Rates and Expectations Models of the Term Structure. *Journal of Political Economy* 87, 1190-1219.
- Tauchen, G., 1986. Finite state markov-chain approximations to univariate and vector autoregressions. *Economics Letters* 20, 177-181.
- Tuzel, S., 2010. Corporate real estate holdings and the cross section of stock returns. *Review of Financial Studies*.
- Wachter, J. A., 2006. A consumption-based model of the term structure of interest rates. *Journal of Financial Economics* 79, 365-399.
- Warusawitharana, M., 2010. The expected real return to equity. Unpublished working paper. Federal Reserve Board. Washington DC.
- Zhang, L., 2005. The value premium. *Journal of Finance* 60, 67-103.

Appendix A: On negative state prices

This appendix presents a result that shows that our approach for preventing state prices from becoming negative is quite general, and thus, that our quantitative results appear quite robust. In particular, it is shown that when setting one of the two real state prices to zero, for instance, by assuming a particular value for one of the two productivity realizations, the value of the productivity realization in the other state is irrelevant for state prices, and thus for bond prices.

To prevent $P(\mathfrak{s}_2|s^t)$ from turning negative in some states s^t , we adjust the marginal productivity term $A_2(1 + x(\mathfrak{s}_1|s^t))$, by having $x(\mathfrak{s}_1|s^t)$ be negative, so that $P(\mathfrak{s}_2|s^t) = 0$. The next proposition shows that in this case, the value given to $x(\mathfrak{s}_2|s^t)$ does not matter at all for the other real state price $P(\mathfrak{s}_1|s^t)$, and thus for bond prices.

Proposition: If

$$P(\mathfrak{s}_2|s^t) = \frac{R_1^I(s^t, \mathfrak{s}_1) - R_2^I(s^t, \mathfrak{s}_1)}{|R|} = 0 \quad (43)$$

then

$$P(\mathfrak{s}_1|s^t) = \frac{1}{R_1^I(s^t, \mathfrak{s}_1)} = \frac{1}{R_2^I(s^t, \mathfrak{s}_1)}, \quad (44)$$

and thus, the values of the two remaining investment returns $R_2^I(s^t, \mathfrak{s}_2)$ and $R_1^I(s^t, \mathfrak{s}_2)$ do not matter for state prices and thus bond prices.

Proof: $P(\mathfrak{s}_2|s^t) = 0$ implies $R_1^I(s^t, \mathfrak{s}_1) = R_2^I(s^t, \mathfrak{s}_1)$. Thus

$$\begin{aligned} |R| &= R_1^I(s^t, \mathfrak{s}_1) R_2^I(s^t, \mathfrak{s}_2) - R_2^I(s^t, \mathfrak{s}_1) R_1^I(s^t, \mathfrak{s}_2) \\ &= R_1^I(s^t, \mathfrak{s}_1) \times [R_2^I(s^t, \mathfrak{s}_2) - R_1^I(s^t, \mathfrak{s}_2)] \\ &= R_2^I(s^t, \mathfrak{s}_1) \times [R_2^I(s^t, \mathfrak{s}_2) - R_1^I(s^t, \mathfrak{s}_2)] \end{aligned} \quad (45)$$

so that

$$P(\mathfrak{s}_1|s^t) = \frac{R_2^I(s^t, \mathfrak{s}_2) - R_1^I(s^t, \mathfrak{s}_2)}{|R|} = \frac{1}{R_1^I(s^t, \mathfrak{s}_1)} = \frac{1}{R_2^I(s^t, \mathfrak{s}_1)}. \quad (46)$$

This proposition implies that when $P(\mathfrak{s}_2|s^t)$ is increased to 0, by lowering $R_2^I(s^t, \mathfrak{s}_1)$ through adjusting $x(\mathfrak{s}_1|s^t)$, the value $x(\mathfrak{s}_2|s^t)$ takes, and thus $R_2^I(s^t, \mathfrak{s}_2)$ and $R_1^I(s^t, \mathfrak{s}_2)$, do not affect $P(\mathfrak{s}_1|s^t)$. An obvious alternative way to set $P(\mathfrak{s}_2|s^t) = 0$, would be to raise $R_1^I(s^t, \mathfrak{s}_1)$. However, this would imply—counterfactually—that a negative productivity shock is associated with a positive innovation in investment growth. In simulations, we also found that in this case, the productivity shocks would need to be larger and that risk premiums vary even more.

Appendix B: Continuous-time model

The capital stock evolves as $dK_{j,t} = (I_{j,t} - \delta_j K_{j,t}) dt$, and the investment cost is given by

$$H_j(I_{j,t}, K_{j,t}) = \left\{ \frac{b_j}{\nu_j} (I_{j,t}/K_{j,t})^{\nu_j} + c_j \right\} K_{j,t}, \quad (47)$$

which is homogenous of degree one in I_j and K_j . The gross profit is given as $A_j K_{j,t}$. The firm uses two capital stocks that enter production separably, so that $j = 1, 2$.

Assume that the state-price process is given as

$$d\Lambda_t = -\Lambda_t r(x_t) dt + \Lambda_t \sigma(x_t) dz_t, \quad (48)$$

where dz_t is a one-dimensional Brownian motion and

$$dx_{i,t} = \mu_{xi}(x_t)dt + \sigma_{xi}(x_t)dz_t \quad \text{for } i = 1, \dots, N, \quad (49)$$

that is, there are N state variables all driven by the same univariate Brownian motion. Assume that the functions $\mu_{xi}(x_t)$, $\sigma_{xi}(x_t)$, $r(x_t)$ and $\sigma(x_t)$, satisfy the regular conditions such that there are solutions for these stochastic differential equations.

The firm maximizes its value

$$V = \max_{\{I_{1t+s}, I_{2t+s}\}} E_t \left\{ \int_t^\infty \left[\sum_{j=1,2} A_j K_{j,t+s} - H_j(I_{j,t+s}, K_{j,t+s}) \right] \frac{\Lambda_{t+s}}{\Lambda_t} ds \right\}. \quad (50)$$

Given the dynamics of Λ_t , it is obvious that the firm's value function V is independent of Λ_t . Following from the Markov property of the state variables x_t , the firm's value function would be a function of (K_{1t}, K_{2t}, x_t) . The Hamilton-Jacobi-Bellman equation is

$$rV = \max_{\{I_{1t}, I_{2t}\}} \left\{ \begin{aligned} & [A_1 K_{1t} - H_1(I_{1t}, K_{1t}) + A_2 K_{2t} - H_2(I_{2t}, K_{2t})] \\ & + (I_{1t} - \delta_1 K_{1t}) V_{K1} + (I_{2t} - \delta_2 K_{2t}) V_{K2} \\ & + \sum \mu_{xi} V_{xi} + \frac{1}{2} \sum \sum \sigma_{xi} \sigma_{xj} V_{xi,xj} + \sum \sigma \sigma_{xi} V_{xi} \end{aligned} \right\}. \quad (51)$$

The first-order conditions are

$$H_{I_j}(I_{j,t}, K_{j,t}) = V_{K_j} \equiv q_{j,t} \quad (52)$$

That is,

$$V_{K_j} = b_j (I_{j,t}/K_{j,t})^{\nu_j-1} \quad (53)$$

and

$$I_{j,t} = \left(\frac{V_{K_j}}{b_j} \right)^{\frac{1}{\nu_j-1}} K_{j,t}, \quad \text{for } j = 1, 2. \quad (54)$$

Because of constant returns to scale in K_t , following Hayashi, it is easy to see that $V(K_{1t}, K_{2t}, x_t) = V_1(K_{1t}, x_t) + V_2(K_{2t}, x_t) = K_{1t}V_{K1}(x_t) + K_{2t}V_{K2}(x_t)$. Thus, it is clear that optimal investment follows an Ito process, $dI_{j,t}/I_{j,t} = \mu_{I_j}(K_{j,t}, x_t) dt + \sigma_{I_j}(K_{j,t}, x_t) dz_t$.

Define realized returns to each of the firm's capital stocks as

$$\frac{dR_j}{R_j} = \frac{A_j K_{j,t} - H_j(I_{j,t}, K_{j,t})}{V_{j,t}} dt + \frac{dV_{j,t}}{V_{j,t}}, \text{ for } j = 1, 2. \quad (55)$$

Dropping the index j , given Hayashi's result and the first-order conditions,

$$\frac{AK_t - H(I_t, K_t)}{V_t} dt + \frac{dV_t}{V_t} = \frac{AK_t - H(I_t, K_t)}{q_t K_t} dt + \frac{dK_t}{K_t} + \frac{dq_t}{q_t}. \quad (56)$$

Using the first-order condition $q_t = H_I(I_t, K_t)$ together with Ito's lemma, the last term of this equation can be written as

$$\frac{dq_t}{q_t} = \frac{dH_I(I_t, K_t)}{H_I(I_t, K_t)} = \frac{H_{II}(I_t, K_t) dI + H_{IK}(I_t, K_t) dK + \frac{1}{2} H_{III}(I_t, K_t) (dI)^2}{H_I(I_t, K_t)}, \quad (57)$$

and given the functional form for $H(\cdot)$, some algebra yields

$$\frac{dq_t}{q_t} = (\nu - 1) \left[\mu_I - (I_t/K_t - \delta) + \frac{1}{2} (\nu - 2) \sigma_I^2 \right] dt + (\nu - 1) \sigma_I dz. \quad (58)$$

Using this result, the return Eq. (31) given in the main text can then be derived.

Appendix C: Different investment volatilities and quadratic adjustment cost

We start with the more general case for which $\sigma_{I_1} \neq \sigma_{I_2}$, with $\sigma_{I_1}, \sigma_{I_2} > 0$. To obtain relatively simple expressions for the diffusion term of the short rate, $\sigma_r(\cdot)$,

$$dr = \mu_r(\cdot) dt + \sigma_r(\cdot) dz, \quad (59)$$

assume, as in the main text, that λ^I and σ_I are constant, so that the only state variables are $I_{1,t}/K_{1,t}$ and $I_{2,t}/K_{2,t}$. Following the same procedure as in the main text

$$\begin{aligned} \sigma_r &= \frac{(\nu_2 - 1)\sigma_{I_2}(\nu_1 - 1)\sigma_{I_1}}{(\nu_2 - 1)\sigma_{I_2} - (\nu_1 - 1)\sigma_{I_1}} \\ &\times \left[\left(\frac{A_2 - c_2}{b_2 \left(\frac{I_{2,t}}{K_{2,t}}\right)^{\nu_2 - 1}} + \left(1 - \frac{1}{\nu_2}\right) \frac{I_{2,t}}{K_{2,t}} \right) - \left(\frac{A_1 - c_1}{b_1 \left(\frac{I_{1,t}}{K_{1,t}}\right)^{\nu_1 - 1}} + \left(1 - \frac{1}{\nu_1}\right) \frac{I_{1,t}}{K_{1,t}} \right) \right], \end{aligned} \quad (60)$$

and at steady state

$$\sigma_r|_{ss} = \frac{(\nu_2 - 1)\sigma_{I_2}(\nu_1 - 1)\sigma_{I_1}}{(\nu_2 - 1)\sigma_{I_2} - (\nu_1 - 1)\sigma_{I_1}} [\bar{R}_2 + \delta_2 - \bar{R}_1 - \delta_1]. \quad (61)$$

Thus, the substantive conclusions from the analysis in the main text go through. In particular, with $[\bar{R}_2 + \delta_2 - \bar{R}_1 - \delta_1] > 0$, $\sigma_r|_{ss} > 0$ and with $[\bar{R}_2 + \delta_2 - \bar{R}_1 - \delta_1] < 0$, $\sigma_r|_{ss} < 0$.

For the quadratic case, $\nu_1 = \nu_2 = 2$, this simplifies to

$$\sigma_r|_{ss} = \frac{\sigma_{I_2}\sigma_{I_1}}{\sigma_{I_2} - \sigma_{I_1}} [\bar{R}_2 + \delta_2 - \bar{R}_1 - \delta_1]. \quad (62)$$

In this case, the market price of risk at steady state equals

$$\sigma|_{ss} = \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} = \frac{\bar{R}_2 - \bar{R}_1}{\sigma_{I,2} - \sigma_{I,1}}. \quad (63)$$

Clearly, in this case, for the normalization $\sigma_{I,2} - \sigma_{I,1} > 0$, a positive equity premium requires $\bar{R}_2 - \bar{R}_1 > 0$. Thus, with equal depreciation rates, $\delta_1 = \delta_2$, the short rate commoves positively with investment, and given the analysis in the main text, short term bonds then have a negative term premium. A positive term premium requires $\bar{R}_2 - \bar{R}_1 < \delta_1 - \delta_2$.

More generally, away from steady state,

$$\begin{aligned} \sigma &= \frac{\mu_2 - \mu_1}{\sigma_2 - \sigma_1} = \frac{\frac{A_2 - c_2}{b_2 \frac{I_{2,t}}{K_t}} - \frac{1}{2} I_{2,t}/K_t + \lambda^{I_2} - \frac{A_1 - c_1}{b_1 \frac{I_{1,t}}{K_t}} + \frac{1}{2} I_{1,t}/K_t - \lambda^{I_1}}{\sigma_{I,2} - \sigma_{I,1}} \\ &= \frac{\left(\frac{A_2 - c_2}{b_2 \frac{I_{2,t}}{K_t}} - \frac{1}{2} I_{2,t}/K_t \right) - \left(\frac{A_1 - c_1}{b_1 \frac{I_{1,t}}{K_t}} - \frac{1}{2} I_{1,t}/K_t \right) + (\lambda^{I_2} - \lambda^{I_1})}{\sigma_{I,2} - \sigma_{I,1}}. \end{aligned} \quad (64)$$

As in the model considered in the text, I/K ratios, expected growth rates and volatility terms all matter.

Appendix D : Sensitivity to investment/inflation process

This appendix presents additional details about the calibration of the investment/inflation process, and shows that model implications for the term structure are not very sensitive to changes in the investment/inflation process.

The benchmark calibration uses a constrained minimization routine to match the population moments of the target VAR with a four-state Markov chain. This cannot be perfectly achieved. In particular, the two slope coefficients of the inflation equation cannot be matched simultaneously. The calibration that is retained produces a reasonably close fit for both of these two coefficients, while matching the remaining moments of the target VAR. In the simulations, this Markov chain is modified because upper and lower bounds on investment/capital ratios are imposed, as described in the main text. This affects somewhat the investment growth process.

Table 9 reports alternative specifications. Specification (A) uses Tauchen's (1986) method to fit the VAR with a four-state Markov chain. As is well known, this procedure has difficulties matching processes with high persistence with a small number of grid points. As shown in Table 9, while there are some differences in the investment/inflation process relative to the Benchmark case, this has only moderate effects on the model implied term structure. In particular mean and standard deviations of the long term yields, $E(y^{(5)})$ and $\sigma(y^{(5)})$, and the slope coefficients from the Fama-Bliss regressions, $\beta^{(2)}$ and $\beta^{(5)}$ are only moderately affected. The mean and standard deviation of the short rate, $E(y^{(1)})$ and $\sigma(y^{(1)})$ in Table 9 are exactly the same as in the benchmark case, because the two adjustment cost curvature parameters, ν_E and ν_S , and the marginal product coefficients A_E and A_S are recalibrated to match first and second moments of the short rate and the market return.

Specification (B) entirely removes the dependence between investment and inflation. This

specification also only produces moderately different outcomes for interest rate behavior.

Table 9: Sensitivity to investment/inflation process

	Benchmark		(A)		(B)		Data Target	
	λ_t^I	λ_t^P	λ_t^I	λ_t^P	λ_t^I	λ_t^P	λ_t^I	λ_t^P
λ_{t-1}^I	.20	.03	.15	-.004	.15	0	.24	.05
λ_{t-1}^P	-.57	.77	-.40	.89	0	.80	-.53	.86
Std(ε_t)	.070	.0121	.071	.011	.073	.016	.0739	.0120
Corr	.23		.25		0		.24	
$E(y^{(1)})$ %	.0529		.0529		.0529			
$E(y^{(5)})$ %	.0588		.0575		.0589			
$\sigma(y^{(1)})$ %	.0298		.0298		.0298			
$\sigma(y^{(5)})$ %	.0217		.0244		.0218			
$\beta^{(2)}$.301		.328		.251			
$\beta^{(5)}$.550		.652		.532			

(A) is based on the method from Tauchen (1986), (B) assumes independence between investment and inflation. $y^{(j)}$ is the yield for a j – period bond, $\beta^{(j)}$ the slope coefficient from a Fama-Bliss regression with j – period bonds.

The matrix of transition probabilities and the vectors of the realizations for the benchmark case are given below. The probability matrix with $\Pr(x_{t+1} = \mathbf{x}_j | x_t = \mathbf{x}_i)$ in column j and row i is

$$\begin{bmatrix} 0.5404 & 0.0000 & 0.4596 & 0.0000 \\ 0.1286 & 0.5468 & 0.0829 & 0.2417 \\ 0.2417 & 0.0829 & 0.5468 & 0.1286 \\ 0.0000 & 0.4596 & 0.0000 & 0.5404 \end{bmatrix},$$

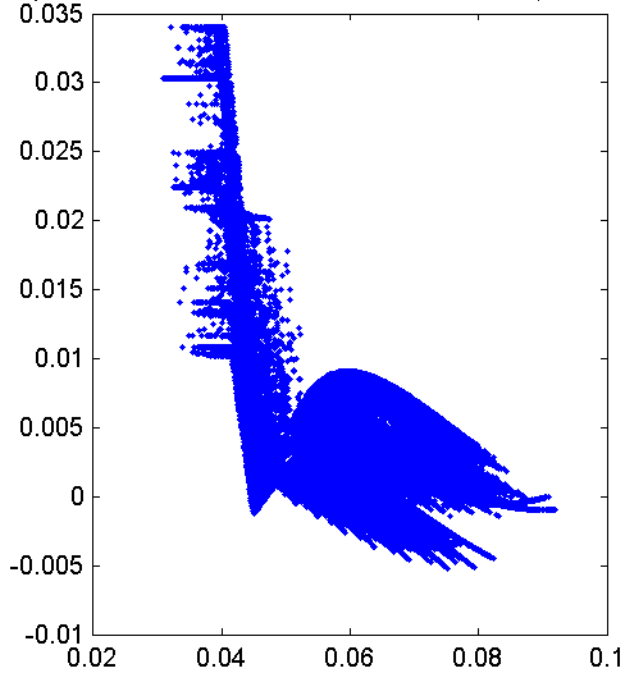
and the corresponding realizations for investment growth and inflation (in logarithm)

$$\begin{bmatrix} -0.0411 \\ -0.0411 \\ 0.1121 \\ 0.1121 \end{bmatrix}, \begin{bmatrix} 0.0131 \\ 0.0522 \\ 0.0205 \\ 0.0595 \end{bmatrix},$$

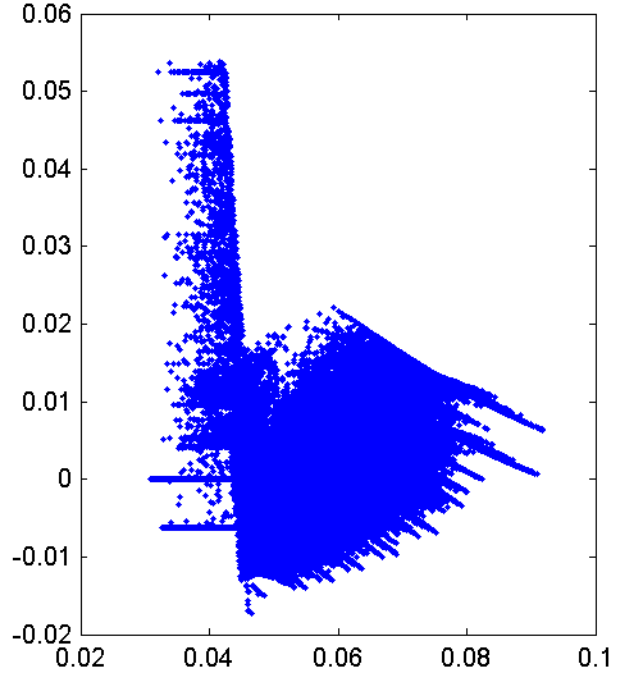
respectively.

Figure 1

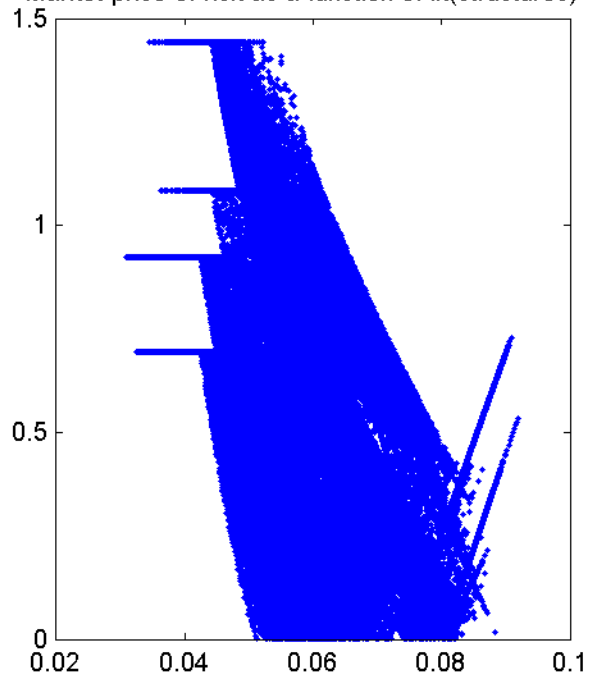
Expected excess return, rx_2 , as a function of $ik(\text{structures})$



Forward premium, fp_2 , as a function of $ik(\text{structures})$



Market price of risk as a function of $ik(\text{structures})$



Equity premium as a function of $ik(\text{structures})$

