Not Only What But also When – A Theory of Dynamic Voluntary Disclosure

PRELIMINARY AND INCOMPLETE

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September 2011

1 Introduction

In this paper, we study a dynamic model of voluntary disclosure of multiple news. Corporate voluntary disclosure is one of the major sources of information in capital markets. The extant theoretical literature on voluntary disclosure focuses on static models in which an interested party (e.g., a firm) may privately observe a single piece of private information (e.g., Dye 1985 and Jung and Kwon 1988). Corporate disclosure environments, however, are characterized by multi-period and multi-dimensional flow of information from the firm to the market. The interaction between these two dimensions plays a critical role. When deciding whether to disclose one piece of information a manager must also consider the possibility of learning and potentially disclosing a new piece of information in the future.

To better understand the dynamic interaction between firms and the capital market, we extend Dye's (1985) voluntary disclosure model with uncertainty about information endowment to a two-period and two-signal setting. Our model demonstrates how dynamic considerations shape the strategy of a privately informed agent and the market reactions to what he releases and when. Our setting is such that absent information asymmetry, the firm's price at the end of the second period is independent of the disclosure time of the firm's private information. Nevertheless, our model shows that in equilibrium, the market price depends not only on what information has been disclosed so far, but also on when it was disclosed. In particular, we show that the price at the end of the second period given disclosure of one signal is higher if the signal is disclosed later in the game. This result might be counter intuitive, as one may expect the market to reward the manager for early disclosure of information, since then he seems less likely to be "hiding something."

The intuition for our finding that the price at the end of the second period given disclosure of one signal is higher if the signal is disclosed later in the game is as follows. Let time be $t \in \{0, 1\}$ and suppose it is now $t = 1$. Consider two histories on the equilibrium-path: in both the manager disclosed a single signal x, but in history 1 he disclosed x at $t = 0$ while in history 2 he did that at $t = 1$. The market price depends on x and on what the market believes about the second signal given the history. Let y denote that second signal. The market considers three possibilities: the manager does not know y, or he does know y and learned it at $t = 1$ or at $t = 0$. Obviously, if the manager did not learn y the market's inference is independent of the observed history. In case the agent learned y at $t = 1$, the market's inference is also independent of the observed history because in both cases he would reveal y if and only if it would increase the market price at $t = 1$ (relative to non-disclosure of y). But what if the agent knew y at $t = 0$? If x is disclosed at $t = 0$ then the market can infer that y is less than x and is small enough that revealing it would not increase the price at $t = 1$. If x is disclosed at $t = 1$, the market additionally infers that in case he knew y already at $t = 0$ but learned x only at $t = 1$, then y is lower than the threshold for disclosure of a single signal at $t = 0$.[1]

On the face of it, one might expect that this additional negative inference about y if x is disclosed later, should lead to more negative beliefs about y. But the opposite is true in equilibrium! Why? There are two effects that affect investors' beliefs in opposite directions. On one hand, the lower threshold implies that the expected y conditional on the agent knowing y is lower. On the other hand, the lower threshold implies that it is less likely that the manger is informed, which increases the expected value of y. This second effect always dominates! The reason is that the agent still discloses at $t = 1$ signals y that exceed the market perception at this time. Hence, in case of the history with late disclosure, investors additionally rule out any y that is above the disclosure threshold at $t = 0$ but below the threshold for disclosure at $t = 1$. Since the disclosure threshold at $t = 1$ equals the average y for all agents' types who do not report at $t = 1$ (including the informed and uninformed), ruling out these types which are lower than the overall average y increases the expectation of y and hence the market price.

To further characterize strategic behavior and market inferences in our model, in Section 4 we characterize threshold equilibria. We show that under suitable conditions a threshold equilibrium exists.[2] We then characterize the threshold equilibrium strategies and the properties of the corresponding equilibrium prices.

1.1 Related Literature

The voluntary disclosure literature goes back to Grossman and Hart (1980), Grossman (1981) and Milgrom (1981), who established the "unraveling result", stating that under certain assumptions (including common knowledge that an agent is privately informed, disclosing is costless and information is verifiable), in equilibrium all types disclose their information. In light of companies

[1] For this simple intuition, we assume that the agent follows a threshold strategy at $t = 0$. Our proof does not make this assumption.

[2] In most of the existing voluntary disclosure literature (e.g., Verrecchia 1983, Dye 1985, Acharya et al. 2011), the equilibrium is unique and is characterized by a threshold strategy. In our model, due to multiple signals, it is not guaranteed, and therefore we provide sufficient conditions for existence (similar to Pae 2005).

propensity to withhold some private information, the literature on voluntary disclosure evolved around settings in which the unraveling result does not prevail. The two major streams of literature where: (*i*) assuming that disclosure is costly (pioneered by Jovanovic 1982 and Verrecchia 1983) and (*ii*) investors' uncertainty about information endowment (pioneered by Dye 1985 and Jung and Kwon 1988). Our model follows Dye (1985) and Jung and Kwon (1988) and extends it to a multi-signal and multi-period setting.

In spite of the vast literature that models voluntary disclosure, very little has been done on multi-period settings and on multi-signals settings. Corporate disclosure environments however, are characterized by multi-period and multi-dimensional flow of information from the firm to the market.[3]

To the best of our knowledge the only papers that study multi-periods voluntary disclosures are Shin (2003 and 2006), who discusses how his single disclosure period setting can be extended to multiple disclosure periods, Einhorn and Ziv (2008) and Beyer and Dye (2011). The setting studied in these papers as well as the dynamic considerations of the agents are very different from ours. Shin (2003, 2006) studies a setting in which the firms may learn a binary signal for each of their projects that may either fail or succeed. In this binary setting, Shin (2003, 2006) studies the "sanitization" strategy, under which the agent discloses only the good news. Einhorn and Ziv (2008) study a setting in which in each period the manager may obtain a single signal about the period's cash flows, where at the end of each period the realized cash flows are publicly disclosed. If the agent chooses to disclose his private signal, he incurs some disclosure costs. Finally, Beyer and Dye (2011) study a reputation model in which the manager may learn a single private signal in each of two periods. The manger can be either "forthcoming" and disclose any information he learns or he may be "strategic." At the end of each period, the firm's signal/cash flow for the period becomes public and the market updates beliefs about the value of the firm and the type of the agent. Importantly, the option to "wait for a better signal" that is behind our main result is not present in any of these papers.

Our paper also adds to the understanding of management's decision to selectively disclose information. Most voluntary disclosure model assume a single signal setting, in which the manager can either disclose all of his information or not disclose at all. The only exceptions that we are aware of, in which agents may learn multiple-signals are Shin (2003, 2006), which we discussed

[3] For example, this gap is pointed out in a survey by Hirst, Koonce and Venkataraman (2008), who write "much of the prior research ignores the iterative nature of management earnings forecasts.""

above, and Pae (2005). The latter considers a single period setting in which the agent can learn up to two signals. We add to his model dynamic considerations, which are again crucial for creating the option value to wait for a better signal.

2 Model setup

We study a two period setting, $t \in \{0, 1\}$, in which an agent (the manager) may receive private signals about his firm's value (his type). The value of the firm, V, is the realization of a normally distributed random variable and without loss of generality we assume that $\tilde{V} \sim N(0, \sigma^2)$. The manager might obtain up to two private signals of the form $\tilde{S}_i = \tilde{V} + \tilde{\varepsilon}_i$ where ε_1 and ε_2 are independent of $\tilde{V}$ and of each other and $\varepsilon_1, \varepsilon_2 \sim N(0, \sigma_\varepsilon^2)$. The probability of obtaining a signal $\tilde{S}_i$ at a given period (given that the signal has not yet been received) is independent of whether the other signal was observed and the realizations of signals. We denote this probability by p. In each period, the manager can publicly disclose all or part of the signals he obtained. We follow Dye (1985) and assume that an uninformed manager can not credibly convey the fact that he did not obtain a signal. Any disclosure is assumed to be truthful and does not impose direct cost on the manager or the firm. The manager's objective is to maximize a weighted average of the firm's price over the two periods. For simplicity and without loss of generality we assume that the manager weighs the prices equally across the two periods. In each period, based on the publicly available information investors set the firm's price to equal its expected value. The publicly available information at time t includes the information that was disclosed and when it was disclosed. We further denote by x the first signal that is disclosed, the time at which x is being disclosed we denote by t_x and the time at which the signal x was observed by the manager we denote by τ_x. We denote the other signal that the manager might have received by y and the time at which it was obtained by the manager by τ_y. We let

$$h(x, t_x, t)$$

denote investors' expectation at time t of the signal y conditional on the fact that only x was disclosed until period t and it was disclosed at time t_x. The structure of the game and all parameters of the model are common knowledge.

From the properties of the joint normal distribution it follows that the conditional expectation of the firm value given that the manager obtained a single signal is given by:

$$E(\tilde{V}|S_1 = s_1) = \beta_1 s_1$$

where $\beta_1 = \frac{\sigma^2}{\sigma^2 + \sigma_\varepsilon^2}$. Note that also $E(\tilde{S}_2|S_1 = s_1) = \beta_1 s_1$.

Following disclosure of two signals the conditional expectation of the firm value is given by

$$E(\tilde{V}|S_1 = s_1, S_2 = s_2) = \beta_2(s_1 + s_2)$$

where $\beta_2 = \frac{\sigma^2}{2\sigma^2 + \sigma_\varepsilon^2}$. Finally, the expectation of the firm value given disclosure of a single signal, x, at $t = t_x$ as calculated at the end of period t is given by

$$\beta_2(x + h(x, t_x, t))$$

Note that $\beta_2 < \beta_1 < 2\beta_2 < 1$ and $\beta_2(1 + \beta_1) = \beta_1$.

Figure 1 summarizes the sequence of events in the model.

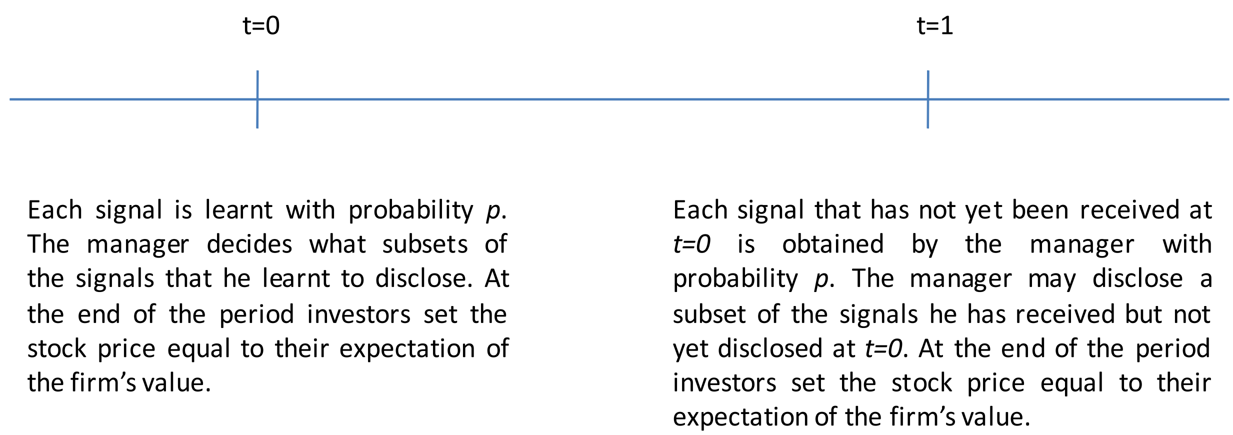

Figure 1: Timeline

3 Properties of any Equilibrium

Multiple equilibria are common in signaling models. In section 4, we identify and analyze a specific class of equilibria based on threshold strategies. In this section, we show that our main result, that the inference about the firm's value at a given point in time depends not only on what information has been disclosed so far but also on when it was disclosed, holds in any equilibrium. In particular, we show that the price at the end of the second period is higher when the manager disclosed a single signal, x, at $t=1$ and none at $t=0$ than when the manager discloses a single signal, x, at $t=0$ and non at $t=1$.

At $t=1$ the number of signals that were disclosed can be zero, one or two. To demonstrate our main result, the only relevant case is when exactly one signal is being disclosed. If both signals were disclosed there is no information asymmetry, so the price is independent of when the signals were disclosed. If no signal was disclosed we cannot condition on the time of disclosure.

We consider two possible equilibrium scenarios. In the first scenario, a signal x is disclosed at $t=0$ and in the other it is disclosed at $t=1$. In both cases, this is the only signal that the agent discloses. At $t=1$, the market sets a price of $\beta_2(x+h(x,0,1))$ in the first scenario and $\beta_2(x+h(x,1,1))$ in the second one. We argue that $h(x,1,1) \geq h(x,0,1)$ so that later disclosure receives a better interpretation. That is, investors' valuation of the firm is higher if the manager discloses x at $t=1$ rather than at $t=0$.

Before proving this claim, we first note that in both scenarios (given disclosure of x at either $t=0$ or $t=1$) the market cannot perfectly tell whether the agent learnt a second signal y or not. To see why, suppose instead that an agent who only knows x never discloses it. Following the disclosure of only x it would become commonly known that the agent hides the second signal. The usual unraveling argument then implies that in equilibrium the agent would disclose also the second signal, y. This precludes a disclosure of a single signal being part of the equilibrium. If instead only an agent who learnt a single signal discloses it, it becomes commonly known that the agent does not know the other signal. In this case, agents with a low signal y would reveal x but not y - contrary to the assumption.

We refer to agents who by the end of $t=1$ know also the second signal y as '*informed*' and those who have not learned y as '*uninformed*'. Formally, the set of informed is given by $\{\tau_y = 0, 1\}$ and those who are *uninformed* by $\{\tau_y > 1\}$. It is useful to define a subset of *informed* agents which we refer to as '*potential disclosers*'. These are informed agents who have not disclosed y at $t=0$ and therefore may potentially disclose y at $t=1$. In some cases we need also to exclude types that would have disclosed y rather than x. This occurs for example, if x is disclosed at $t=0$, when we can rule out the possibility that the agent knew both signals at $t=0$ and $y \in D_0(y|x)$, where $D_0(y|x)$ is the set of signals y that would have been disclosed in $t=0$ if the agent learned both y and x at $t=0$ (this includes y that would have been disclosed either with or without disclosure of x). We denote the set of uniformed agents at $t=1$ by A, and the set of *potential disclosers* by B.

Specifically, we let B_0 denote the set of potential disclosers at $t = 1$ when x is disclosed at $t = 0$ and B_1 the set of potential disclosers for the scenario in which x is disclosed at $t = 1$.

$$\begin{aligned} B_0 = &\{\text{Informed agents who have disclosed only } x \text{ at } t = 0\} \\ &\setminus \left\{ y \in D_0(y|x), \tau_y = 0, \tau_x = 0 \right\} \\ B_1 = &\{\text{Informed agents who have disclosed only } x \text{ at } t = 1\} \\ &\setminus \left\{ \{y > x, \tau_y = 1\} \cup \left\{ y \in D_0(y|x), \tau_y = 0, \tau_x = 0 \right\} \cup \left\{ y \in D_0(y), \tau_y = 0, \tau_x = 1 \right\} \right\} \end{aligned}$$

where $D_0(y)$ is the set of agents with $\tau_y = 0, \tau_x = 1$ that would have disclosed their signal y at $t = 0$.

At $t = 1$, given that they have disclosed x, potential disclosers are myopic in deciding whether to disclose the second signal, y. They will disclose y if it is higher than the market perception about y, which is given by $h(x, \cdot, 1)$. The myopic disclosure policy can be characterized by a set and a distribution over that set, which we denote by $S^f_{A,B}$. Formally, for arbitrary sets A and B, with distributions f^A and f^B respectively, we define $S^f_{A,B}$ as:

$$S^f_{A,B} = A \cup \{B \cap \{(y, \tau_y) : y \leq E_y(S^f_{A,B})\}. \tag{1}$$

Or equivalently:

$$S^f_{A,B} = A \cup B \setminus \{(y, \tau_y) \in B : y > E_y(S^f_{A,B})\}.$$

where $E_y(S^f_{A,B})$ is the expectation of y when calculated over the union of the set A and the set $\{B \cap \{(y, \tau_y) : y \leq E_y(S^f_{A,B})\}$ such that the distribution f assigns a weight to each of the sets and its corresponding distribution according to the relative likelihood of y belonging to each set. To gain better intuition for the definition of $S^f_{A,B}$ consider a Dye (1985) setting in which an agent, whose type y is distributed according to a standard normal distribution, may learn his type with probability p. When learnings his type, the agent needs to decide whether to disclose it. The agent's objective is to maximize investors' beliefs about his type. Both the set of uninformed agents, A^{Dye}, and the set of potential disclosers (informed agents), B^{Dye}, consists of all the real numbers where the distribution over both sets is standard normal. The set $\{B^{Dye} \cap \{(y, \tau_y) : y \leq E_y(S^f_{A,B})\}$ is all values of y such that $y < E_y\left(S^f_{A,B}\right)$. Denoting by $\phi(\cdot)$ and $\Phi(\cdot)$ respectively the pdf and cdf of standard normal distribution, we have $E_y\left(S^f_{A,B}\right) = \frac{(1-p)\int_{-\infty}^{\infty} z\phi(z)dz + p\int_{-\infty}^{E_y\left(S^f_{A,B}\right)} z\phi(z)dz}{(1-p) + p\Phi\left(E_y\left(S^f_{A,B}\right)\right)}$. In such a Dye (1985) setting, investors' beliefs given no disclosure, $E_y\left(S^f_{A,B}\right)$, equal the disclosure threshold of an informed agent.

The definition of $S^f_{A,B}$ is an implicit definition that relies on the existence and uniqueness of a fixed point. We verify this in the following Lemma.

Lemma 1 *Equation* 1 *has a unique solution* $S^f_{A,B}$.

Proof. See appendix ■

Using the definitions of $S^f_{A,B}$ and the sets B_0 and B_1, we can express $h(x,0,1)$ and $h(x,1,1)$ in terms of S^f_{A,B_0} and S^f_{A,B_1} as follows:

$$\begin{aligned} h(x,0,1) &= E(y|y \in S^f_{A,B_0}) \equiv E_y(S^f_{A,B_0}) \\ h(x,1,1) &= E(y|y \in S^f_{A,B_1}) \equiv E_y(S^f_{A,B_1}) \end{aligned} \tag{2}$$

In the following, to simplify notation and readability, we will abuse the notation and omit the reference to the distribution of the sets A, B and S^f_{A,B_1}. That is, for any two sets A, B and their respective distributions we will denote $S^f_{A,B}$ by $S_{A,B}$ and the expectation of the union of these sets given the distributions over the sets by $E_y(A \cup B)$.

A key argument that we will use is the following extension of the minimum principle that appeared first in Acharya, DeMarzo and Kremer (2011):[4]

Lemma 2 *Generalized Minimum Principle*

For any two sets A and B (and their respective distributions) we have:

(i) $E_y(A \cup B) \geq E_y(S_{A,B})$[5]

(ii) $B' \supseteq B" \Rightarrow E_y(S_A, B") \geq E_y(S_A, B')$

(iii) Suppose that $B' \supseteq B"$ *and every* $y \in B' \backslash B"$ *satisfies* $y > E_y(S_A, B")$*. Then* $S_A, B" = S_A, B'$

While we use this Lemma in a specific context it is important to note that the above Lemma holds for arbitrary sets A and B and distributions over A and B. We provide a formal proof in the appendix but the logic can be demonstrated through three simple examples:

Examples: In all three examples, we consider two disjoint sets. Each element in a set is multi-dimensional, but we are interested only in one dimension of the element - call it the value of y. Suppose that for set A y is uniformly distributed on $[0,1]$, i.e., $y_A \sim U[0,1]$.

1. Suppose that B is such that $y_B \sim U[0,1]$. In this case $E_y(A \cup B) = 0.5$ while $E_y(S_{A,B}) < 0.5$ because $S_{A,B}$ is defined as $A \cup B$ excluding some high types in B.

[4] They established point (i) of the Lemma below.

[5] $E_y(A \cup B) = E_y(S_{A,B})$ if and only if $E_y(S_{A,B})$ is greater than the highest element in B.

2. Suppose that $B'' = \emptyset$ and B is as defined in example 1 above. In this case we have $S_{A,B''} = A$ and $E_y(S_{A,B''}) = 0.5 > E_y(S_{A,B})$

3. Suppose that $B'' = \emptyset$ but B' is a unit mass distributed uniformly on $[0.5, 1]$. In this case we have $S_{A,B"} = S_{A,B'} = A$.

Based on Lemma 2, we argue that the interpretation of a disclosed signal when there was no new disclosure deteriorates over time. The following Lemma formalizes it.

Lemma 3 $h(x, 0, 1) \leq h(x, 0, 0)$.

Proof Consistent with our notation earlier, let the set A denote the set of uniformed agents who disclosed x at $t = 0$ and did not learn y by the end of $t = 1$. Let B_0 denote the set of agents who disclosed x at $t = 0$ and become informed of y by the end of $t = 1$. The claim follows from Lemma 2 as $h(x, 0, 0) = E_y(A \cup B_0)$ and $h(x, 0, 1) = E_y(S_{A,B_0})$. **QED**

The above Lemma implies the following Corollary.

Corollary 1 *A manager that has disclosed x at $t = 0$ is myopic with respect to the decision to release y. That is, conditional on disclosing x at $t = 0$ the manager reveals also y at $t = 0$ if and only if $y > h(x, 0, 0)$.*

We now turn to our main result.

Theorem 1 $h(x, 1, 1) \geq h(x, 0, 1)$

We provide the proof of the Theorem in the Appendix but describe here our strategy to prove the Theorem. We prove this by way of contradiction. A simple way would have been to argue that if $h(x, 0, 1) > h(x, 1, 1)$ then $B_0 \supset B_1$ which based on equation (2) and part (*ii*) of Lemma 2 leads to a contradiction. Since we do not rely on the equilibrium structure (e.g., a threshold strategy) we use a slightly more involved argument. We assume by contradiction that $h(x, 0, 1) > h(x, 1, 1)$ and find a set $\widehat{B}_0 \supset B_1$ such that also $\widehat{B}_0 \supseteq B_0$ and $\forall y \in \widehat{B}_0 \backslash B_0$ we have $y > E_y(S_{A,B_0}) = h(x, 0, 1)$. Then, based on part (*iii*) of Lemma 2 we obtain a contradiction.

4 A Threshold Equilibrium

The objective of this section is to demonstrate the existence of a threshold equilibrium under suitable conditions. In a static model with a single signal such a result would be trivial since the payoff upon disclosure is increasing in the manager's type while the payoff upon non disclosure is fixed. Hence, if a given type chooses to disclose his type so would a higher type. This simple argument is not applicable in our dynamic setting. The reason is that an agent's expected payoff upon non-disclosure in the first period also increases in his type. Moreover, the relation between the expected payoff of an agent that discloses a signal in the first period and his type is not straight forward. This complicates the analysis and requires few interim steps before establishing existence of a threshold equilibrium. Our proof strategy is to first derive prices that would occur if the market believes that the agent follows a threshold strategy. For these prices, we then show that under suitable conditions the agent's expected payoff upon disclosure in $t = 0$ is increasing faster in his type, x, as compared to his expected payoff upon non-disclosure in $t = 0$. Therefore, given these prices the agent's best response would indeed be to follow a threshold strategy.

We define a threshold strategy in our dynamic setting with two signals in the following way.

Definition 1 *Denote the information set of an agent by* $\{s_1', s_2\}$ *where* $s_i \in \{\mathcal{R}, \emptyset\}$ *and* $s_i = \emptyset$ *implies that the agent has not learnt this signal yet. We say that the equilibrium is a threshold equilibrium if an agent with information set* $\{s_1, s_2\}$ *who discloses* s_1 *at* $t \in \{0, 1\}$ *discloses any* $s_1' > s_1$ *when his information set is* $\{s_1', s_2\}$.[6]

Since the equilibrium reporting strategy in $t = 1$ is always a threshold strategy as defined above, we will focus on an informed agent's disclosure decision at $t = 0$. We first assume (and later confirm) that there exists a threshold equilibrium in which an agent that learns a signal x at $t = 0$ discloses it at $t = 0$ if and only if $x > x^*$. It proves convenient to partition the set of agents that learn at $t = 0$ a signal $x \geq x^*$ into the following three subsets: (i) agents that learn only x at $t = 0$, (ii) agents that learn both signals at $t = 0$ but the signal y (where $y < x$) is sufficiently high such that if the agent doesn't disclose y at $t = 0$ he will disclose y at $t = 1$, and (iii) agents that learn both signals at $t = 0$ but the signal y is sufficiently low such that if the agent doesn't disclose y at $t = 0$ he will not disclose y at $t = 1$.

[6] While we do not know whether a non-threshold equilibrium exists, one can show that it is always the case that the equilibrium reporting strategy of the second period is a threshold strategy as defined above.

We start by discussing agents in subset (i). If such an agent discloses x at $t=0$ he will disclose y at $t=1$ only if $y \geq h(x,0,1)$ (in case he learns y). On the other hand, if the agent does not disclose at $t=0$ he benefits from two "real options." The first option value will be realized if he learns at $t=1$ a sufficiently high value of y such that at $t=1$ he will disclose only y and conceal x (for $y > y^H(x)$). This increases his payoff at $t=1$ relative to the case in which he discloses x at $t=0$. The second option value will be realized if the manager does not learn y at $t=1$ or if he learns a sufficiently low y ($y < h(x,1,1)$) such that he does not disclose it. In this case, since $h(x,1,1) > h(x,0,1)$ (see Theorem 1) the manager's payoff at $t=1$ is higher than his payoff would have been had he disclosed x at $t=0$. In order for a partially informed agent to disclose x at $t=0$ the expected value of his two real options should be (weakly) lower than the decrease in the price at $t=0$ relative to the price given disclosure of x at $t=0$. This implies that $h(0) < \beta_2(x + h(x,0,0))$.

Formally, if the agent decides to disclose x at $t=0$ his expected payoff is

$$E_{t=0}(U|\tau_x = 0, \tau_y \neq 0, t_x = 0, x) = \beta_2(x + h(x,0,0)) + E_y[\max\{\beta_2(x + h(x,0,1)), \beta_2(x+y)\}|x]$$
$$= \beta_2(x + h(x,0,0)) + (1-p)\beta_2(x + h(x,0,1))$$
$$+ p\beta_2\left[\left(x + \int_{-\infty}^{h(x,0,1)} h(x,0,1) f(y|x)\,dy\right) + \int_{h(x,0,1)}^{\infty} (x+y) f(y|x)\,dy\right].$$

If he withholds information at $t=0$ his expected payoff is[7]

$$E_{t=0}(U|\tau_x = 0, \tau_y \neq 0, t_x \neq 0, x) = h(0) + E_y[\max\{\beta_2(x + h(x,1,1)), \beta_2(x+y), \beta_2(y + h(y,1,1))\}|x]$$
$$= h(0) + (1-p)\beta_2(x + h(x,1,1))$$
$$+ p\beta_2\left[\int_{-\infty}^{h(x,1,1)} (x + h(x,1,1) f(y|x)\,dy) + \int_{h(x,1,1)}^{y^H(x)} (x+y) f(y|x)\,dy + \int_{y^H(x)}^{\infty} (y + h(y,1,1) f(y|x)\,dy)\right].$$

Such an agent prefers to disclose x at $t=0$ over not disclosing it if

$$E_{t=0}(U|\tau_x = 0, \tau_y \neq 0, t_x = 0, x) \geq E_{t=0}(U|\tau_x = 0, \tau_y \neq 0, t_x \neq 0, x). \tag{3}$$

Next, we consider agents in subset (ii). Such agents, whose second signal $y > h(x,1,1)$, will disclose y at $t=1$ regardless of whether they did or did not disclose x at $t=0$. Therefore, such agents will not benefit from any of the real options that agents in subset (i) might benefit.

Finally, agents in subset (iii), whose second signal $y < h(x,1,1)$, will not disclose y at $t=1$. Such agents will always benefit at $t=1$ from the fact that $h(x,1,1) > h(x,0,1)$. So they trade off

[7] The agent considers $E_y[\max\{\beta_2(x + h(x,1,1)), \beta_2(x+y), \beta_2(y + h(y,1,1)), h(1)\}|x]$ where $h(1)$ is the price at $t=1$ when the agent hasn't made any disclosure. However, for any $x > x^*$ an agent that did not disclose at $t=0$ is better off disclosing x at $t=1$ over not disclosing at all. Therefor, we omit $h(1)$ in the agent's expected utility.

higher price at $t=0$ against lower price at $t=1$. Such an agent will disclose x at $t=0$ if

$$\beta_2\left(x+h\left(x,0,0\right)\right)+\beta_2\left(x+h\left(x,0,1\right)\right)\geq h\left(0\right)+\beta_2\left(x+h\left(x,1,1\right)\right). \tag{4}$$

Later in the paper, we demonstrate some characteristics of prices that always hold under a threshold disclosure strategy. As mentioned at the beginning of this section, the major challenge in proving existence of a threshold equilibrium is to show that for an agent that learns only x at $t=0$ the expected payoff from disclosing x at $t=0$ is increasing in x faster than his expected payoff from not disclosing x at $t=0$. That is showing that $LHS-RHS$ of inequality 3 is increasing in x. It turns out that given the above mentioned characteristics of prices that always hold, a sufficient condition for $LHS-RHS$ of inequality 3 to increase in x is that $LHS-RHS$ of inequality 4 is increasing in x, i.e., that

$$\frac{\partial}{\partial x}h\left(x,0,0\right)+\frac{\partial}{\partial x}h\left(x,0,1\right)\geq\frac{\partial}{\partial x}h\left(x,1,1\right)-1.$$

Since there are few steps we need to take prior to proving existence of a threshold equilibrium, we outline the structure of the remainder of this section. In order to characterize the prevailing prices in our dynamic setting if the market believes that the agent follows a threshold strategy it is useful to study a variant of a Dye (1985) setting. In Section 4.1, we study a variant of Dye (1985) in which the disclosure threshold of the agent is determined exogenously and is stochastic. Equipped with the insights from the variant static model, we characterize in section 4.2 the prices that would occur if the market believes that the agent follows a threshold strategy. The characteristics of the prices derived in section 4.2 set the ground for Section 4.3 where we establish the existence of a threshold equilibrium under suitable conditions. Finally, in Section 4.4 we further characterize the equilibrium.

4.1 A Variant of a Static Model

We briefly present and discuss few properties of a static voluntary disclosure setting similar to Dye (1985) and Jung and Kwon (1988). These properties will be later used in characterizing prices in the dynamic setting and in proving existence of a threshold equilibrium. Assume that an agent's type (his firm's value) is the realization of $\tilde{s}\sim N(\mu,\sigma^2)$ and with probability p the agent learns this value. If the agent learns the realization of $\tilde{s}$ he may choose to disclose it. We are interested in investors' beliefs about the firm value given no disclosure by the agent. In particular, we assume that the agent's strategy is to disclose s if and only if $s \geq z$, where z is some exogenously determined disclosure threshold. Note that unlike Dye (1985) and Jung and Kwon (1988), we are not looking at an equilibrium strategy, but rather on some exogenously determined disclosure threshold strategy. We will refer to this setting as a *Dye setting with exogenous disclosure threshold*. Let's denote the investors' expectation of the firm value given no disclosure and given the agent's disclosure threshold is z by $h^{stat}(\mu, z)$. It is easy to see that:

1. $h^{stat}(\mu + \Delta, z + \Delta) = h^{stat}(\mu, z) + \Delta$ for any constant Δ; this implies that $h_1^{stat}(\mu, z) + h_2^{stat}(\mu, z) = 1$.

2. The minimum principle we discussed in section 3 implies that $h^{stat}(\mu, z)$ satisfies $z^* = \arg\min_z h^{stat}(\mu, z) \Longleftrightarrow z^* = h^{stat}(\mu, z^*)$. This implies that the equilibrium disclosure threshold in standard Dye (1985) and Jung and Kwon (1988) minimizes $h^{stat}(\mu, z)$.

Figure 2 plots $h^{stat}(\mu, z)$ for standard normal distribution with $p = 0.5$.

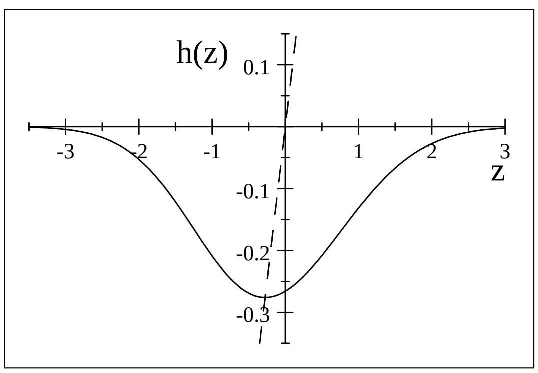

Figure 2: Price Given No-Disclosure in a Dye Setting with Exogenous Disclosure Threshold z

Note that for all $z < h^{stat}(\mu, z)$ ($z > h^{stat}(\mu, z)$) the price given no disclosure, $h^{stat}(\mu, z)$, is decreasing (increasing) in z. Further analysis shows that for $p < 0.95$ the slope of $h^{stat}(\mu, z)$ with respect to z is always higher than -1. We will later use this lower bound of the slope.

For the analysis of our dynamic model it will prove useful to consider a variant of this model. The variant is still a static model but the threshold for disclosure depends on μ and the agent follows a random disclosure policy. In particular, with probability λ_i, $i \in \{1, .., K\}$, where $\sum_{i=1}^{K} \lambda_i = p$, the agent discloses only if his type is above $z_i(\mu)$. The reasons we consider a disclosure threshold that depends on μ is that in our dynamic setting investors update their beliefs about the undisclosed signal based on the value of the disclosed signal. The reason we consider a random disclosure policy is as follows. In our dynamic setting, when by $t = 1$ the agent disclosed a single signal investors do not know whether the agent learnt a second signal and if so, whether he learnt it at $t = 0$ or at $t = 1$. Since the agent follows different disclosure thresholds at the two possible dates investors' beliefs about the agent's disclosure threshold are stochastic.

We next analyze some properties of the static setting with random threshold disclosure policy. Let us denote by $h^{stat}(\mu, \{z_i(\cdot)\})$ the conditional expectation of the type given the disclosure thresholds, $z_i(\cdot)$.

Lemma 4 *Suppose that: (i) for all i* $z_i(\mu) \leq h^{stat}(\mu, \{z_i(\cdot)\})$, *(ii)* $z_i'(\mu) \in [0, c]$ *and (iii)* $p \leq 0.95$. *Then* $\frac{d}{d\mu} h^{stat}(\mu, \{z_i(\cdot)\}) \in (1 - c, 2)$

The intuition for the random case, in which $K > 1$, is a little complicated and therefore we defer it to the Appendix where we formally prove the Lemma. In order to provide the basic intuition for the result, we analyze the particular case in which the disclosure strategy is non-random, i.e., $K = 1$. We start by providing the two simplest examples, for the cases where $z'(\mu) = 1$ and $z_i'(\mu) = 0$. These examples are useful in demonstrating the basic logic and how it can be analyzed using Figure 2. These two examples also provide most of the intuition for the case with no restriction on $z_i'(\mu)$, which is presented in example 3.

Examples:

1. Assume that $z'(\mu) = 1$ (and $K = 1$). From point 1 above we know that $\frac{d}{d\mu} h^{stat}(\mu, z(\mu)) = \frac{\partial}{\partial\mu} h^{stat}(\mu, \{z(\mu)\}) + z'(\mu) * \frac{\partial}{\partial z} h^{stat}(\mu, \{z(\mu)\}) = 1$. The intuition can be demonstrated using figure 2. A unit increase in μ (keeping $z_i(\cdot)$ constant) shifts the entire graph both upwards and right by one unit. However, since also $z_i(\cdot)$ increases by a unit, the overall effect is an increase in $h^{stat}(\mu, \{z(\mu)\})$ by one unit.

2. Assume that $z_i'(\mu) = 0$ and the agent discloses his signal if it is higher than z^*, i.e., $z_i(\mu) = z^*$. From point 1 above we know that $\frac{\partial}{\partial\mu} h^{stat}(\mu, z^*) + \frac{\partial}{\partial z^*} h^{stat}(\mu, z^*) = 1$ and therefore $\frac{\partial}{\partial\mu} h^{stat}(\mu, z^*) = 1 - \frac{\partial}{\partial z^*} h^{stat}(\mu, z^*)$. We also know that since $z^* < h^{stat}(\mu, \{z^*\})$ we have $\frac{\partial}{\partial z^*} h^{stat}(\mu, z^*) \in (-1, 0)$. Hence, we conclude that $\frac{\partial}{\partial\mu} h^{stat}(\mu, z^*) \in (1, 2)$. The intuition can be demonstrated using figure 2. The effect of a unit increase in μ can be presented as a sum of two effects: (i) a unit increase in the disclosure threshold z as well as a shift of the entire graph

both to the right and upwards by one unit and (ii) a unit decrease in the disclosure threshold z (as $z_i'(\mu) = 0$). The first effect is similar to the first example we had and therefore increases $h^{stat}(\mu, \{z_i(\cdot)\})$ by one. The second effect increases $h^{stat}(\mu, \{z_i(\cdot)\})$ by the absolute value of the slope of $h^{stat}(\mu, \{z_i(\cdot)\})$. So, in summary we have $\frac{\partial}{\partial \mu} h^{stat}(\mu, z^*) = 1 - \frac{\partial}{\partial z^*} h^{stat}(\mu, z^*)$. Moreover, $\frac{\partial}{\partial \mu} h^{stat}(\mu, z^*) \in [1, 2]$.

3. The general case for $K = 1$. Assume that $z'(\mu) = c$, which is a more general case that generalizes both of the examples above. Following a similar logic, we conclude that $\frac{d}{d\mu} h^{stat}(\mu, \{z(\cdot)\}) = \frac{\partial}{\partial \mu} h^{stat}(\mu, z(\cdot)) + c * \frac{\partial}{\partial z} h^{stat}(\mu, z(\cdot)) = 1 + (c-1) \frac{\partial}{\partial z} h^{stat}(\mu, z(\cdot))$.[8]

4.2 Prices Given a Threshold Disclosure Strategy

In this section, we assume the existence of a threshold equilibrium in which a manager who learns only x at $t = 0$ discloses it if and only if $x \geq x^*$. We will derive some characteristics of prices that are consistent with such disclosure strategy.

Recall two observations that we discussed earlier. First, at $t = 1$, the agent behaves myopically and his strategy at $t = 1$ follows a threshold strategy. Second, for any $x \geq x^*$ the price at $t = 0$ given no disclosure, $h(0)$, is lower than the price given disclosure of x. That is $h(0) < \beta_2 (x + h(x, 0, 0))$.

Another intuitive observation is that an agent who learns both signals at $t = 0$ ($\tau_x = \tau_y = 0$), where $y < x$, and discloses x at $t = 0$, behaves myopically with respect to the disclosure of his signal y. In particular, such a type discloses also y at $t = 0$ if and only if $y \geq h(x, 0, 0)$. This has been established in section 3, where we have also shown that for $y \in (h(x, 0, 1), h(x, 0, 0))$ the agent discloses y at $t = 1$ and if $y < h(x, 0, 1)$ he never discloses y. That is, conditional on disclosing x at $t = 0$ that agent is myopic in both periods with respect to the disclosure of y.

Next, we characterize the slopes of $h(x, 0, 0)$, $h(x, 1, 1)$ and $h(x, 0, 1)$. This will be useful later, when we show that the agent's expected payoff from disclosing his signal at $t = 0$ is increasing in his signal faster than his expected payoff from concealing his signal at $t = 0$.

Claim 1 *Suppose there exists a threshold equilibrium in which and agent that learns only x at $t = 0$ discloses it if and only if $x \geq x^*$. Then, the following are upper and lower bounds for the slopes of*

[8] Recall that $\frac{\partial}{\partial \mu} h^{stat}(\mu, z(\cdot)) = 1 - \frac{\partial}{\partial z} h^{stat}(\mu, z(\cdot))$.

$h(x,0,0)$, $h(x,1,1)$ *and* $h(x,0,1)$.

$$\frac{\partial}{\partial x}h(x,0,0)\begin{cases} =\beta_1 & if \quad h(x,0,0)<x \\ \in(2\beta_1-1,\beta_1) & if \quad h(x,0,0)>x \text{ (if such case exists)} \end{cases}$$
$$\frac{\partial}{\partial x}h(x,1,1)\begin{cases} =\beta_1 & if \quad h(x,1,1)<x^* \text{ (if such case exists)} \\ \in(2\beta_1-1,2\beta_1) & if \quad h(x,1,1)>x^* \end{cases}$$
$$\frac{\partial}{\partial x}h(x,0,1)\begin{cases} =\beta_1 & if \quad h(x,0,1)<x \\ \in(2\beta_1-1,\beta_1) & if \quad h(x,0,1)>x \text{ (if such case exists)} \end{cases}$$

Proof of Claim 1

We start by analyzing $h(x,0,0)$.

As we showed in Section 3, for any x that is disclosed at $t=0$ such that $h(x,0,0)<x$ (the non-binding case)[9], if $\tau_y=0$ the agent is myopic with respect to the disclosure of y and discloses it whenever $y\geq h(x,0,0)$. This makes the analysis of the effect of an increase in x on $h(x,0,0)$ qualitatively similar to the analysis of an increase in the mean of the distribution in a standard Dye (1985) and Jung and Kwon (1988) equilibrium, in which the increase in both equilibrium beliefs and disclosure threshold is identical to the increase in the mean. This case is captured by example 1 of Section 4.1. The quantitative difference in our dynamic setting is that a unit increase in x increases investors' beliefs about y by β_1 (rather than by 1) and therefore also increases the beliefs about y and the disclosure threshold by β_1. As a result, in our dynamic setting for $h(x,0,0)<x$ we have $h'(x,0,0)=\beta_1$.[10]

In the binding case, i.e., for all x such that $h(x,0,0)>x$ (if such $x>x^*$ exists) we know that if $\tau_y=0$ then $y<x$. This case is captured by example 3 of Section 4.1. In particular, an increase in x increases the beliefs about y at a rate of β_1 while the increase in the constraint/disclosure threshold ($y<x$) is at a rate of 1. Therefore, this is a particular case of example 3 of Section 4.1 in which we increase the mean by β_1 and $z'(\mu)=c=\frac{1}{\beta_1}$. From example 3 we know that an increase in the beliefs about y given a unit increase in x (which is equivalent to an increase of β_1 in μ in example 3) is given by $\beta_1\left(1+(c-1)\frac{\partial}{\partial z}h^{stat}(\mu,z(\cdot))\right)$. Substituting $c=\frac{1}{\beta_1}$ and rearranging terms yields

$$h'(x,0,0)=\beta_1+(1-\beta_1)\frac{\partial}{\partial z}h^{stat}(\mu,z(\cdot)).$$

Since $\frac{\partial}{\partial z}h^{stat}(\mu,z(\cdot))\in(-1,0)$ we have $\frac{\partial}{\partial x}h(x,0,0)\in[2\beta_1-1,\beta_1]$.

[9] We use the term non-binding to indicate that the constraint $y<x$ is not binding. The reason is that since $h(x,0,0)<x$ the constraint $y<h(x,0,0)$ also implies that $y<x$.

[10] Since both the beliefs about y and the disclosure threshold increase at the same rate, the probability that the agent learnt y at $t=0$ but did not disclose it is independent of x.

Since the analysis of how $h(x,1,1)$ and $h(x,0,1)$vary with x is more involved and more technical we defer it to the Appendix, where we prove the remainder of Claim 1. The reason these cases are more complicated is that when pricing the firm at $t=1$ investors do not know whether the agent learnt y at $t=0$ or at $t=1$ (in case the agent did in fact learn y). Investors' inference about y depends on when the agent learnt it, and therefore the analysis of $h(x,1,1)$ and $h(x,0,1)$ requires analysis of a stochastic disclosure threshold. Lemma 4 will be useful in conducting this analysis.

Given the characterization of a threshold equilibrium that we have developed so far, we are now ready to establish the existence of a threshold equilibrium.

4.3 Existence of a threshold equilibrium

Proposition 1 *For $\beta_1 > 0.5$ and $p < 0.95$ there exists a threshold equilibrium in which an agent that learns at $t=0$ only one signal, x, discloses it at $t=0$ if and only if $x \geq x^*$. If the agent learns two signals at $t=0$ and one of them is greater than x^* he makes a disclosure at $t=0$. In particular, he may choose to disclose at $t=0$ both signals or just the higher one. Disclosing a single signal $x < x^*$ at $t=0$ is not part of the equilibrium disclosure strategy.*[11]

Proof. The sketch of the proof is as follows. We show that if the highest signal learnt by an agent at $t=0$ is sufficiently high he will disclose it at $t=0$ (and if he learnt a second signal he sometimes discloses it as well). If his highest signal is sufficiently low the agent will not make a disclosure at $t=0$. Finally, using the properties of the slopes of the various prices that we derived, we show that the difference between the agent's expected payoff at $t=0$ from disclosing a signal and his expected payoff at $t=0$ from not disclosing the signal is increasing in the signal. This is formalized in the following lemma.

Lemma 5 (*a*) *For sufficiently high (low) realizations of x, an agent that learns a single signal, x, at $t=0$ ($\tau_x=0, \tau_y \neq 0$) discloses (does not disclose) x at $t=0$.*
(*b*) *For sufficiently high (low) realizations of x, an agent that learns both signals at $t=0$ ($\tau_x = \tau_y = 0$) and does not disclose y at $t=0$ discloses (does not disclose) x at $t=0$.*
(*c*) *On the equilibrium path, the difference between the agent's expected payoff if he discloses x at $t=0$ and if he does not disclose at $t=0$ is increasing in x.*

[11] We believe that the threshold equilibrium exists for a wider set of (all) parameters, however, for tractability reasons we restrict the set of parameters for which we show the existence of a threshold equilibrium.

■

The proof of the Lemma is in the appendix.

4.4 Further characterization of the equilibrium

While we have already established the existence of a threshold equilibrium under suitable conditions, we have not yet discussed how x^*, the disclosure threshold at $t = 0$ for an agent that learnt a single signal, is determined. We complete this analysis below.

In most signaling models, and in particular voluntary disclosure models, the agent's private information consists of a single signal. In such settings, the disclosure threshold equals the signal for which the manager is indifferent between disclosing and not disclosing his signal. In our richer setting, the difference between types is multidimensional, and therefore the simple indifference condition used in the standard models does not apply. We next discuss how the disclosure threshold is determined.

At the beginning of Section 4, when discussing the manager's trade-offs we partitioned the set of agents that make a disclosure at $t = 0$ into subsets $(i) - (iii)$. We will use the same partition in order to describe how the threshold for disclosure of a single signal at $t = 0$ is determined.

For a given x if an agent in subset (iii) prefers to disclose x at $t = 0$ then it is easy to see that every agent in subset (ii) strictly prefers to disclose x at $t = 0$. It is not easy, however, to determine whether the fact that an agent in subset (iii) prefers to disclose x at $t = 0$ over not disclosing it implies that also a type in subset (i) prefers disclosure of x at $t = 0$ over non-disclosure at $t = 0$. The reason is that a type in subset (i) that does not disclose x at $t = 0$ may benefit from either one of the real options, or none of them, while a type in subset (iii) benefits for sure from just one of the options (the increased price at $t = 1$).

To obtain an equilibrium with a threshold for disclosure of a single signal at $t = 0$ we set x^* to equal the lowest value of x for which all agents with $x = x^*$ from all subsets $(i) - (iii)$ weakly prefer to disclose x^* at $t = 0$ over not disclosing at $t = 0$. Note that the binding constraint might be either equation (3) or equation (4). Since there are agents that strictly prefer to disclose x^* at $t = 0$ over not disclosing at $t = 0$ (these are agents in subset (ii) and agents in either subset (i) or (iii)) the price given disclosure of $x < x^*$ at $t = 0$, which is off the equilibrium path, must be sufficiently low to prevent the above types from disclosing $x < x^*$ at $t = 0$. This implies that a necessary condition for a threshold equilibrium in which x^* is the lowest value of x for which all agents weakly prefer to disclose x^* at $t = 0$ prices must exhibit a discontinuity at x^*.

Appendix

Proof Lemma 1

For a constant c let $S^c_{A,B} \quad = A \cup \{B \cap \{(y, \tau_y) : y \leq c\}$. For $c \to -\infty$ we have that $E_y(S^c_{A,B}) > c$ and for $c \to \infty$ we have that $E_y(S^c_{A,B}) < c$ so by continuity we can find c^* so that $E_y(S^{c^*}_{A,B}) = c^*$.

Now suppose by way of contradiction that there are multiple solutions. Specifically, $c' < c$" so that $E_y(S^{c'}_{A,B}) = c', E_y(S^{c"}_{A,B}) = c$". When we compare $S^{c'}_{A,B}$ to $S^{c"}_{A,B}$ we note that $S^{c"}_{A,B} \supset S^{c'}_{A,B}$ and that for $(y, \tau_y) \in S^{c"}_{A,B} \backslash S^{c'}_{A,B}$ we have $y < E_y(S^{c"}_{A,B})$. This implies that $S^{c"}_{A,B}$ can be represented as a union of $S^{c'}_{A,B}$ where the average $c' < c$" and a set of types that are lower than c". It implies that $E_y(S^{c"}_{A,B}) < c$" and we get a contradiction. QED

Proof of Lemma 2

1. When comparing $S_{A,B}$ to $A \cup B$ we note that we have excluded above average types for which $y > E_y(S_{A,B})$. This results in lower average type.

2. Suppose first that there exists $(y, \tau_y) \in S_{A,B"} \backslash S_{A,B'}$. Since $B' \supseteq B''$ it must be that these $(y, \tau_y) \in B' \cap B''$. From the definition of $S_{A,B}$ since $(y, \tau_y) \in S_{A,B''}$ we conclude that $E_y(S_{A,B''}) > y$. Since $(y, \tau_y) \notin S_{A,B'}$, we conclude that $E_y(S_{A,B'}) < y$ which implies the claim. Hence, we will assume that $S_{A,B'} \supseteq S_{A,B"}$ and we consider $(y, \tau_y) \in S_{A,B'} \backslash S_{A,B"}$; this implies $y < E_y(S_{A,B'})$. Hence, all the elements $(y, \tau_y) \in S_{A,B'} \backslash S_{A,B"}$ have y that is below the average in $S_{A,B'}$ which implies that $E_y(S_{A,B"}) \geq E_y(S_{A,B'})$ **.**

3. Consider the set $S_{A,B"}$, and note that it satisfy the definition for $S_{A,B'}$ given in (1) Hence, the claim follows from uniqueness that was proven in Lemma 1. **QED**

Proof of Theorem 1

Step 1 If $h(x, 0, 1) > h(x, 1, 1)$ then if x is disclosed at time $t = 1$ then the agent could not have known both signals at $t = 0$.

Proof: We know that x is being disclosed with positive probability if it is the only signal known at $t = 0$. Let I denote the payoff for such an agent, who learnt only x at $t = 0$, from disclosing x at $t = 0$ and II his payoff from not disclosing at $t = 0$

$$I = \beta(x + h(x, 0, 0)) + E_y [\max \{\beta(x + h(x, 0, 1)), \beta(x + y)\}]$$
$$II = h(0) + E_y [\max \{\beta(x + h(x, 1, 1)), \beta(y + h(y, 1, 1)), \beta(x + y), h(1)\}]$$

where $h(0)$ and $h(1)$ are the prices at the end of $t = 0$ and $t = 1$ respectively, given that no disclosure was made until time t..

We know that for some x we have that $I - II \geq 0$**.** Consider an agent who knows both signals at $t = 0$ and prefers to disclose just x at $t = 1$. Such an agent knows at time $t = 0$ that he will disclose x and not disclose y at $t = 1$. So, for this to happen it must be that $II\prime - I\prime >= 0$ where:

$$\begin{array}{l} I' = \beta(x + h(x,0,0)) + \beta(x + h(x,0,1)) \\ II' = h(0) + \beta(x + h(x,11)) \end{array}$$

This leads to contradiction as $h(x,0,1) > h(x,1,1), \Rightarrow I' - II' > I - II > 0$

The above provide us with a simple characterization of B_1. Let ND denotes the set of signals that are not disclosed at $t = 0$ if this is the only signal the agent knows. We argue that

Step 2 If $h(x,0,1) > h(x,1,1)$ then $B_1 = \{\tau_y = 0,1\} \cap \{y < x\} \backslash \{(y,\tau_y) : \tau_y = 0, y \notin ND\}$

Proof: If the agent learnt y at $t = 0$ then based on step1 we conclude that this was the only signal he knew at the time so it must be that $y \in ND$. Since $x + h(x,1,1)$ is increasing in x we conclude that $y \leq x$.

Consider the set:

$$\widehat{B}_0 = B_0 \cup \{\{y < x\} \cap \{\tau_y = 0,1\}\}$$

Clearly $\widehat{B}_0 \supset B_1$, $\widehat{B}_0 \supseteq B_0$

Step 3 $\forall y \in \widehat{B}_0 \backslash B_0$, $y > h(x,0,1)$

Proof: We need to consider only agents that become informed. The claim follows from **Corollary 1** showing that an informed agent reveals y if and only if $y > h(x,0,0) \geq h(x,0,1)$.

From Lemma 2 (iii) we know that $S_{A\widehat{B}_0} = S_{AB_0}$ and from Lemma 2 (ii) we know that $E_y(S_{AB_1}) \geq E_y\left(S_{A\widehat{B}_0}\right) = E_y(S_{AB_0})$. This contradicts the assumption that $h(x,0,1) > h(x,1,1)$.

QED

Proof of Lemma 4

By applying Bayes role, $h^{stat}(\mu, \{z_i(\cdot)\})$ is given by:

$$h^{stat}(\mu, \{z_i(\cdot)\}) = \frac{(1-p)\mu + \sum_{i=0}^{K} \lambda_i \int_{-\infty}^{z_i(\mu)} y\phi(y|\mu)\, dy}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(\mu)|\mu)}.$$

Taking the derivative of $h^{stat}(\mu, \{z_i(\cdot)\})$ with respect to μ and applying some algebraic manipulation yields:

$$\frac{d}{d\mu} h^{stat}(\mu, \{z_i(\cdot)\}) = 1 + \frac{\sum_{i=0}^{K} \lambda_i (z_i'(\mu) - 1)\, \phi(z_i(\mu)|\mu)\left(z_i(\mu) - h^{stat}(\mu, \{z_i(\cdot)\})\right)}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(\mu)|\mu)}.$$

We start by proving the supremum of this derivative

Given that $z_i'(\mu) \geq 0$ and $z_i(\mu) \leq h^{stat}(\mu, \{z_i(\cdot)\})$ for all $i \in \{1,..,K\}$ we have

$$\frac{d}{d\mu} h^{stat}(\mu, \{z_i(\cdot)\}) \leq 1 + \frac{\sum_{i=0}^{K} \lambda_i \phi(z_i(\mu)|\mu)\left(z_i(\mu) - h^{stat}(\mu, \{z_i(\cdot)\})\right)}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(\mu)|\mu)}$$
$$\leq 1 + \max_{\substack{z_i \leq h(x) \\ i \in \{1,...K\}}} \frac{\sum_{i=0}^{K} \lambda_i \phi(z_i(\mu)|\mu)\left(z_i(\mu) - h^{stat}(\mu, \{z_i(\cdot)\})\right)}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(\mu)|\mu)}$$

Due to symmetry, for all $i \in \{1,..,K\}$ the maximum is achieved at $z_i(\mu) = z^*(\mu)$. To see this, note that the FOC of the maximization with respect to $z_i(\mu)$ is

$$0 = \left(\phi'(z_i(\mu)|\mu)\left(h^{stat}(\mu, \{z_i(\cdot)\}) - z_i(\mu)\right) - \phi(z_i(\mu)|\mu)\right)\left((1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(\mu)|\mu)\right)$$
$$- \left(\sum_{i=1}^{K} \lambda_i \phi(z_i(\mu)|\mu)\left(h^{stat}(\mu, \{z_i(\cdot)\}) - z_i(\mu)\right)\right)\phi(z_i(\mu)|\mu)$$

Since $\phi'(z_i(\mu)|\mu) = -\alpha(z_i(\mu) - \mu)\phi(z_i(\mu)|\mu)$ (for some constant $\alpha > 0$), this simplifies to

$$-\alpha(z_i(\mu) - \mu)\left(h^{stat}(\mu, \{z_i(\cdot)\}) - z_i(\mu)\right) = \frac{\sum_{i=0}^{K} \lambda_i \phi(z_i(\mu)|\mu)\left(z_i(\mu) - h^{stat}(\mu, \{z_i(\cdot)\})\right)}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(\mu)|\mu)} + 1$$

In the range $z_i(\mu) \leq h^{stat}(\mu, \{z_i(\cdot)\}) \leq \mu$, the LHS is decreasing in $z_i(\mu)$.[12] The RHS is the same for all i. Therefore, the unique solution to this system of FOC is for all $z_i(\mu)$ to be equal (and note that the maximum is achieved at an interior point since at $z_i(\mu) = h^{stat}(\mu, \{z_i(\cdot)\})$ the LHS is zero and the RHS is positive; and as $z_i(\mu)$ goes to $-\infty$ the LHS goes to $+\infty$ while the RHS is bounded). This implies that the example we discussed following the statement of the Lemma also provides an upper bound. The lower bound can be concluded in a similar way by observing that if we want to minimize the slope we will again choose the same $z_i(\mu)$ for all i and therefore our example provides also a lower bound.

QED

Proof of Claim 1

The case of $h(x, 0, 0)$ has been proved right bellow the Claim. We analyze the cases of $h(x, 1, 1)$ and $h(x, 0, 1)$ bellow.

Next we analyze $h(x, 1, 1)$

[12] Since $z_i(x) \leq h(x, \{z_i(\cdot)\})$ also $h(x, \{z_i(\cdot)\}) \leq E[x|y] = \beta_1 x$.

When an agent discloses $x > x^*$ at $t = 1$ investors know that $\tau_x = 1$ (otherwise the agent would have disclosed x at $t = 0$). Investors' beliefs about the manager's other signal at $t = 1$ is set as a weighted average of three scenarios: $\tau_y = 0$, $\tau_y = 1$ and $\tau_y > 1$. We start by describing the disclosure thresholds conditional on each of the three scenarios.

(i) If $\tau_y > 1$ the agent cannot disclose y and therefore the disclosure threshold is not relevant. In the pricing of the firm conditional on $\tau_y > 1$ investors use $E(y|x)$ which equals $\beta_1 x$.

(ii) If $\tau_y = 1$ investors know that $y < h(x, 1, 1)$ and also that $y < x$. We need to distinguish between the binding case and the non-binding case. In the non-binding case, where $h(x, 1, 1) \leq x$, investors know that $y < h(x, 1, 1)$, so conditional on $\tau_y = 1$investors set their beliefs as if the manger follows a disclosure threshold of $h(x, 1, 1)$. In the binding case, where $h(x, 1, 1) > x$, investors know that $y < x$, so it is equivalent to a disclosure threshold of x.

(iii) If $\tau_y = 0$ investors know that $y < x^*$ (where $x^* \leq x$) and also $y < h(x, 1, 1)$. Here again we should distinguish between a non-binding case in which $h(x, 1, 1) < x^*$ (if such case exists) and a binding case in which $h(x, 1, 1) > x^*$. In the non-binding case the disclosure threshold is $h(x, 1, 1)$. In the binding case the disclosure threshold is x^*, which is independent of x.

The next Lemma provides an upper and lower bound for $\frac{\partial}{\partial x} h(x, 1, 1)$. The proof of the Lemma, which is provided in the appendix, uses the disclosure thresholds for each of the three scenarios above. This Lemma holds also for $h(x, 0, 1)$.

Lemma 6 *For $\beta_1 > 0.5$ and $p < 0.95$*

$$\frac{\partial}{\partial x} h(x, 1, 1) \in (2\beta_1 - 1, 2\beta_1).$$

We next show that for the particular case in which $h(x, 1, 1) < x^*$ (if such case exists) $h'(x, 1, 1) = \beta_1$.

$h(x, 1, 1)$ is a weighted average of the beliefs about y over the three scenarios $\tau_y = 0$, $\tau_y = 1$ and $\tau_y > 1$. That is, we can write

$$h(x, 1, 1) = \lambda_0 h_0 + \lambda_1 h_1 + (1 - \lambda_0 - \lambda_1) h_2,$$

where $\lambda_i = \Pr(\tau_y = i|ND_y)$ and $h_i = E(y|\tau_y = i, ND_y)$ for $i = 0, 1$ and $i = 2$ represents the case of $\tau_y > 1$. ND_y stands for No-Disclosure of y (where x was disclosed at $t = 1$). Assume that If $h'(x, 1, 1) = \beta_1$ then the for both $\tau_y = 0$ and for $\tau_y = 1$ an increase in x increases both the disclosure threshold $h(x, 1, 1)$ and the expectation of y given no-disclosure, $E(y|\tau_y = i, ND_y)$, at a rate of β_1. Therefore the probabilities λ_i are independent of x. In addition, for $i = 0, 1, 2$ we have $h'_i = \beta_1$. Computing $h'(x, 1, 1)$ and incorporating the fact that $\lambda'_i = 0$ for all i yields

$$h'(x, 1, 1) = \lambda_0 h'_0 + \lambda_1 h'_1 + (1 - \lambda_0 - \lambda_1) h'_2 = \beta_1.$$

The above only showed consistency of $h'(x, 1, 1) = \beta_1$. Following the same line of logic one can preclude any other value of $h'(x, 1, 1)$. For brevity, we omit this part of the proof .

Finally, we analyze $h(x, 0, 1)$

Recall that Lemma 6 applies also to $h(x, 0, 1)$. However, for $h(x, 0, 1)$ we can show tighter bounds.

We first show that for the case where $h(x, 0, 1) < x$ we have $h'(x, 0, 1) = \beta_1$.

If $h(x, 0, 1) < x$ (the non-binding case) then when pricing the firm at $t = 1$ investors know that if the agent learnt y (at either $t = 0$ or $t = 1$) then $y < h(x, 0, 1)$. If the agent did not learn y then investors use in their pricing $E(y|x) = \beta_1 x$. So, the beliefs about y is a weighted average of $E(y|y < h(x, 0, 1))$ and $E(y|x) = \beta_1 x$. This is similar to a Dye (1985) and Jung and Kwon (1988) setting and therefore, in equilibrium we have $h'(x, 0, 1) = \beta_1$.

Next we show that for x such that $h(x, 0, 1) > x$ (if such case exists) $h'(x, 0, 1) \in (2\beta_1 - 1, \beta_1)$.

The argument is similar to the one we made in the proof that $h'(x, 0, 0) \in (2\beta_1 - 1, \beta_1)$ for x such that $h(x, 0, 0) > x$. First note that for $h(x, 0, 1) > x$ investors' beliefs about y conditional on that the agent learnt y is independent on whether he learnt y at $t = 0$ or at $t = 1$. Moreover, given that $\tau_y \leq 1$ investors know that $y < x$. So from investors' perspective, it doesn't matter if the agent learnt y at $t = 0$ or at $t = 1$. Their pricing, $h(x, 0, 1)$, will reflect a weighted average between $E(y|y < x)$ and $E(y|\tau_y > 1, x) = \beta_1 x$. From here on the proof is qualitatively the same as in the proof for $h'(x, 0, 0) \in (2\beta_1 - 1, \beta_1)$, where the only quantitative difference is the probability that the agent learnt y.

QED Claim 1

Proof of Lemma 6

In this proof we use a slightly different notation, as part of the proof is more general than our setting.

Suppose that x and y have joint normal distribution and the agent is informed about y with probability p and uninformed with probability $1 - p$. Conditional on being informed the agent's disclosure strategy is assumed to be as follows: with probability λ_i, $i \in \{1, .., K\}$, he discloses if his type is above $z_i(x)$, where the various $z_i(x)$ are determined exogenously such that $z_i(x) \leq h(x, \{z_i(\cdot)\})$ for all i (which always holds in our setting). Note that $\sum_{i=1}^{K} \lambda_i = p$. Let's denote the conditional expectation of y given x and given the disclosure thresholds, $z_i(x)$, by $h(x, \{z_i(\cdot)\})$.

By applying Bayes role, $h(x)$ is given by:

$$h(x, \{z_i(\cdot)\}) = \frac{(1-p) E[y|x] + \sum_{i=0}^{K} \lambda_i \int_{-\infty}^{z_i(x)} y \phi(y|x) \, dy}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(x)|x)}.$$

Taking the derivative of $h(x, \{z_i(\cdot)\})$ with respect to x and applying some algebraic manipulation (recall that $\frac{\partial E[y|x]}{\partial x} = \beta_1$) yields:

$$h'(x, \{z_i(\cdot)\}) = \beta_1 + \frac{\sum_{i=0}^{K} \lambda_i (z_i'(x) - \beta_1) \phi(z_i(x)|x) (z_i(x) - h(x, \{z_i(\cdot)\}))}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(x)|x)}. \tag{5}$$

We start by proving the supremum of $h'(x)$.

Given that $z_i'(x) \geq 0$ and $(z_i(x) - h(x, \{z_i(\cdot)\})) \leq 0$ for all $i \in \{1, .., K\}$ we have

$$\begin{aligned} h'(x, \{z_i(\cdot)\}) &\leq \beta_1 + \frac{\beta_1 \sum_{i=0}^{K} \lambda_i \phi(z_i(x)|x) (h(x, \{z_i(\cdot)\}) - z_i(x))}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(x)|x)} \\ &\leq \beta_1 + \max_{\substack{z_i \leq h(x) \\ i \in \{1, ...K\}}} \frac{\beta_1 \sum_{i=0}^{K} \lambda_i \phi(z_i|x) (h(x, \{z_i(\cdot)\}) - z_i)}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i|x)} \end{aligned}$$

Due to symmetry, for all $i \in \{1, .., K\}$ the maximum is achieved at $z_i(x) = z^*(x)$. To see this, note that the FOC of the maximization with respect to $z_i(x)$ is

$$\begin{aligned} 0 &= \left(\phi'(z_i(x)|x) (h(x, \{z_i(\cdot)\}) - z_i(x)) - \phi(z_i(x)|x)\right) \left((1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(x)|x)\right) \\ &- \left(\sum_{i=1}^{K} \lambda_i \phi(z_i(x)|x) (h(x, \{z_i(\cdot)\}) - z_i(x))\right) \phi(z_i(x)|x) \end{aligned}$$

Since $\phi'(z_i(x)|x) = -\alpha(z_i(x) - \beta_1 x) \phi(z_i(x)|x)$ (for some constant $\alpha > 0$), this simplifies to

$$-\alpha(z_i(x) - \beta_1 x)(h(x, \{z_i(\cdot)\}) - z_i(x)) = \frac{\sum_{i=0}^{K} \lambda_i \phi(z_i(x)|x) (h(x, \{z_i(\cdot)\}) - z_i(x))}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(x)|x)} + 1$$

In the range $z_i(x) \leq h(x, \{z_i(\cdot)\}) \leq \beta_1 x$, the LHS is decreasing in $z_i(x)$.[13] The RHS is the same for all i. Therefore, the unique solution to this system of FOC is for all $z_i(x)$ to be equal (and note that the maximum is achieved at an interior point since at $z_i(x) = h(x)$ the LHS is zero and the RHS is positive; and as $z_i(x)$ goes to $-\infty$ the LHS goes to $+\infty$ while the RHS is bounded).

[13] Since $z_i(x) \leq h(x, \{z_i(\cdot)\})$ also $h(x, \{z_i(\cdot)\}) \leq E[x|y] = \beta_1 x$.

Let $z^*(x)$ be the maximizing value. Then

$$h'(x,\{z_i(\cdot)\}) \le \beta_1 + \frac{\beta_1 \sum_{i=0}^{K} \lambda_i \phi(z^*(x)|x)(h(x,\{z_i(\cdot)\}) - z^*(x))}{(1-p) + p\Phi(z^*(x)|x)}$$
$$= \beta_1 + \frac{p\beta_1 \phi(z^*(x)|x)(h(x,\{z_i(\cdot)\}) - z^*(x))}{(1-p) + p\Phi(z^*(x)|x)}.$$

The right hand side of the above inequality is identical to the slope in a Dye setting with exogenous disclosure threshold with probability of being uninformed $(1-p)$ and an exogenously determined determined disclosure threshold of $z^*(x)$, where the disclosure threshold does not change in x. In such a setting, we can think of the effect of a marginal increase in x as the sum of two effects. The first effect is a shift by β_1 in both the distribution and the disclosure threshold. This will increase $h(x)$ by β_1. The second effect is a decrease in the disclosure threshold by β_1 (as the disclosure threshold does not change in x). Since $z^*(x) < \beta_1 x$ we are in the decreasing part of the beliefs about y given no disclosure (to the left of the minimum beliefs). Therefore, the decrease in the disclosure threshold increases the beliefs about y by the change in the disclosure threshold times the slope of the beliefs about y given no disclosure. Since for $p < 0.95$ the slope of the beliefs about y given no disclosure is greater than -1, the latter effect increases the beliefs about y by less than β_1. The overall effect is therefore smaller than $2\beta_1$.

Next we prove the infimum of $h'(x)$.

Equation (5) capture a general case with any number of potential disclosure strategies. In our particular case $K = 1$ where $i = 0$ represents the case of $\tau_y = 0$ and $i = 1$ represents the case of $\tau_y = 1$. So, in our setting equation (5) can be written as

$$h'(x,\{z_i(\cdot)\}) = \beta_1 + \frac{\lambda_0 (z_0'(x) - \beta_1)\phi(z_0(x)|x)(z_0(x) - h(x,\{z_i(\cdot)\}))}{(1-p) + \sum_{i=0}^{1} \lambda_i \Phi(z_i(x)|x)}$$
$$+ \frac{\lambda_1 (z_1'(x) - \beta_1)\phi(z_1(x)|x)(z_1(x) - h(x,\{z_i(\cdot)\}))}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(x)|x)}$$

When calculating $h(x,1,1)$ and $h(x,0,0)$ in our setting, the disclosure threshold, $z_i(x)$, in any possible scenario (the binding and non-binding case for both $\tau_y = 0$ and $\tau_y = 1$) takes one of the following three values: $h(x,\cdot,\cdot)$, x or x^*. Note that whenever $z_i(x) = h(x,\{z_i(\cdot)\})$ we have $\frac{(z_i'(x) - \beta_1)\phi(z_i(x)|x)(z_i(x) - h(x,\{z_i(\cdot)\}))}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(x)|x)} = 0$.

For the remaining two cases ($z_i(x) = x$ and $z_i(x) = x^*$), for all $i \in \{0,1\}$ we have $z_i'(x) \le 1$ and $(z_i(x) - h(x,\{z_i(\cdot)\})) \le 0$. This implies

$$h'(x) \ge \beta_1 - \frac{(1-\beta_1)\sum_{i=0}^{K} \lambda_i \phi(z_i(x)|x)(h(x,\{z_i(\cdot)\}) - z_i(x))}{(1-p) + \sum_{i=0}^{K} \lambda_i \Phi(z_i(x)|x)}.$$

Using the same symmetry argument for the first order condition as before, $h'(x, \{z_i(\cdot)\})$ is minimized for some $z^{\min}(x)$ and hence

$$h'(x, \{z_i(\cdot)\}) \geq \beta_1 + \frac{p(1-\beta_1)\phi\left(z^{\min}(x)|x\right)\left(h(x, \{z_i(\cdot)\}) - z^{\min}(x)\right)}{(1-p) + p\Phi\left(z^{\min}(x)|x\right)}.$$

The right hand side of the above inequality is identical to the slope in a Dye setting with exogenous disclosure threshold in which: the probability of being uninformed is $(1-p)$, the exogenously determined disclosure threshold is $z^{\min}(x)$ and $\frac{\partial}{\partial x} z^{\min}(x) = 1$. In such a setting, we can think of the effect of a marginal increase in x as the sum of two effects. The first is a shift by β_1 in both the distribution and the disclosure threshold. This will increase $h(x)$ by β_1. The second effect is an increase in the disclosure threshold by $(1-\beta_1)$ (as the disclosure threshold increases by 1). Since $z^{\min}(x) < \beta_1 x$ we are in the decreasing part of the beliefs about y given no disclosure (to the left of the minimum beliefs). Therefore, the increase in the disclosure threshold decreases the beliefs about y by the change in the disclosure threshold, $(1-\beta_1)$, times the slope of the beliefs about y given no disclosure. Since for $p < 0.95$ the slope of the beliefs about y given no disclosure is greater than -1 the latter effect decreases the beliefs about y by less than $(1-\beta_1)$. The overall effect is therefore greater than $\beta_1 - (1-\beta_1) = 2\beta_1 - 1$.

QED Lemma 6

Proof of Lemma 5

We start by analyzing the partially informed agent, i.e., $(\tau_x = 0, \tau_y \neq 0)$ and then move to the fully informed agent.

Partially informed agent $(\tau_x = 0, \tau_y \neq 0)$

First note that for sufficiently low realizations of x the agent is always better off not disclosing it at $t = 0$, as they can "hide" behind uninformed agents. Next we establish that for sufficiently high realizations of the only private signal that the agent obtains at $t = 0$ he will disclose it at $t = 0$.

Lemma 7 *Consider an agent that obtains a single signal x at $t = 0$. In the threshold equilibrium, the difference between the agent's expected payoff (as calculated at $t = 0$) from disclosing his signal at $t = 0$ and from not disclosing it at $t = 0$ is increasing in x. That is,*

$$\frac{\partial}{\partial x}\left(E\left(U|\tau_x = 0, \tau_y \neq 0, t_x = 0\right) - E\left(U|\tau_x = 0, \tau_y \neq 0, t_x \neq 0, x \geq x_D\right)\right) > 0$$

and

$$\frac{\partial}{\partial x}\left(E\left(U|\tau_x=0,\tau_y\neq 0,t_x=0\right)-E\left(U|\tau_x=0,\tau_y\neq 0,t_x\neq 0,x<x_D\right)\right)>0.$$

Proof. We start by showing that the Proposition holds for the case where $x \geq x_D$ (i.e., $\beta_2\left(x+h\left(x,1,1\right)\right) \geq h(1)$). For simplicity of disposition, we partition the support of x into two cases: realizations of x for which $\beta_2\left(x+h\left(x,1,1\right)\right) \geq h(1)$ and for which $\beta_2\left(x+h\left(x,1,1\right)\right) < h(1)$.[14]

Case I - $\beta_2\left(x+h\left(x,1,1\right)\right) \geq h(1)$

Rewriting $E\left(U|\tau_x=0,\tau_y\neq 0,t_x=0,x\right)-E\left(U|\tau_x=0,\tau_y\neq 0,t_x\neq 0,x\geq x_D\right)$ yields

$$\beta_2\left[x+h\left(x,0,0\right)+h\left(x,0,1\right)-h\left(x,1,1\right)\right]-h\left(0\right)$$
$$+p\beta_2\left[\int_{y^*(x)}^{\infty}\left(y-h\left(x,0,1\right)\right)f\left(y|x\right)dy-\int_{y^1(x)}^{\infty}\left(y-h\left(x,1,1\right)\right)f\left(y|x\right)dy-\int_{y^H(x)}^{\infty}\left(h\left(y,1,1\right)-x\right)f\left(y|x\right)dy\right]$$

The derivative of this expression with respect to x has the same sign as

$$D=1+\frac{\partial}{\partial x}\left(h\left(x,0,0\right)+h\left(x,0,1\right)-h\left(x,1,1\right)\right)+p\left[A+B+C\right]$$

where

$$A=\frac{\partial}{\partial x}\int_{y^*(x)}^{\infty}\left(y-h\left(x,0,1\right)\right)f\left(y|x\right)dy$$
$$B=-\frac{\partial}{\partial x}\int_{y^1(x)}^{\infty}\left(y-h\left(x,1,1\right)\right)f\left(y|x\right)dy$$
$$C=-\frac{\partial}{\partial x}\int_{y^H(x)}^{\infty}\left(h\left(y,1,1\right)-x\right)f\left(y|x\right)dy.$$

To evaluate this derivative we will use the following equations which are easy to obtain:

$$\frac{\partial}{\partial x}y^*\left(x\right)=\frac{\partial}{\partial x}h\left(x,0,1\right),$$
$$\frac{\partial}{\partial x}f\left(y|x\right)=-\beta_1\frac{\partial}{\partial y}f\left(y|x\right),$$
$$\frac{\partial}{\partial x}\left(F\left(y\left(x\right)|x\right)\right)=f\left(y\left(x\right)|x\right)\left(\frac{\partial}{\partial x}y\left(x\right)-\beta_1\right).$$

Next we analyze the three terms A, B, and C. Note that the derivative with respect to the limits of integrals is zero for all cases because of the definition of the three cutoffs. Hence we get:

$$A=-\frac{\partial h\left(x,0,1\right)}{\partial x}\left(1-F\left(y^*\left(x\right)|x\right)\right)-\beta_1\int_{y^*(x)}^{\infty}\left(y-h\left(x,0,1\right)\right)\frac{\partial}{\partial y}f\left(y|x\right)dy.$$

[14] Note that on the equilibrium path we are always in case I, i.e., $\beta_2\left(x+h\left(x,1,1\right)\right)\geq h(1)$.

Note that by integrating $\int_{y^*(x)}^{\infty} (y - h(x,0,1)) \frac{\partial}{\partial y} f(y|x)\, dy$ by parts w.r.t. y we got:

$$\begin{aligned}
&\int_{y^*(x)}^{\infty} (y - h(x,0,1)) \frac{\partial}{\partial y} f(y|x)\, dy \\
&= -(y^*(x) - h(x,0,1)) f(y^*(x)|x) - \int_{y^*(x)}^{\infty} f(y|x)\, dy = -(1 - F(y^*(x)|x)).
\end{aligned}$$

Plugging back to A we get

$$A = -\left(\frac{\partial h(x,0,1)}{\partial x} - \beta_1\right)(1 - F(y^*(x)|x)).$$

Next, we calculate B:

$$\begin{aligned}
B &= \int_{y^1(x)}^{\infty} \frac{\partial h(x,1,1)}{\partial x} f(y|x)\, dy + \beta_1 \int_{y^1(x)}^{\infty} (y - h(x,1,1)) \frac{\partial}{\partial y} f(y|x)\, dy \\
&= \frac{\partial h(x,1,1)}{\partial x}\left(1 - F\left(y^1(x)|x\right)\right) - \beta_1\left(1 - F\left(y^1(x)|x\right)\right) \\
&= \left(\frac{\partial h(x,1,1)}{\partial x} - \beta_1\right)\left(1 - F\left(y^1(x)|x\right)\right)
\end{aligned}$$

Finally, we calculate C:

$$\begin{aligned}
C &= \left(1 - F\left(y^H(x)|x\right)\right) + \beta_1 \int_{y^H(x)}^{\infty} (h(y,1,1) - x) \frac{\partial}{\partial y} f(y|x)\, dy \\
&= \left(1 - F\left(y^H(x)|x\right)\right) - \beta_1 \int_{y^H(x)}^{\infty} \frac{\partial h(y,1,1)}{\partial y} f(y|x)\, dy
\end{aligned}$$

Substituting A, B and C back to the whole derivative and re-arranging terms yields:

$$\begin{aligned}
D &= 1 + \frac{\partial}{\partial x}(h(x,0,0) + h(x,0,1) - h(x,1,1)) \\
&\quad - p\left[\begin{array}{c} \left(\frac{\partial h(x,0,1)}{\partial x} - \beta_1\right)(1 - F(y^*(x)|x)) + \left(\frac{\partial h(x,1,1)}{\partial x} - \beta_1\right)\left(1 - F\left(y^1(x)|x\right)\right) + \\ \left(1 - F\left(y^H(x)|x\right)\right) - \beta_1 \int_{y^H(x)}^{\infty} \frac{\partial h(y,1,1)}{\partial y} f(y|x)\, dy \end{array}\right] \\
&= (1-p)\left(1 + \frac{\partial}{\partial x}(h(x,0,0) + h(x,0,1) - h(x,1,1))\right) \\
&\quad + p\left[\begin{array}{c} 1 + \frac{\partial h(x,0,0)}{\partial x} + \frac{\partial h(x,0,1)}{\partial x} F(y^*(x)|x) + \beta_1(1 - F(y^*(x)|x)) - \frac{\partial h(x,1,1)}{\partial x} F\left(y^1(x)|x\right) \\ -\beta_1\left(1 - F\left(y^1(x)|x\right)\right) + 1 - F\left(y^H(x)|x\right) - \beta_1 \int_{y^H(x)}^{\infty} \frac{\partial h(y,1,1)}{\partial y} f(y|x)\, dy \end{array}\right] \\
&= (1-p)\left(1 + \frac{\partial}{\partial x}(h(x,0,0) + h(x,0,1) - h(x,1,1))\right) \\
&\quad + p\left[\begin{array}{c} 1 + \frac{\partial h(x,0,0)}{\partial x} + \frac{\partial h(x,0,1)}{\partial x} F(y^*(x)|x) - F(y^*(x)|x)\beta_1 - \frac{\partial h(x,1,1)}{\partial x} F\left(y^1(x)|x\right) \\ +F\left(y^1(x)|x\right)\beta_1 + \left(1 - F\left(y^H(x)|x\right)\right) - \beta_1 \int_{y^H(x)}^{\infty} \frac{\partial h(y,1,1)}{\partial y} f(y|x)\, dy \end{array}\right]
\end{aligned}$$

Rearranging yields:

$$
\begin{aligned}
D = &(1-p)\left(1+\frac{\partial}{\partial x}\left(h\left(x,0,0\right)+h\left(x,0,1\right)-h\left(x,1,1\right)\right)\right)\\
&+p\left[1+\frac{\partial h\left(x,0,0\right)}{\partial x}+\left(\frac{\partial h\left(x,0,1\right)}{\partial x}-\beta_1\right)F\left(y^*\left(x\right)|x\right)-\left(\frac{\partial h\left(x,1,1\right)}{\partial x}-\beta_1\right)F\left(y^1\left(x\right)|x\right)\right]\\
&+p\beta_1\int_{y^H(x)}^{\infty}\frac{1}{\beta_1}-\frac{\partial h\left(y,1,1\right)}{\partial y}f\left(y|x\right)dy
\end{aligned}
$$

Since $\frac{\partial h(x,0,1)}{\partial x}\leq\beta_1$ (see Claim 1) and $F\left(y^1\left(x\right)|x\right)\geq F\left(y^*\left(x\right)|x\right)$ we have

$$
\begin{aligned}
D \geq &(1-p)\left(1+\frac{\partial}{\partial x}\left(h\left(x,0,0\right)+h\left(x,0,1\right)-h\left(x,1,1\right)\right)\right)\\
&+p\left[1+\frac{\partial h\left(x,0,0\right)}{\partial x}+\left(\frac{\partial h\left(x,0,1\right)}{\partial x}-\frac{\partial h\left(x,1,1\right)}{\partial x}\right)F\left(y^1\left(x\right)|x\right)\right]\\
&+p\beta_1\int_{y^H(x)}^{\infty}\frac{1}{\beta_1}-\frac{\partial h\left(y,1,1\right)}{\partial y}f\left(y|x\right)dy\\
&=\left(1-p\left(1-F\left(y^1\left(x\right)|x\right)\right)\right)\left(1+\frac{\partial}{\partial x}\left(h\left(x,0,0\right)+h\left(x,0,1\right)-h\left(x,1,1\right)\right)\right)+\\
&+p\left(1-F\left(y^1\left(x\right)|x\right)\right)\left(1+\frac{\partial h\left(x,0,0\right)}{\partial x}\right)+p\beta_1\int_{y^H(x)}^{\infty}\frac{1}{\beta_1}-\frac{\partial h\left(y,1,1\right)}{\partial y}f\left(y|x\right)dy\\
&=\left(1-p\left(1-F\left(y^1\left(x\right)|x\right)\right)\right)\left(1+\frac{\partial}{\partial x}\left(h\left(x,0,0\right)+h\left(x,0,1\right)-h\left(x,1,1\right)\right)\right)\\
&+p\int_{y^1(x)}^{\infty}\left(1+\frac{\partial h\left(x,0,0\right)}{\partial x}\right)f\left(y|x\right)dy+p\int_{y^H(x)}^{\infty}1-\beta_1\frac{\partial h\left(y,1,1\right)}{\partial y}f\left(y|x\right)dy\\
&=\left(1-p\left(1-F\left(y^1\left(x\right)|x\right)\right)\right)\left(1+\frac{\partial}{\partial x}\left(h\left(x,0,0\right)+h\left(x,0,1\right)-h\left(x,1,1\right)\right)\right)+\\
&+p\int_{y^1(x)}^{y^H(x)}\left(1+\frac{\partial h\left(x,0,0\right)}{\partial x}\right)f\left(y|x\right)dy+p\int_{y^H(x)}^{\infty}2+\frac{\partial h\left(x,0,0\right)}{\partial x}-\beta_1\frac{\partial h\left(y,1,1\right)}{\partial y}f\left(y|x\right)dy\\
&\geq\left(1-p\left(1-F\left(y^1\left(x\right)|x\right)\right)\right)\left(1+\frac{\partial}{\partial x}\left(h\left(x,0,0\right)+h\left(x,0,1\right)-h\left(x,1,1\right)\right)\right)\\
&+p\int_{y^H(x)}^{\infty}2+\frac{\partial h\left(x,0,0\right)}{\partial x}-\beta_1\frac{\partial h\left(y,1,1\right)}{\partial y}f\left(y|x\right)dy
\end{aligned}
$$

So, the following are sufficient conditions to prove the Lemma for case I. For all x:

1. $\frac{\partial}{\partial x}h\left(x,0,0\right)+\frac{\partial}{\partial x}h\left(x,0,1\right)\geq\frac{\partial}{\partial x}h\left(x,1,1\right)-1$

2. $\frac{\partial h(y,1,1)}{\partial y}\leq\left(2+\frac{\partial h(x,0,0)}{\partial x}\right)\frac{1}{\beta_1}$ for any $y>x$

Case II - $x<x_D$ (i.e., $\beta_2\left(x+h\left(x,1,1\right)\right)<h(1)$)

The analysis of Case I was for generic bounds of the integrals $h\left(x,0,1\right)$ and $y^H\left(x\right)$. The difference between Case I and Case II is that the price given no disclosure of y (if the agent does not obtain a signal y or obtains a low realization of y) is $h(1)$ in Case II and $\beta_2(x + h(x,1,1))$ in Case I. This causes the expected payoff of the agent in Case II to be less sensitive to x than in Case I, and therefore $\frac{\partial}{\partial x}(E(U|\tau_x = 0, \tau_y \neq 0, t_x = 0) - E(U|\tau_x = 0, \tau_y \neq 0, t_x \neq 0, x \geq x_D)) > 0$ implies that also $\frac{\partial}{\partial x}(E(U|\tau_x = 0, \tau_y \neq 0, t_x = 0) - E(U|\tau_x = 0, \tau_y \neq 0, t_x \neq 0, x < x_D)) > 0$.

To summarize, conditions 1 and 2 above are sufficient for both cases and therefore for the Proposition as a whole.

Claim 1 shows that condition 2 above holds.

So, it is only left to show that condition 1 holds. Since $\beta_1 > \frac{1}{2}$, the LHS of condition 1 is greater than $2(2\beta_1 - 1) > 0$ and the RHS is less than $2\beta_1 - 1$. Therefore the condition holds.[15]

■

Fully informed agent ($\tau_x = \tau_y = 0$)

The only case we still haven't analyzed is the case of a fully informed agent that learns both signals at $t = 0$ ($\tau_x = \tau_y = 0$) and y is sufficiently low such that it will not be disclosed.

The analysis bellow shows that such an agent whose signal x is sufficiently high will disclose at least one signal at $t = 0$. In particular, for low realizations of y (such that y will not be disclosed also at $t = 1$) if x is sufficiently high the agent will disclose it at $t = 0$.

Claim 2 *Assume an agent that learnt both signals at $t = 0$ and the realization of y is such that he does not disclose y. For sufficiently high realizations of x the agent prefers to disclose x at $t = 0$ over not disclosing x at $t = 0$.*

Proof. We need to show that

$$\beta_2[x + h(x,0,0)] + \beta_2[x + h(x,0,1)] > h(0) + \beta_2[x + h(x,1,1)].$$

Rearranging yields

$$\beta_2[x + h(x,0,1)] - h(0) > \beta_2[h(x,1,1) - h(x,0,0)].$$

Since $h(x,0,1)$ is not decreasing for sufficiently high x the LHS of the above inequality, $\beta_2[x + h(x,0,1)] - h(0)$, goes to infinity as x goes to infinity. Therefore, it is sufficient to show that $h(x,1,1) - h(x,0,0)$ is bounded. Both $h(x,1,1)$ and $h(x,0,0)$ are lower than $\beta_2 x$. From the minimum principle we know that $h(x,0,0)$ is higher than the price given no disclosure in a Dye (1985), Jung

[15] We conjecture that the condition holds also for $\beta_1 < 0.5$, however, we have not yet been able to prove that.

and Kwon (1988) setting where $y \sim N(\beta_1 x, Var(y|x))$. The price given no disclosure in such a setting is $\beta_1 x - Cons$, so $h(x,0,0) > \beta_1 x - Cons$. Hence, given that $h(x,1,1) < \beta_1 x$ we have $h(x,1,1) - h(x,0,0) < Const$. QED ■

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