Efficient Sovereign Default

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ABSTRACT

Sovereign debt crises are associated with severe output and consumption losses for the debtor country and with reductions in payments for the creditors. Moreover, such crises are accompanied by trade disruptions that lead to a sharp fall in the imports of intermediate inputs. Here I study the efficient risk-sharing arrangement between a sovereign borrower and foreign lenders in a production economy where the sovereign government cannot commit and has some private information. I show that the ex-ante efficient arrangement involves outcomes that resemble sovereign default episodes in the data. These outcomes are ex-post inefficient, in the sense that if the borrower and the lenders could renegotiate the terms of their agreement, committing not to do it again in the future, then both could be made better off. The resulting efficient allocations can be implemented with non-contingent defaultable bonds and active maturity management. Defaults and periods of temporary exclusion from international credit markets happen along the equilibrium path and are essential to supporting the efficient allocation. Furthermore, as in the data, interest rate spreads increase and the maturity composition of debt shifts toward short-term debt as the indebtedness of the sovereign borrower increases.

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1. Introduction

In this paper, I show that the main aspects of sovereign debt crises can be rationalized as part of the efficient risk-sharing arrangement between a sovereign borrower and foreign lenders in a production economy with informational and commitment frictions. I show that the efficient allocation can be implemented as the equilibrium outcome of a sovereign debt game with non-contingent defaultable debt of multiple maturities. Defaults and *ex-post* inefficient outcomes along the equilibrium path are not a pathology. Rather, they serve the purpose of supporting the *ex-ante* efficient outcome. *Ex-post* inefficient debt crises in the model exhibit many of the characteristics of those in the data.

Specifically, the debt crises in the model are consistent with four key aspects observed in the data. First, sovereign debt crises are associated with severe output and consumption losses for the debtor country. Second, they coincide with trade disruptions. In particular, the drop in imports of intermediate goods is very large relative to the drop that occurs in episodes with recessions of similar magnitudes but without default. Third, after a default economic activity eventually recovers and these recoveries are accompanied by large trade surpluses. Moreover, eventually there is a partial repayment of the defaulted debt, after which the country regains access to international credit markets. Fourth, as the crisis approaches, interest rate spreads on sovereign debt rise and the maturity of this debt gets shorter.[1]

I develop a model to study the optimal risk-sharing arrangement between a sovereign borrower and a large number of foreign lenders. The environment has three main ingredients. First, motivated by the data, I consider a production economy in which imported intermediate inputs are used in production. Second, the sovereign government cannot commit to repaying its debt and the only recourse available to the lenders to ensure repayment is the threat of exclusion from future borrowing, lending, and trade. Third, the sovereign borrower has some private information about the state of the domestic economy.[2]

In the baseline economy, the source of private information is the relative productivity

[1] In Appendix A, I document the first two features of the data and discuss an extensive literature which also documents these key aspects of the data.

[2] The main results of the paper remain valid if I consider an environment in which the incentive problem arises because of moral hazard. For instance, if the sovereign government can take a costly hidden action that increases the probability of higher future realization of aggregate productivity as in Atkeson (1991). Results are available upon request.

of the domestic non-tradable sector. One interpretation of this assumption is that the government has more information about the domestic economy than the foreign lenders and has control over the released statistics and other sources of information. Rogoff (2011) suggests that the lack of explicit state contingencies in international debt contracts can be explained by the fact that sovereign borrowers have enormous discretion over the creation of statistics to be used for indexation of the contingent claims. For simplicity, I consider the extreme case where the foreign lenders have no information about the state of the domestic economy. The main insights of the paper should carry through in the less extreme case in which foreigner lenders receive some noisy signals about the state of the domestic economy. I also show that it is possible to reinterpret the baseline economy as one in which the source of private information is a *taste* shock that affects the marginal utility of consumption of the sovereign borrower, as in Atkeson and Lucas (1992). This reinterpretation is useful for establishing some of the technical results.

To derive the model's implications I proceed in two steps. First, I solve the optimal contracting problem subject to the restrictions imposed by lack of commitment and private information. This problem gives the efficient allocations for output, consumption, imports, and exports. I then implement these allocations as a *sustainable equilibrium* outcome of a sovereign debt game with non-contingent defaultable debt of multiple maturities. I use this implementation to interpret some of the features of these allocations as debt crises and to derive the implications for the maturity composition of debt and for interest rate spreads.

Consider first the optimal contracting problem. At the beginning of any period, the lenders supply intermediate goods to the borrower. Then the productivity of the non-tradable sector is realized. The borrower uses the imported intermediates and domestic labor to produce a non-tradable domestic consumption good and an export good. The export good can be thought of as repayment of past debt. Since the sovereign borrower is risk averse and the foreign lenders are risk neutral, absent contracting frictions the lenders would completely insure the borrower. In particular, when the productivity of the domestic non-tradable sector is low, more resources would be devoted to non-tradables and repayments in the form of exports would be lower. Moreover, the realization of the shock would have no effect on the continuation of the allocation. Private information and the borrower's lack of commitment

limit such insurance.

To understand how the insurance must be limited by the presence of private information, consider an adverse shock to productivity. If the lenders completely insured such a shock, it would be in the borrower's best interest to always claim to have low productivity and devote fewer resources to repaying the lenders, thus eliminating any possibility of insurance. Therefore, in order to make insurance payments to borrowers with currently low productivity shocks incentive compatible, there must be a cost associated with claiming to have low productivity. In a dynamic model, lenders can impose such a cost by reducing the continuation value of the borrower through lowering the borrower's future consumption levels.

In this contracting problem, the lack of commitment interacts with the incentive problem. In particular, when the continuation value of the borrower is low, the borrower is tempted to deviate from the efficient allocation by increasing current consumption by not repaying the amount prescribed and then living in autarky thereafter. To prevent such an outcome, the lenders must provide a sufficiently low amount of intermediate goods so that this kind of deviation is unprofitable.

Enforcing such an outcome is also costly for the lenders. The reason is that, by lending little when the continuation value of the borrower is low, the lender is depressing the production level of the borrower and hence limiting the ability of the borrower to repay the pre-existing debt. Enforcing a continuation value for the borrower close to autarky is *ex-post* inefficient. That is, if the borrower and the lenders could renegotiate the terms of their agreement, committing not to do it again in the future, then both could be made better off. By increasing the borrower's value when it is close to autarky, it is possible to avoid the drop in imported intermediate inputs which depresses production and reduces the ability of the sovereign borrower to repay the lenders. The necessity of providing incentives *ex-ante* requires that these *ex-post* inefficient outcomes happen along the equilibrium path with strictly positive probability. I also show that such *ex-post* inefficiencies are recurrent.

Notice that the interaction between private information and lack of commitment is key for generating this *ex-post* inefficiency. With lack of commitment but without private information, it is optimal to backload consumption and increase the continuation value of the borrower. Thus, the outcomes are *ex-post* efficient. With private information but with

commitment, the statically efficient amount of production can always be sustained; thus, there is no cost to the lenders associated with lowering the borrower's continuation value.

Next, I describe how the model can generate the features of output, consumption, imports, and exports that occur during and after debt crises. The proximate cause of a debt crisis is a sufficiently long string of low productivity shocks which lead the borrower's continuation value to decrease until it is close to the value of autarky. The lack of commitment implies that the imports of intermediates must drop to prevent a deviation by the borrower. This drop in imports reduces output, consumption, and the repayments made to the lenders. Once the economy receives a high productivity shock in the non-tradable sector, output increases, the borrower shifts some resources to the traded sector and uses these resources to run a large trade surplus to repay the foreign lenders. These repayments result in the gradual increase of the borrower's continuation value and hence consumption, production, and imported intermediate inputs used in production also increase in the future.

To interpret these outcome paths as debt crises, I then turn to implementing an efficient allocation as a sustainable equilibrium outcome of a sovereign debt game between a sovereign borrower, competitive foreign lenders, and private domestic agents. There are several ways in which one can implement the efficient allocation. The specific elements I choose are motivated by three key facts about sovereign debt. First, in the data, the vast majority of sovereign and external debt comes in the form of non-contingent debt (see Rogoff (2011) for a discussion). Second, default episodes are infrequent. Third, defaults happen when the sovereign is highly indebted.

Motivated by these facts, I restrict the set of securities that the sovereign borrower can use to non-contingent bonds of multiple maturities. The sovereign borrower has the option to *default*, which I define as suspending the principal and coupon payments specified by the bond contracts. The borrower is excluded from credit markets until at least a partial repayment to the bond holders is made. Finally, the sovereign government can impose a tariff on the imports of intermediate goods, capturing the idea that the sovereign government cannot commit to repay foreign lenders.

Along the equilibrium path that supports an efficient allocation, defaults are recurrent but infrequent and happen when the borrower is highly indebted. In particular, they happen

only when the continuation value for the sovereign is close to the value of autarky. Defaults and periods of temporary exclusion from international credit markets occur at the same time as the *ex-post* inefficient outcomes prescribed by the efficient allocation. In all the other periods, the sovereign government repays in full its debt obligations.

When there is full repayment, in the absence of contingent debt, the state contingent returns implied by the efficient allocation are replicated by exploiting the variation in the price of long-term debt after a shock. After the realization of a low productivity shock, the continuation value for the sovereign borrower decreases and the probability that there will be a default in the near future goes up. The increase in the likelihood of a future default reduces the value of the outstanding long-term debt. This reduction results in a capital gain for the borrower and provides some debt relief after an adverse shock. If the maturity composition of debt is appropriately chosen, this mechanism can exactly replicate the state-contingent returns implied by the efficient allocation.

Along the path approaching default, the maturity composition of the sovereign debt shifts toward short term debt. This shift occurs because, when the probability of future default is high, the price of the long-term debt is more sensitive to shocks. Therefore, a lower long-term debt holding is needed to replicate the debt relief that is implicit in the efficient allocation after a low realization of productivity in the non-tradable sector. Since the overall indebtedness of the sovereign borrower is increasing along the path approaching a default, it must be that the amount of short term debt issued goes up as the probability of default increases. Thus, the maturity composition shortens as indebtedness increases, as is true in the data.

Furthermore, an equilibrium outcome path that supports an efficient allocation is consistent with the evidence on interest rate spreads in emerging economies. In particular, long term spreads are generally higher than short term spreads. During debt crises, the term spread curve inverts, as documented by Broner, Lorenzoni, and Schmukler (2010) and Arellano and Ramanarayanan (2012).

Finally, note that the policy implications of this paper differ from those that can be drawn from the existing literature. First, because *ex-post* inefficient outcomes are part of the efficient allocation, interventions by a supranational authority aimed at reducing the

inefficiencies in a sovereign default episode are not beneficial from an *ex-ante* perspective. Moreover, choosing arrangements that are hard to restructure *ex-post* is consistent with *ex-ante* efficiency. One interpretation of the results in my model is that lack of coordination is desirable. In this sense, attempts by, say, international organizations to coordinate lenders during debt restructuring may lead to worse outcomes from an *ex-ante* point of view. Second, the increasing share of short term debt when a sovereign borrower accumulates external debt can be optimal when only non-contingent defaultable debt of multiple maturities is available. This feature of the data is often blamed as one of the main causes of sovereign or external debt crises. For instance, see Rodrik and Velasco (1999). In sharp contrast, my model generates an endogenous shortening of the maturity structure of the debt. My analysis suggests that interventions that penalize the issuance of short term debt might negatively affect welfare.

Related Literature This paper is related to several strands of literature. First, it is related to the quantitative incomplete market literature on sovereign default. Following Eaton and Gersovitz (1981), recent contributions include Aguiar and Gopinath (2006), Arellano (2008), Hatchondo and Martinez (2009), Benjamin and Wright (2009), Yue (2010) and Chatterjee and Eyigungor (2012a). Defaults are associated with an additional cost in the form of lower endowments in subsequent periods. Mendoza and Yue (2012) assume that this extra cost is driven by a collapse in imports of intermediate goods, as endogenously obtained in this paper. None of these papers can address why the sovereign borrower and the lenders do not renegotiate the terms of the contracts in order to avoid the additional cost associated with default. Even models with explicit renegotiations - such as Yue (2010) and Benjamin and Wright (2009) - impose that the borrower first defaults and incurs the cost in terms of foregone output before the renegotiation can start. I contribute to this strand of the literature by endogenizing the additional default cost and providing a rationale for why it is incurred along the equilibrium path.

This paper makes a contribution to the dynamic contracting literature. Green (1987), Thomas and Worrall (1990) and Atkeson and Lucas (1992) consider environments with only private information. Kehoe and Levine (1993), Thomas and Worrall (1994), Kocherlakota (1996), Kehoe and Perri (2002, 2004), Albuquerque and Hopenhayn (2004), and Aguiar,

Amador, and Gopinath (2009) consider economies with only lack of commitment. In this paper, I combine both contracting frictions and show that their interaction in a production economy can generate cycles with *ex-post* inefficient outcomes.

I build on the seminal contribution in Atkeson (1991) who considers an optimal contracting problem with both lack of commitment and an incentive problem due to the presence of moral hazard. He does not emphasize the role of *ex-post* inefficient outcomes in supporting the efficient arrangement and instead focuses on the downward sloping portion of the utility possibility frontier. Another paper that combines both frictions is Atkeson and Lucas (1995). In their model, there is no cost in terms of production associated with lower continuation values for the borrower. Hence, there are no *ex-post* inefficiencies along the efficient allocation. In parallel work, Ales, Maziero, and Yared (2012) consider both frictions in a similar environment. They allow for the principal to replace the agent; this feature of the environment rules out *ex-post* inefficiencies.

The theme that *ex-post* inefficiencies on-path are necessary to support the *ex-ante* optimal arrangement in economies with incentive problems has been explored in various contexts. For instance, see Green and Porter (1984), Phelan and Townsend (1991), and Yared (2010). In the context of firm dynamics with credit frictions, Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), and DeMarzo and Fishman (2007) show that private information and limited liability can lead to inefficient liquidation. Hopenhayn and Werning (2008) have a similar result for an optimal contracting problem with lack of commitment and private information about the outside option of the agent. A novel feature of my paper is that there is no termination of the risk-sharing relationship. The optimal allocation has periods of temporary autarky (which are *ex-post* inefficient), but cooperation eventually restarts after the domestic economy recovers.

Finally, this paper contributes to the literature that studies the maturity composition of sovereign debt under default risk. Broner, Lorenzoni, and Schmukler (2010) and Arellano and Ramanarayanan (2012) document that the maturity composition of sovereign debt gets shorter when a default is more likely. Moreover, Rodrik and Velasco (1999) find that the ratio of short term debt to reserves is a robust predictor of an external debt crisis. The excessive reliance on short-term debt on the verge of a sovereign debt crisis is often blamed as one

of the main causes of the crisis itself. Models with roll-over risk, such as Cole and Kehoe (2000), provide a rationale for such a prediction because short-term debt is more prone to roll-over risk.[3] I contribute to this literature by showing that the negative correlation between the maturity of outstanding debt and the notional value of debt (and hence the probability of a crisis) emerges as a way to support the efficient allocation. Managing the maturity composition of debt serves to replicate state contingent returns as in Kreps (1982), Angeletos (2004) and Buera and Nicolini (2004). In a related paper, Arellano and Ramanarayanan (2012) endogenize the maturity composition of debt in an Eaton and Gersovitz (1981) type of model. Consistent with my findings, the maturity composition of debt shortens when the probability of default is high. The difference between their paper and mine is that they cannot assess the efficiency of such an equilibrium outcome.

The rest of the paper is organized as follows. In section 2, I describe the baseline environment and its reinterpretation as a taste shock economy. In section 3, I define an efficient allocation. In section 4, I characterize the efficient allocation and show that the *ex-ante* efficient allocation calls for *ex-post* inefficient outcomes. In section 5, I implement the efficient allocation with non-contingent defaultable debt and active maturity management. In section 6, I present an illustrative numerical example and relate the model to the evidence about sovereign default episodes. Finally, section 7 concludes and discusses potential extensions.

2. Environment

In this section, I lay out the baseline environment in which the source of private information is the productivity of the non-tradable sector. Under appropriate conditions, I provide a reinterpretation of this economy as a taste shock economy as in Atkeson and Lucas (1992). Because this formulation is much more tractable, I will use it in the rest of the paper.

A. Baseline Environment: Productivity Shock Economy

Time is discrete and indexed by $t = 0, 1, ...$ There are three types of agents in the economy: a large number of homogeneous domestic households, a benevolent domestic government, and a large number of foreign lenders. There are three goods: a domestic

[3] See Phelan (2004) for a different view.

consumption good (non-traded), an export good, and an intermediate good. There are two sources of uncertainty. They are a shock to the relative productivity of the non-tradable (domestic consumption) sector and a public randomization device, ξ_t, which is distributed uniformly over the interval $[0, 1]$ and is *iid* over time.

Preferences All agents are infinitely lived. The stand-in domestic agent values a stochastic sequence of consumption of the domestic good, $\{y_t\}_{t=0}^{\infty}$, according to

$$(1) \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(y_t)$$

where the period utility function $U : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing and strictly concave, and satisfies standard conditions and $\beta \in (0, 1)$ is the discount factor. The government is benevolent and so it shares the same preferences as the domestic households.

Foreign lenders are risk neutral and they value consumption of the export good . They discount the future with a discount factor $q \in (0, 1)$, which should be thought of as the inverse of the risk-free interest rate in international credit markets. I allow the discount factor β and q to differ, but I will restrict myself to the case where $q \geq \beta$; that is, the domestic households discount the future at a (weakly) higher rate than the international interest rate.

Endowments and Technology Foreign lenders have a large endowment of the intermediate good. They have access to a technology that transforms one unit of the intermediate good into one unit of the export good so that the relative price between the export and the intermediate good is fixed at one.

Each domestic agent is endowed with one unit of labor in each period. There is a domestic production technology that transforms the intermediate good and labor into domestic consumption good, y, and foreign consumption good, y^*, as follows:

$$(2) \quad y \leq zF(m_1, \ell_1)$$

$$(3) \quad y^* \leq F(m_2, \ell_2)$$

where

$$(4) \quad m_1 + m_2 \leq m, \quad \ell_1 + \ell_2 \leq 1$$

and m_1 and m_2 are the units of the intermediate good allocated to the production of the domestic and export good respectively, m is the total amount of intermediates used domestically, and ℓ_1 and ℓ_2 are the units of domestic labor allocated to domestic and export production respectively. The production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ has constant returns to scale; it is such that $F(0,1) > 0$, so that strictly positive output can be produced in autarky; and it satisfies the Inada condition $\lim_{m \to 0} F_m(m, \ell) = +\infty \; \forall \ell > 0$, as well as other standard conditions. Let $f(m) = F(m, 1)$. The relative productivity of the domestic sector, $z \in Z \equiv \left\{z^1, z^2, ..., z^N\right\}$ with $N < \infty$, is distributed according to a probability distribution μ and it is *iid* over time. Without loss generality, let $z^i < z^j$ if $i < j$. By properties of constant returns to scale technology, the technological restrictions imposed by (2)-(4) can be summarized by the following aggregate resource constraint

$$(5) \quad \frac{y}{z} + y^* \leq f(m)$$

along with the non-negativity conditions on y and y^*.

Timing The timing of events within the period is as follows:

1. The public randomization device $\xi_t \in [0, 1]$ is realized;
2. Foreign lenders supply intermediate goods $m_t \geq 0$;
3. z_t is realized according to μ;
4. Real activity occurs: production, consumption, and exporting take place.

Let $s_t = (\xi_t, z_t)$ and $s^t = (s_0, s_1, ..., s_t)$. An allocation for this economy is a stochastic process $\mathbf{x}_z \equiv \{m(s^{t-1}, \xi_t), y(s^t), y^*(s^t)\}_{t=0}^{\infty}$. An allocation $\mathbf{x}_z$ is feasible if it satisfies (5).

Information Foreign lenders observe the amount of intermediate goods that the country imports, m, and the amount of exports, y^*. Moreover, they can observe the amount of

resources, m_1 and ℓ_1, employed in the domestic consumption (non-tradable) sector. They cannot see the amount of output produced with the inputs because the realization of z is privately observed by the domestic government. From (5), they can use their information about m and y^* to infer y/z but not y and z separately. Next, I will show how under appropriate assumptions I can relabel the variables in this economy to obtain an equivalent taste shock formulation.

B. Reinterpretation: Taste Shock Economy

Suppose that the period utility function displays constant relative risk aversion:

$$(6) \quad U(y) = \frac{y^{1-\gamma}}{1-\gamma}$$

with $\gamma > 1$. Under this assumption, I can rewrite the baseline environment as an economy with two goods - a final and an intermediate good - where the stand-in domestic household is subject to a taste shock $\theta_t \in \Theta \equiv \{\theta^1, \theta^2, ..., \theta^N\} = \{(z^N)^{1-\gamma}, (z^{N-1})^{1-\gamma}, ..., (z^1)^{1-\gamma}\}$, which is *iid* over time and is privately observed by the domestic agent. For notational convenience, let $\theta_L = \theta^1$ and $\theta_H = \theta^N$. The taste shock affects the domestic agent's marginal utility of consumption in a multiplicative fashion; a higher θ_t makes current consumption more valuable. A high taste shock corresponds to a low productivity shock in the original baseline economy. Intuitively, after either a high taste shock or a low productivity shock in the non-tradable sector, the marginal utility of imported intermediates used for domestic consumption is high. Define

$$(7) \quad c = \frac{y}{z} \quad \text{and} \quad \theta = z^{1-\gamma}$$

where c is domestic consumption and θ is a taste shock. With some abuse of notation, let $s_t = (\xi_t, \theta_t)$. Under (6), I can write the preferences for a stand-in domestic agent over a stochastic sequence $\{c_t(s^t)\}_{t=0}^{\infty}$ as

$$(8) \quad \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) \theta_t U\left(c_t(s^t)\right)$$

From (5), the resource constraint for this economy can be written as:

$$\text{(RC)}\quad c(s^t) + y^*(s^t) \leq f\left(m(s^{t-1}, \xi_t)\right)$$

where y^* are exports as in the productivity shock formulation. An allocation for this taste shock economy is a stochastic process $\mathbf{x} \equiv \{m(s^{t-1}, \xi_t), c(s^t), y^*(s^t)\}_{t=0}^{\infty}$. The allocation is feasible if it satisfies (RC). Clearly, if $\mathbf{x}$ is feasible then $\mathbf{x}_z = \{m(s^{t-1}, \xi_t), c(s^t)\theta_t^{1/(1-\gamma)}, y^*(s^t)\}_{t=0}^{\infty}$ is feasible for the baseline economy[4] and viceversa.

In the rest of the paper, I will present the results using this taste shock formulation. I assume that the primitives of the taste shock economy satisfy the following conditions:

ASSUMPTION 1. $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is C^2, strictly increasing, has constant returns to scale and is such that $F(0,1) > 0$ and $\lim_{m\to 0} F_m(m, \ell) = +\infty \ \forall \ell > 0$. Furthermore, $f'(m) > 0$ and $f''(m) < 0$. $U : \mathbb{R}_+ \to \mathbb{R}$ is C^2, strictly increasing and strictly concave, satisfies $\lim_{c\to 0} U'(c) = \infty$, $\lim_{c\to\infty} U'(c) = 0$ and is such that $\theta_L U(0) + \beta v^{**} < \theta_L U\left(f(0)\right) + \beta v_a$ where $m^* \equiv f'^{-1}(1)$ and $v^{**} \equiv \left[\theta_H U(f(m^*)) + \beta v_a\right] / \left[\theta_H(1-\beta) + \beta\right]$. Finally, $q \in [\beta, \min\{\beta/\theta_L, 1\})$.

3. Efficient Allocation

In this section, I first define a (constrained) efficient allocation under the two contracting frictions. Then, I show that it solves a nearly recursive problem.

A. Definition

Private information and lack of commitment by the sovereign borrower impose constraints in addition to the resource constraint (RC) which an allocation must satisfy in order to be implementable. Consider first the restriction imposed by the fact that θ is privately observed by the borrower. By the *revelation principle*, it is without loss of generality to focus on the direct revelation mechanism in which the sovereign borrower reports his type. Define the continuation utility for the sovereign borrower associated with the allocation $\mathbf{x}$

[4] This statement is true under the requirement that $y^* \geq 0$. In characterizing the efficient allocation for the taste shock economy, I abstract from this constraint for simplicity. This is not affecting any of the results in the paper as this constraint is potentially binding outside the region of interest.

after history s^t (according to truth-telling) as:

$$(9) \quad v(s^t) \equiv \sum_{j=1}^{\infty} \sum_{s^{t+j}} \beta^{j-1} \Pr(s^{t+j}|s^t) \theta_{t+j} U(c(s^{t+j}))$$

An allocation $\mathbf{x}$ is incentive compatible if and only if it satisfies the following (temporary) *incentive compatibility constraint* $\forall t, s^{t-1}, \xi_t, \theta_t, \theta'$:

$$\text{(IC)} \quad \theta_t U(c(s^{t-1}, \xi_t, \theta_t)) + \beta v(s^{t-1}, \xi_t, \theta_t) \geq \theta_t U\left(c(s^{t-1}, \xi_t, \theta')\right) + \beta v(s^{t-1}, \xi_t, \theta')$$

That is, after any history, there are no gains from the borrower reporting $\theta' \neq \theta_t$.

Second, because the sovereign borrower lacks commitment, to be implementable an allocation $\mathbf{x}$ must satisfy the following *sustainability constraint* $\forall t, s^t$:

$$\text{(SUST)} \quad \theta_t U(c(s^t)) + \beta v(s^t) \geq \theta_t U(f(m(s^{t-1}, \xi_t))) + \beta v_a$$

where v_a is the value of autarky

$$(10) \quad v_a \equiv \frac{\sum_{\theta \in \Theta} \mu(\theta) \theta U(f(0))}{1 - \beta} = \frac{\mathbb{E}(\theta) U(f(0))}{1 - \beta}$$

That is, after any history, the borrower cannot gain from increasing his consumption by failing to export $y^*(s^t)$ and living in autarky forever after. As is standard in the literature, I assume that after this observable deviation, the borrower is punished with autarky. This entails two forms of punishment. First, the sovereign borrower cannot access credit markets to obtain insurance. Second, he suffers a loss in production because he cannot use imported intermediate goods.

A feasible allocation $\mathbf{x}$ is said to be *efficient* if it maximizes the present value of net transfers to the foreign lenders, $y^* - m$, subject to (RC), (IC), (SUST), and a participation constraint for the borrower:

$$\text{(PC)} \quad \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) \theta_t U(c(s^t)) \geq v_0$$

for some feasible initial level of promised utility $v_0 \in [v_a, \bar{v}]$, with $\bar{v} \equiv \lim_{c\to\infty} \frac{\mathbb{E}(\theta)U(c)}{1-\beta}$. An efficient allocation solves:

$$\text{(J)} \quad J(v_0) = \max_{\mathbf{x}} \sum_{t=0}^{\infty} \sum_{s^t} q^t \Pr(s^t) \left[y^*(s^t) - m(s^{t-1}, \xi_t) \right]$$

subject to (RC), (PC), (IC) and (SUST). I will refer to $J : [v_a, \bar{v}] \to \mathbb{R}$ as the *Pareto frontier*.

B. Near Recursive Formulation

The problem in (J) admits a nearly recursive formulation using the borrower's promised utility, v, as a state variable. From $t \geq 1$, an efficient allocation solves the following recursive problem for $v \in [v_a, \bar{v}]$:

$$\text{(P)} \quad B(v) = \max_{m(\xi), c(\xi,\theta), v'(\xi,\theta)} \int_0^1 \left\{ f\left(m(\xi)\right) - m(\xi) + \sum_{\theta\in\Theta} \mu(\theta) \left[-c(\xi,\theta) + qB(v'(\xi,\theta)) \right] \right\} d\xi$$

subject to

$$\text{(11)} \quad \int_0^1 \left\{ \sum_{\theta\in\Theta} \mu(\theta) \left[\theta U(c(\xi,\theta)) + \beta v'(\xi,\theta) \right] \right\} d\xi = v$$

$$\text{(12)} \quad \theta U(c(\xi,\theta)) + \beta v'(\xi,\theta) \geq \theta U(c(\xi,\theta')) + \beta v'(\xi,\theta') \quad \forall \xi, \forall \theta, \theta'$$

$$\text{(13)} \quad \theta U(c(\xi,\theta)) + \beta v'(\xi,\theta) \geq \theta U\left(f(m(\xi))\right) + \beta v_a \quad \forall \xi, \forall \theta$$

$$\text{(14)} \quad v'(\xi,\theta) \geq v_a \quad \forall \xi, \forall \theta$$

where $B(v)$ is the maximal present discounted value of net transfers, $y^* - m = f(m) - c - m$, that the foreign lenders can attain subject to the constraint that the recursive allocation delivers a value of v to the sovereign borrower (the *promise keeping constraint*), (11), a recursive version of the incentive compatibility constraint, (12), a recursive version of the sustainability constraint, (13), and the fact that continuation utility must be greater than the value of autarky, (14). The function B traces out the *utility possibility frontier*.

At $t = 0$, for all $v_0 \in [v_a, \bar{v}]$ the problem in (J) can be expressed as

$$\text{(15)} \quad J(v_0) = \max_{m(\xi), c(\xi,\theta), v'(\xi,\theta)} \int_0^1 \left\{ f\left(m(\xi)\right) - m(\xi) + \sum_{\theta\in\Theta} \mu(\theta) \left[-c(\xi,\theta) + qB(v'(\xi,\theta)) \right] \right\} d\xi$$

subject to (12), (13), (14), and the participation constraint

$$(16)\quad \int_0^1 \left\{ \sum_{\theta\in\Theta} \mu(\theta) \left[\theta U(c(\xi,\theta)) + \beta v'(\xi,\theta)\right] \right\} d\xi \geq v$$

The difference between J and B is that, in B, I require that the allocation delivers *exactly* the promised utility $v \in [v_a, \bar{v}]$ to the sovereign borrower - see (11). This is because for $t \geq 1$ the promise keeping constraint serves to maintain incentives from previous periods. In contrast, in the definition of the Pareto frontier J, the participation constraint (16) requires that the sovereign borrower receives *at least* v. In many applications, this asymmetry is irrelevant because the participation constraint in (J) is binding. This is not the case here because $B(v)$ has an increasing portion, as I will later show.

The constraint set in (P) is not necessarily convex because of the presence of $U \circ f(m)$, a concave function, on the right hand side of the sustainability constraint (13). Thus, randomization may be optimal. The programming problem in (P) can be represented as follows:

$$(\text{P'})\quad B(v) = \max_{\zeta\in[0,1], v_1, v_2 \in [v_a,\bar{v}]} \zeta \hat{B}(v_1) + (1-\zeta)\hat{B}(v_2) \quad \text{s.t.} \quad \zeta v_1 + (1-\zeta)v_2 = v$$

where

$$(\hat{\text{P}})\quad \hat{B}(v) = \max_{m, c(\theta), v'(\theta)} \sum_{\theta\in\Theta} \mu(\theta) \left[f(m) - m - c(\theta) + qB(v'(\theta))\right]$$

subject to

$$(17)\quad \sum_{\theta\in\Theta} \mu(\theta) \left[\theta U(c(\theta)) + \beta v'(\theta)\right] = v$$

$$(18)\quad \theta U(c(\theta)) + \beta v'(\theta) \geq \theta U(c(\theta')) + \beta v'(\theta') \quad \forall \theta, \theta'$$

$$(19)\quad \theta U(c(\theta)) + \beta v'(\theta) \geq \theta U(f(m)) + \beta v_a \quad \forall \theta$$

$$(20)\quad v'(\theta) \geq v_a \quad \forall \theta$$

$\hat{B}$ is the maximal value that the lenders can attain *without* using randomization in the

current period and using the convexified value for B to evaluate the continuation value. For any $v \in [v_a, \bar{v}]$, the value of $B(v)$ can be obtained from $\hat{B}$ using (P') where, without loss of generality, the randomization is between two values. It is possible to rule out randomization as part of the efficient allocation by making an additional assumption as in Aguiar, Amador, and Gopinath (2009).

ASSUMPTION 2. Define $H : [U(f(0)), U(f(m^*))] \to \mathbb{R}$ as $H(\underline{u}) \equiv C(\underline{u}) - f^{-1} \circ C(\underline{u})$ with $C = U^{-1}$. H is concave.

If Assumption 2 is satisfied[5], then randomization is not optimal, $B(v) = \hat{B}(v)$ for all $v \in [v_a, \bar{v}]$ and the solution to $(\hat{\text{P}})$ is unique and continuous in the borrower's promised utility (see the appendix for a proof). When Assumption 2 does not hold, it is not guaranteed that the maximizer of $(\hat{\text{P}})$ is unique. I will assume that this is the case. In the next section, I characterize the solution to (P) using the equivalent representation given by (P')-$(\hat{\text{P}})$.

4. Characterization: Optimality of Ex-Post Inefficiencies

In this section, I establish that under certain conditions an efficient allocation has cyclical periods with *ex-post* inefficient outcomes which resemble a sovereign default episode in the data. First, I show that the value function for the lenders (the utility possibility frontier) has an upward sloping portion. I call this the *region with ex-post inefficiencies* because both agents could do better. Second, I show that under appropriate assumptions the process for the borrower's continuation value implied by the efficient allocation transits with strictly positive probability to the region with *ex-post* inefficiencies. Finally, I show that if $q > \beta$, there is a unique non-degenerate limiting distribution and the region with *ex-post* inefficiencies is part of its support. Thus, cycles with *ex-post* inefficient outcomes persist in the long-run.

A. Preliminaries

Before moving to the main results of this section, I first establish some preliminary results. The constraint set in $(\hat{\text{P}})$ can be simplified as follows:

[5] Assumption 2 is satisfied if the curvature in U and f is low.

LEMMA 1. Under Assumption 1: (i) only local upward incentive compatibility constraints bind at a solution to $(\hat{\text{P}})$; (ii) if $(m, c(\theta), v'(\theta))$ is incentive compatible and sustainable for θ_H, then it satisfies the sustainability constraint for all $\theta \in \Theta$.

Proof. Appendix. □

Part (i) states that the relevant incentive compatibility constraints are the ones for which the borrower of type θ_i wants to report being of type θ_{i+1} where $\theta_{i+1} > \theta_i$. This is because the efficient allocation provides more current consumption after a higher taste shock. This result is standard; see for instance Thomas and Worrall (1990). Part (ii) states that the only relevant sustainability constraint is the one for the highest taste shock type, θ_H. This is not standard. Models with lack of commitment and no incentive problem typically display the opposite binding pattern.

The next proposition establishes three properties of the efficient allocation that I will later use.

PROPOSITION 1. *Under Assumption 1, the efficient allocation is such that:*

(i) There are distortions in production. Let m^* be the statically efficient level of intermediates, i.e. m^* such that $f'(m^*) = 1$. There exists $v^* \in (v_a, \bar{v})$ such that $m(v) = m^*$ for all $v \geq v^*$, $m(v) < m^*$ for all $v \in [v_a, v^*)$, and in particular $m(v_a) = 0$. Moreover, if Assumption 2 holds, then $m(v)$ is strictly increasing in v over $[v_a, v^*]$.

(ii) The efficient allocation is dynamic: $\forall v \in [v_a, \bar{v}]$, $c(v, \theta_H) > c(v, \theta_L)$ and $v'(v, \theta_H) < v'(v, \theta_L)$.

(iii) There is subsidization across states. Let $b(v, \theta) \equiv y^*(v, \theta) - m(v) + qB(v'(v, \theta))$ be the lenders' value after the realization of θ. $\forall v \in [v_a, \bar{v}]$ and for all $\theta' > \theta$, $b(v, \theta) \geq b(v, \theta')$. In particular, $b(v, \theta_L) > b(v, \theta_H)$.

Proof. Appendix. □

Part (i) states that low levels of promised utility for the borrower are associated with imported intermediates that are below the statically efficient level, m^* such that $f'(m^*) = 1$. When the continuation value for the borrower is low, imports must be low to satisfy the

sustainability constraint. Whenever the sustainability constraint is binding $m < m^*$. In particular, at autarky it must be that $m(v_a) = 0$. In fact, if the foreign lenders supplied any $m > 0$, the sovereign government could unilaterally achieve a life-time utility of $U(f(m)) + \beta v_a > U(f(0)) + \beta v_a = v_a$. Thus, only $m = 0$ is consistent with the promise keeping and sustainability constraints at autarky. On the other hand, for continuation values high enough, $v \geq v^*$, the threat of autarky after an observable deviation is sufficiently harsh that the statically efficient amount of intermediate imports can be supported, $m(v) = m^*$ for all $v \geq v^*$. If Assumption 2 is satisfied, it can be shown that m is actually strictly increasing in the borrower's promised value for $v \in [v_a, v^*]$. Part (ii) states that the efficient allocation is *dynamic*, in the sense that it uses variation in the borrower's continuation utility to provide incentives, thus allowing for higher consumption after the realization of a higher taste shock. Part (iii) shows that the present value of payments received by the lenders is state-contingent; there is debt relief when the borrower has a high marginal utility of consumption. Thus, the efficient allocation provides some insurance, albeit imperfect.

B. Optimality of Ex-Post Inefficiencies

I now turn to the main result of this section: an efficient allocation calls for *ex-post* inefficient outcomes with strictly positive probability, provided that a sufficient condition is satisfied.

Region with Ex-Post Inefficiencies The next proposition establishes that the utility possibility frontier is upward sloping for borrower values that are close to autarky.

PROPOSITION 2. *[Region with ex-post inefficiencies]* $\exists\ \tilde{v} \in (v_a, v^*)$ *such that* $B(v)$ *is strictly increasing over* $[v_a, \tilde{v})$ *and decreasing over* $[\tilde{v}, \bar{v}]$.

Proof. Appendix. □

I refer to the interval $[v_a, \tilde{v})$ as the *region with ex-post inefficiencies* because for all $v \in [v_a, \tilde{v})$ the lenders can attain a strictly higher value by providing higher utility to the borrower. This is because supporting a continuation value for the borrower that is close to the autarkic level requires that a very low level of intermediate goods is employed in

production so that the sustainability constraint (19) is satisfied. This depresses production and, consequently, the repayments that the lenders can receive in the period. In particular, when the borrower's value is close to autarky, intermediates are close to zero (see Proposition 1 part (i)). Thus, because of the Inada condition on f, the marginal return from additional intermediates is large enough that the benefit from extra production that can be obtained by increasing the borrower's continuation value is larger than the cost to the lender of providing the additional value. Therefore, both agents can be made better off from autarky and B is upward sloping in a neighborhood of v_a. In contrast, for sufficiently high promised values, $v \geq v^*$, the statically efficient level of intermediates can be supported. For such promised values, increasing the borrower's value is costly and has no benefit for the lenders and so B is strictly decreasing for $v \geq v^*$. Therefore, because of the concavity of B, the utility possibility frontier must peak at some $\tilde{v} \in (v_a, v^*)$. Over the interval $[\tilde{v}, \bar{v}]$, which I will refer to as the *efficient region*, B is decreasing.

In the rest of the paper, I will assume that randomization may only occur in the region with *ex-post* inefficiencies:

ASSUMPTION 3. An efficient allocation is such that if randomization is optimal, there exists $v_r \in (v_a, \tilde{v})$ such that (i) for $v \in (v_a, v_r)$, it is optimal to randomize between v_a and v_r, and (ii) if $v \geq v_r$ then there is no randomization.

Assumption 3 states that if there is a *randomization region* (linear portion of B), this is given by the set $[v_a, v_r]$ with $v_r < \tilde{v}$. In this case, for all $v \in [v_a, v_r]$, let

$$\zeta(v) = \frac{v_r - v}{v_r - v_a} \tag{21}$$

be the probability that continuation utility after randomization is equal to v_a. With probability $1 - \zeta(v)$ the post-randomization continuation value is equal to v_r. This pattern for randomization is what I find in any computed example.

The Efficient Allocation Transits to the Region with Ex-Post Inefficiencies Any efficient allocation starts in the efficient region because the participation constraint, (PC), in (J) can hold as an inequality. For any borrower value, v, in the region with *ex-post*

inefficiencies, (PC) does not bind and $J(v) = J(\tilde{v}) = B(\tilde{v}) > B(v)$. It is optimal for the lenders to promise at least $\tilde{v}$ to the borrower. Instead, for v in the efficient region (PC) in (J) binds and $J(v) = B(v)$. These results are illustrated in Figure 2. The question now is: Does an efficient allocation transit to the region with *ex-post* inefficiencies after some history? Or is the efficient region an ergodic set?

Provided that a *sufficient* condition is satisfied, the continuation of any efficient allocation transits to the region with *ex-post* inefficiencies after a sufficiently long (but finite) string of realizations of θ_H. The essential piece of the argument is to show that following a realization of θ_H, the continuation utility is strictly lower than the current one: $v'(v, \theta_H) < v$. This is not obvious because there is a tension between two countervailing forces. First, there is an *incentive effect* that calls for lowering $v'(v, \theta_H)$ below v. This is because lowering the continuation utility after a high taste shock helps to separate types and to provide more current consumption when the marginal utility of consumption is high. Second, there is a countervailing *commitment effect*: lowering the continuation utility tightens future sustainability constraints. As is standard in economies with only lack of commitment, there is a motive to backload payments to the sovereign borrower in order to relax future sustainability constraints and allow for lower production distortions in the future.

Two point support For simplicity, in the rest of this section, I consider the case with $N = 2$, $\Theta \equiv \{\theta_L, \theta_H\}$. Let $c_H(v) = c(v, \theta_H)$, $v'_H(v) = v'(v, \theta_H)$, $c_L(v) = c(v, \theta_L)$, and $v'_L(v) = v'(v, \theta_L)$. Let $x = (c_L, c_H, v'_L, v'_H, m)$ be the solution to $(\hat{\mathrm{P}})$ for some v in the efficient region for which randomization is not optimal, $B(v) = \hat{B}(v)$. In order to see why the continuation value after a high taste shock must be lower than the borrower's current value and to understand the tensions in the model, consider the following variation: for some $\varepsilon \in \mathbb{R}$ sufficiently close to zero, decrease v'_H and v'_L by ε/β and increase c_H and c_L such that both $U(c_H)$ and $U(c_L)$ increase by ε. This variation satisfies the relevant incentive compatibility constraint and the promise keeping constraint under the normalization $\mathbb{E}(\theta) = 1$. Moreover, it also relaxes the relevant sustainability constraint because it delivers a higher value after the realization of θ_H. This value increases by $(\theta_H - 1)\,\varepsilon$. Therefore, the amount of intermediates can be increased by $\varepsilon_m = \varepsilon(\theta_H - 1)/\left[\theta_H U'(f(m)) f'(m)\right]$. Since x is optimal, the change in

the lenders' value from this variation must be equal to zero:

$$(22)\quad 0 = \frac{\Delta B}{\varepsilon} \approx -\left[\frac{\mu_H}{U'(c_H)} + \frac{\mu_L}{U'(c_L)}\right] - \frac{q}{\beta}\left[\mu_H B'(v'_H) + \mu_L B'(v'_L)\right] + \frac{(\theta_H - 1)\left(f'(m) - 1\right)}{\theta_H U'(f(m))f'(m)}$$

where the first term in square brackets is the cost of providing more consumption in the current period, the second is the benefit (or cost if v'_j is in the region with *ex-post* inefficiencies and $B'(v'_j) > 0$) of reducing continuation values, and the last term is the benefit from relaxing the current sustainability constraint. Using the fact that (see Lemma 7 in the appendix for the derivation)

$$(23)\quad B'(v) = -\left[\frac{\mu_H}{U'(c_H)} + \frac{\mu_L}{U'(c_L)}\right] + \frac{f'(m) - 1}{U'(f(m))f'(m)}$$

the expression in (22) can be rearranged as follows:

$$(24)\quad B'(v) = \frac{q}{\beta}\left[\mu_H B'(v'_H) + \mu_L B'(v'_L)\right] + \frac{f'(m) - 1}{\theta_H U'(f(m))f'(m)}$$

or equivalently, using the fact that $B'(v) \leq 0$ and $\beta \leq q$, as:

$$(25)\quad \left[B'(v_H) - B'(v)\right] \geq \mu_L\left[B'(v'_H) - B'(v'_L)\right] - \frac{\beta}{q}\frac{f'(m) - 1}{\theta_H U'(f(m))f'(m)}$$

Equation (25) illustrates the two forces operating in the model. The first term in square brackets on the right hand side of (25) stands in for the incentive effect, while the second term stands in for the commitment effect. First notice that by the concavity of B, if the right hand side of (25) is positive, then it must be that $v'_H(v) < v$. By Proposition 1 part (ii), $v'_L > v'_H$ and thus the first term on the right hand side of (25) is strictly positive. Absent any commitment problem, $f'(m) = 1$, the second term on the right hand side of (25) is equal to zero. Therefore the right hand side is positive and consequently $v'_H(v) < v$. When the sustainability constraint binds, $f'(m) > 1$, the second term on the right hand side of (25), $-\left[f'(m) - 1\right] / \left[\theta_H U'(f(m))f'(m)\right]$, is negative; it is then not obvious that the right hand side of (25) is positive. Thus, in this case it is not guaranteed that $v'_H(v) < v$. The next assumption guarantees that this is indeed the case.

ASSUMPTION 4. $\Theta = \{\theta_L, \theta_H\}$ and either (i) the difference between θ_L and θ_H is sufficiently large or (ii) μ_H is sufficiently small.

LEMMA 2. Under Assumptions 1, 3, and 4, $\forall v \in [\tilde{v}, \bar{v}]$ $v'_H(v) < v$.

Proof. Appendix. □

Suppose that either (i) the difference between θ_H and θ_L is sufficiently large or (ii) the probability of being in state θ_H is sufficiently small. Intuitively, if $\theta_H - \theta_L$ is sufficiently large, the benefit of separating the two types is large. It is very cheap to satisfy the promise keeping constraint by providing consumption when $\theta = \theta_H$. To provide a large spread in current consumption across types in an incentive compatible way, i.e. such that $\beta [v'_L - v'_H] \geq \theta_L [U(c_H) - U(c_H)]$, it is necessary to have a large spread in continuation values, $v'_L - v'_H$. Thus, the first term in the right hand side of (25) is large. Moreover, if μ_H is small, the cost of tightening future sustainability constraints by reducing the continuation value after θ_H is small from an *ex-ante* perspective. Inspecting (25), if μ_H is low, then the first term on the right hand side is again large. Thus, if either (i) $\theta_H - \theta_L$ is sufficiently large or (ii) μ_H is sufficiently small, the benefits from lowering $v'_H(v)$ below v by relaxing the incentive compatibility constraint and the current sustainability constraint (incentive effect) are larger than the costs that arise from higher production distortions in the future after a high taste shock (commitment effect). In the appendix, I show how Lemma 2 can be extended to the general case with $N \geq 2$.

Under the assumptions in Lemma 2, for all v in the efficient region, $v'_H(v)$ lies *strictly* below the 45 degree line, as illustrated in Figure 3. Let $\Delta \equiv \min_{v \in [\tilde{v}, \bar{v}]} \{v - v'_H(v)\}$. By continuity of $v'_H(v)$, it follows that $\Delta > 0$. Thus, starting from any $v_0 \in [\tilde{v}, \bar{v})$, after a sequence of t consecutive realizations of θ_H, the borrower's continuation value is less than $v_0 - \Delta t$. Thus, after a sufficiently long string $\theta^T = (\theta_H, \theta_H, ..., \theta_H)$ with $T \leq (v_0 - \tilde{v})/\Delta$ finite, the continuation utility transits to the region with *ex-post* inefficiencies, $[v_a, \tilde{v})$. The next proposition summarizes the argument above:

PROPOSITION 3. *Under Assumptions 1, 3 and 4, an ex-ante efficient allocation transits to the region with ex-post inefficiencies with strictly positive probability.*

Proof. It follows from Lemma 2 and the discussion above. □

Role of the Main Ingredients The interaction between lack of commitment and private information is key to having *ex-post* inefficient outcomes happening along the path. Both lack of commitment and the fact that intermediates are used in production are crucial to generating an upward sloping portion of the utility possibility frontier. However, these two features alone cannot generate *ex-post* inefficient outcomes associated with an *ex-ante* efficient allocation. Without an incentive problem, any continuation of an efficient allocation is itself efficient. Thus, an efficient allocation never transits to the region with *ex-post* inefficiencies of the utility possibility frontier. See Aguiar, Amador, and Gopinath (2009) for this result in a related environment.

Private information alone generates a downward drift of the continuation utility - see Thomas and Worrall (1990) and Atkeson and Lucas (1992) - but does not generate *ex-post* inefficiencies because with commitment there is no connection between low continuation values and production in the economy. The statically efficient amount of production can always be sustained. Low continuation values for the borrower only have distributional effects in that the lenders can appropriate larger shares of total undistorted production. Also in this case, continuations of efficient allocations are always on the Pareto frontier.

Both contracting frictions are needed to obtain *ex-post* inefficient outcomes as part of the *ex-ante* optimal arrangement (Proposition 3). Lack of commitment is crucial for having an upward sloping portion of the utility possibility frontier (Proposition 2); private information is crucial for having the efficient allocation to transit to the region with *ex-post* inefficiencies (Lemma 2).[6]

C. Long-Run Properties

In this section, I show that if the sovereign borrower is more impatient than the foreign lenders, $\beta < q$, then any efficient allocation converges to a unique non-degenerate stationary distribution. Moreover, under the assumptions in Lemma 2, *ex-post* inefficient outcomes are

[6]Notice that if preferences are of the form $U(c, g; \theta) = U(c) + \theta G(g)$ where for instance c is private consumprion and g is public consumption and θ is private information but c is observable, then in this case the efficient region is an ergodic set. If there is one lever other than the continuation utility to use in order to provide incentives, then continuations of efficient allocation are efficient.

part of the long-run behavior of the economy.

For expositional simplicity, I consider again the case with a two point support.[7] As a first step to establishing the existence of a non-degenerate stationary distribution, I show that under Assumption 1 autarky is a reflecting point and not an absorbing state.

LEMMA 3. Under Assumption 1, at $v = v_a$ after θ_H it must be that $c_H(v_a) = f(0)$ and $v'_H(v_a) = v_a$. Instead, after θ_L it must be that $c_L(v_a) < f(0)$ and $v'_L(v_a) > v_a$.

Proof. Appendix. □

Lemma 3 characterizes the efficient allocation when the borrower's value is equal to autarky. If the borrower draws θ_H, his consumption is equal to production in autarky, $f(0)$, and his continuation value is equal to autarky. When θ_L is drawn, the borrower's valuation of current consumption is low. Therefore it is efficient to deliver the value of autarky, $\theta_L U(f(0)) + \beta v_a$, by providing lower consumption in the current period, $c_L(v_a) < f(0)$, and increasing the borrower's continuation value, $v'_L(v_a) > v_a$. Thus, autarky is not absorbing.

The next lemma completes the characterization of the law of motion for promised utility implied by the efficient allocation:

LEMMA 4. Under Assumptions 1 and 3, (i) for all $v \in [v_a, \tilde{v})$, $v'_L(v) \geq \tilde{v}$ and $v'_L(\tilde{v}) > \tilde{v}$. (ii) If $\beta = q$ then $\forall v \in (\tilde{v}, \bar{v}]$, $v'_L(v) > v$. (iii) Instead, if $\beta < q$ then there exists $\bar{v}_q \in (\tilde{v}, \bar{v})$ such that for all $v > \bar{v}_q$, $v'_L(v) < v$.

Proof. Appendix. □

Part (i) states that if v is in the region with *ex-post* inefficiencies, it transits to the efficient region the first time that θ_L is drawn, $v'_L(v) \geq \tilde{v}$ for all $v \in [v_a, \tilde{v}]$. For borrower values in the efficient region, I have to consider two cases. If $\beta = q$, part (ii) establishes that $v'_L(v) > v$ for all v. This is because lenders and the sovereign borrower discount the future at the same rate and it is optimal for incentive provision to increase continuation utility after θ_L is drawn. Finally, if the borrower is more impatient than the lenders, $\beta < q$, part (iii)

[7] Lemma 3 and Proposition 4 also hold for the general case with $|\Theta| \geq 2$. In particular, $c(v_a, \theta) = f(0)$ and $v'(v_a, \theta) = v_a$ for all $\theta \in \Theta \backslash \{\theta_L\}$, see the appendix.

states that there is a threshold, $\bar{v}_q$, after which it is optimal to have $v'_L(v) < v$. The relative impatience of the borrower eventually dominates the incentive benefits from backloading payments after θ_L.

Under Assumption 4, the laws of motion for the continuation utility, v'_L and v'_H, are shown in Figure 3 for the case $\beta < q$. They define a unique ergodic set for promised utility. The following proposition establishes this result:

PROPOSITION 4. *Under Assumptions 1, 3, and 4, if $\beta < q$, then any efficient allocation converges to a unique non-degenerate stationary distribution, Ψ^*. Moreover, $[v_a, \tilde{v}] \cap Supp\Psi^* \neq \emptyset$.*

Proof. Appendix. □

If $\beta < q$ then, by Lemma 4, continuation utility does not grow without bound. Moreover, by Lemma 2, after a sufficiently long - and finite - string of draws of θ_H, continuation utility transits to the region with *ex-post* inefficiencies. The region with *ex-post* inefficiencies - and the value of autarky in particular - is not an absorbing state. Lemmas 3 and 4 imply that whenever θ_L is drawn, then the continuation is back in the efficient region. Thus, there is sufficient "mixing" that the existence of a unique limiting distribution is guaranteed.

The limiting distribution has perpetual cycles that transit in and out of the region with *ex-post* inefficiencies. This feature differentiates my environment from related dynamic contracting problems such as Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007), and Hopenhayn and Werning (2008) which also have *ex-post* inefficiencies along the path. In all of these papers, in the long-run either the incentive problem disappears or there is an inefficient termination of the venture between the principal and the agent. In contrast, here the incentive problem does not disappear in the long-run and there is no termination of the risk-sharing relationship. The optimal allocation has periods of temporary autarky, but cooperation eventually restarts after the domestic economy recovers. This is because the sovereign borrower is the owner of the domestic production technology that can be operated also in autarky.

When $\beta = q$, I cannot establish that a stationary distribution exists without imposing an exogenous upper bound on consumption. Preliminary results suggest that, in this case, it is still true that *ex-post* inefficiencies persist in the long-run.

Summing up, in this section I showed that *ex-post* inefficiencies are part of the *ex-ante* efficient arrangement for the economy I consider. In the next section, I provide an implementation of the efficient allocation, associating defaults with these *ex-post* inefficient outcomes.

5. Implementation with Non-Contingent Debt and Maturity Management

In this section, I show that any efficient allocation can be implemented as a *sustainable equilibrium* outcome of a sovereign debt game where the set of securities available to the sovereign borrower is restricted to *non-contingent defaultable bonds of multiple maturities*. A *default* is defined as an episode in which the sovereign borrower makes a lower payment than what is specified in the bond contract. Along the equilibrium outcome path that supports an efficient allocation, defaults and periods of temporary exclusion from international credit markets occur at the same time as the inefficient outcomes.

There are several ways one could implement the efficient allocation. For instance, I could assume that the sovereign government can issue securities contingent on its report about the state of the economy as in Sleet (2004) and Sleet and Yeltekin (2006). Alternatively, I could consider one period debt which is nominally non-contingent but it is understood that the sovereign will not repay the full face value of the debt after certain shocks. Partial repayments introduce *de facto* implicit state contingencies in the bond contract. This is what Grossman and Van Huyck (1988) term *excusable default.*

The specific elements that I choose are motivated by three key facts about sovereign debt. First, in the data, the vast majority of sovereign and external debt comes in the form of non-contingent debt (see Rogoff (2011) for a discussion). Second, default episodes are infrequent events. Third, defaults happen when the sovereign is highly indebted. My implementation is consistent with these three facts: only non-contingent debt is available and there are recurrent but infrequent excusable defaults on path only when the sovereign's continuation value is low (in the region with *ex-post* inefficiencies). In all the other periods, I replicate the state contingent returns implied by the efficient allocation by exploiting the variation in the price of long-term debt, which is determined by default probabilities, after the realization of a shock. This allows me to derive implications for the optimal maturity

structure of sovereign debt.

In the rest of this section, I first describe the sovereign debt game and define a sustainable equilibrium. Next, I show how I can support any efficient allocation as a sustainable equilibrium outcome. Finally, I show that the equilibrium outcome path for bond holdings and prices is qualitatively consistent with the evidence.

A. Sovereign Debt Game

Consider a game between competitive (non-strategic) foreign lenders (bond holders and exporters of the intermediate good), domestic firms, and a benevolent domestic government (the only strategic player). The sovereign government can issue two types of *non-contingent defaultable* bonds: *a one period bond*, b_S, (or foreign reserves if $b_S < 0$) and a *consol*, $b_L \geq 0$. One unit of the one period bond promises to pay one unit of the final good tomorrow in exchange for q_S units of the final good today. The consol is a perpetuity that promises to pay a coupon of one unit of the final consumption good in every period starting tomorrow in exchange for q_L units of the final good today.

The government (borrower) cannot commit to satisfy the terms of the bond contracts. It has the option to *default*: to pay less than what is contractually specified. In particular, the borrower can choose any level of repayment from a set $\mathbf{r} = \{1, r_1, r_2, ..., r_k, 0\} \subset [0, 1]$. Let the repayment decision of the borrower at time t be $\delta_t \in \mathbf{r}$. If $\delta_t = 1$, there is no default and the borrower repays in full both the one period debt and the coupon payment for the consol. If $\delta_t = r_k \in (0, 1)$, the holders of the one-period bond receive r_k units of the tradable final good, while holders of the consol receive $r_k/(1-q)$ units of the final good per unit of debt and the borrower has no further obligations.[8] I will refer to $r_k \in (0, 1)$ as the recovery rate. Finally, if $\delta_t = 0$ there is no repayment in the current period. In this case, the borrower cannot access international credit markets. In the next period, he can choose a repayment policy δ_{t+1} for the notional amount of today's debt obligations. Interest payments are forgiven.

In defining an efficient allocation for the economy, I assumed that observable deviations can be punished with autarky forever after. See the definition of the sustainability

[8] Alternatively, I could have assumed that if r_k is chosen the holders of the consol receive a coupon payment of r_k and r_k units of the newly issued consol. This alternative specification does not alter any result.

constraint (SUST). Consistent with this, I assume that lenders can deny access to savings to the borrower. This assumption is common in the literature; see Atkeson (1991) and Aguiar, Amador, and Gopinath (2009), among others.

Furthermore, the government can tax the payments made by domestic firms to foreign exporters for the intermediate goods at a rate $\tau_t \in [0, 1]$. Thus, foreign exporters receive an after tax payment of $p_t(1 - \tau_t)$ per unit of intermediate good sold, where p_t is the price of the intermediates in terms of the final good. The role of this tax is to ensure that private agents choose imports that are consistent with the sustainability constraint. This is related to the necessity of capital income taxes in the implementation for the efficient allocation in economy with lack of commitment in Kehoe and Perri (2004) and Aguiar, Amador, and Gopinath (2009).

Informally, the sequence of events within the period is the following:

1. The public randomization device $\xi_t \in [0, 1]$ is realized;
2. Foreign lenders set a price for intermediate inputs p_t;
3. Domestic competitive firms choose m_t;
4. θ_t is realized and privately observed by the domestic government;
5. The government picks a policy $\pi_t = (\delta_t, \mathbf{b}_{t+1}, \tau_t)$ that consists of a repayment rule δ_t, new bond holdings, $\mathbf{b}_{t+1} = (b_{S,t+1}, b_{L,t+1})$ if $\delta_t \neq 0$ (if $\delta_t = 0$ then $\mathbf{b}_{t+1} = \mathbf{b}_t$) and a tariff on imported intermediates, τ_t ;
6. Bond prices $\mathbf{q}_t = (q_{S,t}, q_{L,t})$ are consistent with foreign lenders' optimality.

The equilibrium concept I use is an extension to an environment with private information[9] of the *sustainable equilibrium* (SE) concept developed in Chari and Kehoe (1990). Formally, for all $t \geq 0$ let $h^t = (h^{t-1}, \xi_t, p_t, m_t, \pi_t)$ be a public history up to period t and let $h^{-1} = \mathbf{b}_0 = (b_{S,0}, b_{L,0})$ be the initial outstanding debt. It is also convenient to define the following public histories when agents take action: $h_p^t = (h^{t-1}, \xi_t)$, $h_m^t = (h^{t-1}, \xi_t, p_t)$, $h_\sigma^t = (h^{t-1}, \xi_t, p_t, m_t)$, and let H_p^t, H_m^t and H_σ^t be the space of all possible such histories. The price of the intermediate good, p, the allocation rule for m, the strategy for the government,

[9] See Sleet (2004) and Sleet and Yeltekin (2006) for a similar extension to a macro-policy game with private information.

σ, and the price of bonds, $\mathbf{q}$, can be written as:

$$p = \{p_t\}_{t=0}^{\infty}, p_t : H_p^t \to \mathbb{R}_+ \tag{26}$$

$$m = \{m_t\}_{t=0}^{\infty}, m_t : H_m^t \to \mathbb{R}_+ \tag{27}$$

$$\sigma = \{\sigma_t\}_{t=0}^{\infty}, \sigma_t = (\delta_t, \mathbf{b}_{t+1}, \tau_t) : H_\sigma^t \times \Theta \to \mathbf{r} \times (\mathbb{R} \times \mathbb{R}_+) \times [0,1] \tag{28}$$

$$\mathbf{q} = \{q_{S,t}, q_{L,t}\}_{t=0}^{\infty} \ , q_{S,t} \ , q_{L,t} : H^t \to \mathbb{R}_+ \tag{29}$$

Problem of the Government Taking as given p, m, and the price schedule for bonds, $\mathbf{q}$, after any history $(h_\sigma^t, \theta) \in H_\sigma^t \times \Theta$, the strategy for the government, σ, solves the following problem:

$$W(h_\sigma^t, \theta) = \max_{c,\pi=\left(\delta, b'_{ST}, b'_{LT}, \tau\right)} \theta U(c) + \beta \mathbb{E}\left[W(h_\sigma^{t+1}, \theta_{t+1}) | h_\sigma^t, g\right] \tag{30}$$

subject to, if there is no default (i.e. $\delta = 1$)

$$c + (b_{S,t} + b_{L,t}) \leq y(\tau) + q_{S,t}(h_\sigma^t, \pi) b'_S + q_{L,t}(h_\sigma^t, \pi)(b'_L - b_{L,t}) \tag{31}$$

or, if there is partial repayment (i.e. $\delta = r_k$)

$$c + \left(b_{S,t} + \frac{b_{L,t}}{1-q}\right) r_k \leq y(\tau) + q_{S,t}(h_\sigma^t, \pi) b'_S + q_{L,t}(h_\sigma^t, \pi) b'_L \tag{32}$$

or, if there is default without any partial repayment (i.e. $\delta = 0$)

$$c \leq y(\tau) \quad \text{and} \quad (b'_S, b'_L) = (b_{S,t}, b_{L,t}) \tag{33}$$

where $y(\tau)$ is the amount of resources that are available to the sovereign borrower after production, repayments of intermediates, and the collection of the tariff revenue:

$$y(\tau) = F(m_t, 1) - p_t m_t + \tau p_t m_t \tag{34}$$

In (33), I impose the restriction that after $\delta_t = 0$ there is a temporary exclusion from international credit markets.[10]

Bond Prices and Other Equilibrium Objects The equilibrium prices, p and $\mathbf{q}$, and the allocation rule, m, must satisfy the following conditions. The price of the imported intermediate, $p_t : H_p^t \to \mathbb{R}_+$, must be consistent with optimization by competitive foreign lenders that take the tariff level as given:

$$1 = \mathbb{E}\left[p_t(h_p^t)\left(1 - \tau_t(h_\sigma^t, \theta_t)\right)|h_p^t\right] \tag{35}$$

The allocation rule for the quantity of foreign intermediate goods, $m_t : H_m^t \to \mathbb{R}_+$, satisfies the optimality condition for the representative domestic competitive firm

$$F_m(m_t(h_m^t), 1) = p_t(h_p^t) \tag{36}$$

Finally, bond prices $q_{S,t}, q_{L,t} : H^t \to \mathbb{R}_+$ are consistent with the maximization problem of the risk-neutral foreign lenders that discount the future at a rate q given the government repayment policy. For the one period bond, if $b_{S,t+1} \geq 0$, it must be that

$$q_{S,t}(h^t) = q\mathbb{E}\left[\chi_{S,t+1}(h^{t+1})|h^t\right] \tag{37}$$

where $\chi_{S,t+1}$ is the *ex-post value of short-term debt*:

$$\chi_{S,t+1}(h^{t+1}) = \begin{cases} 1 & \text{if } \delta_{t+1} = 1 \\ r_k & \text{if } \delta_{t+1} = r_k \\ q\mathbb{E}\left[\chi_{S,t+2}(h^{t+2})|h^{t+1}\right] & \text{if } \delta_{t+1} = 0 \end{cases} \tag{38}$$

When $\delta_{t+1} = 0$, $q\mathbb{E}\left[\chi_{S,t+2}(h^{t+2})|h^{t+1}\right]$ can be interpreted as the secondary market value of defaulted debt. If instead $b_{S,t+1} < 0$, $q_{S,t}(h^t)$ can take on two values. If the government can

[10] In the background, as in Amador, Aguiar, and Gopinath (2009), the stand-in domestic household supplies labor inelastically and receives lump sum transfers (or taxes if negative), LS_t, from the government. His budget constraint is $c_t = w_t + LS_t$, where $w_t = F_\ell(m_t, 1)$ is the competitive wage rate. (31)-(33) represent the combined budget constraints of the benevolent government and the stand-in household.

save abroad, $q_{S,t}(h^t) = q$. Instead, if the government cannot save, I adopt the convention that $q_{S,t}(h^t) = \infty$. Finally, the price for the consol must be such that

$$(39)\quad q_{L,t}(h^t) = q\mathbb{E}\left[\chi_{L,t+1}(h^{t+1})|h^t\right]$$

where $\chi_{L,t+1}$ is the *ex-post value of the consol* given by

$$(40)\quad \chi_{L,t+1}(h^{t+1}) = \begin{cases} 1 + q_{L,t+1}(h^{t+1}) & \text{if } \delta_{t+1} = 1 \\ \frac{r_k}{1-q} & \text{if } \delta_{t+1} = r_k \\ q\mathbb{E}\left[\chi_{L,t+2}(h^{t+2})|h^{t+1}\right] & \text{if } \delta_{t+1} = 0 \end{cases}$$

Equilibrium Definition Given a set of recovery rates $\mathbf{r}$ and initial outstanding debt $\mathbf{b}_0$, a *sustainable equilibrium* (SE) is a strategy for the government, σ, a price rule for the foreign intermediate good, p, price rules for the government bonds, q_S and q_L, and an allocation rule for the intermediate good, m, such that p, m and q_S and q_L satisfy (35), (36), (37) and (39) given σ, and $\forall(h^t_\sigma, \theta)$, σ is a solution to (30) taking p, m, q_S and q_L as given. The associated equilibrium outcome path is denoted by $\mathbf{y} = (\mathbf{x}, \boldsymbol{\pi}, \mathbf{p})$ where $\mathbf{x} = \{m(s^{t-1}, \xi_t), c(s^t)\}_{t=0}^{\infty}$, $\boldsymbol{\pi} = \{\delta(s^t), b_L(s^t), b_S(s^t), \tau(s^t)\}_{t=0}^{\infty}$ and $\mathbf{p} = \{p(s^{t-1}, \xi_t), \mathbf{q}(s^t)\}_{t=0}^{\infty}$.

B. Implementation

Suppose that $\Theta = \{\theta_L, \theta_H\}$ and $\beta < q$ so that the borrower's value is bounded in the long-run by $\bar{v}_q < \bar{v}$. In the rest of this section, I show that an efficient allocation $\mathbf{x}$ can be implemented as a sustainable equilibrium outcome of the sovereign debt game.[11] Assume that $\mathbf{x}$ satisfies Assumption 3 and the following properties:

ASSUMPTION 5. (i) For all v, $v'_H(v) < v$ and in particular, there exists a $\underline{v} \in (v_a, \bar{v})$ such that $v'_H(v) = v_a$ for all $v \leq \underline{v}$. If randomization is optimal, $\underline{v} < v_r$. (ii) $v'_L(v)$ is strictly increasing and $v'_H(v)$ is strictly increasing for all $v \geq \underline{v}$.

[11] If $\mathbf{y}$ is an equilibrium outcome, its associated real allocation satisfies (IC) and (SUST). Thus, given an initial level of indebtedness, $B_0 = q_{ST0}b_{ST0} + q_{LT0}b_{LT0}$, the government's *ex-ante* value of any equilibrium outcome is bounded from above by the value associated with the efficient allocation that delivers B_0 to the lenders.

Part (i) of the assumption implies that starting from any v there is a strictly positive probability of reaching autarky. That is, starting from any $v \in [v_a, \bar{v})$ after a sufficiently long - but finite - string of high taste shocks, the continuation value is equal to v_a. Part (ii) requires that v'_L and v'_H are monotone increasing. Both (i) and (ii) are satisfied in my simulations (see Figure 3 for an example). Part (i) of Assumption 5 can be dispensed with, but it allows for an easier exposition of the proof for the next proposition:

PROPOSITION 5. *Assume that Assumptions 1 and 3 hold, $\Theta = \{\theta_L, \theta_H\}$, and $\beta < q$. If $\mathbf{x}$ is an efficient allocation that satisfies Assumption 5, then there exist a set of recovery rates, $\mathbf{r}$, an initial debt position, $\mathbf{b}_0$, a strategy for the benevolent government, σ, prices, p and $\mathbf{q}$, and an allocation rule, m, such that: (i) $(\sigma, p, m, \mathbf{q})$ is a SE given $\mathbf{r}$ and $\mathbf{b}_0$, and (ii) $\mathbf{x}$ is the real allocation associated with the equilibrium outcome path.*

The proof of the proposition consists of two main steps. First, I construct the *on-path* default rule, bond holdings, tariffs, and prices that support the efficient allocation. Second, I show that I can find out-of-path behavior that prevents deviation from the constructed plan.

Mapping Between Efficient Allocation and Equilibrium Objects on Path I now construct the candidate equilibrium outcome path that implements an efficient allocation $\mathbf{x}$. Since the efficient allocation can be represented by a time-invariant function of borrower's continuation utility and exogenous shocks, the *on-path* repayment rule, bond holdings, tariffs, and prices can also be expressed as a function of on-path continuation utility for the borrower. In particular, the repayment policy, tariff, and intermediate prices are functions of the post-randomization value:

$$(41)\quad \bar{\delta} : [v_a, \bar{v}] \times \Theta \to \mathbf{r} \quad \text{and} \quad \bar{\tau}, \bar{p} : [v_a, \bar{v}] \to \mathbb{R}$$

Bond holdings and prices are functions of the continuation value (for the next period):

$$(42)\quad \bar{q}_S, \bar{q}_L, \bar{b}_S, \bar{b}_L : [v_a, \bar{v}] \to \mathbb{R}$$

An outcome path $\mathbf{y}$ can be recovered in the natural way from (41), (42), and the law of motion for v from the efficient allocation.

The steps to construct the candidate equilibrium outcome path $\mathbf{y}$ from an efficient allocation $\mathbf{x}$ are: (i) define the repayment policy; (ii) use the repayment policy in the optimality conditions for the foreign lenders to calculate equilibrium bond prices; (iii) choose short and long term debt to match the total value of debt (lenders' value) after a realization of θ implied by the efficient allocation, i.e. for $v \geq v_r$

$$(43) \quad b(v,\theta) = f(m(v)) - c(v,\theta) - m(v) + qB(v'(v,\theta))$$

and finally (iv) use the optimality conditions for the domestic firms and the lenders to get tariffs and prices for the intermediate good.

Consider first the repayment policy which is consistent with the fact that defaults are infrequent and they happen only when the borrower is highly indebted. The borrower defaults only when his continuation value post-randomization is in $[v_a, v_r]$. If it is not optimal to randomize, let $v_r = v_a$. For all the other borrower values, there is full repayment. That is:

$$(44) \quad \bar{\delta}(v,\theta) = 1 \quad \text{if } v \in (v_r, \bar{v}] \text{ for all } \theta$$

$$(45) \quad \bar{\delta}(v,\theta) < 1 \quad \text{if } v \in [v_a, v_r] \text{ for all } \theta$$

I will refer to $[v_a, v_r]$ as the *default region* and to $(v_r, \bar{v}]$ as the *no-default region*. If randomization is optimal, by Assumption 3, for pre-randomization value $v \in [v_a, v_r]$, the post-randomization value is equal to either v_a (with probability $\zeta(v)$) or v_r (with probability $1 - \zeta(v)$). Thus, there are four relevant outcomes for the repayment policy in the default region:

$$(46) \quad \bar{\delta}(v,\theta) = \begin{cases} 0 & \text{if } v = v_a \text{ and } \theta = \theta_H \\ r_{rH} & \text{if } v = v_r \text{ and } \theta = \theta_H \\ r_{aL} & \text{if } v = v_a \text{ and } \theta = \theta_L \\ r_{rL} & \text{if } v = v_r \text{ and } \theta = \theta_L \end{cases}$$

where $\bar{\delta}(v_a, \theta_H) = 0$ because from Lemma 3 it follows that, when the borrower's value is autarky, there are no capital flows: $m(v_a) = 0$ and $c(v_a, \theta_H) = f(0)$. It is worth noting that when $v = v_r$ and $\theta = \theta_H$ then there is a partial repayment today (r_{rH} is generally greater than zero) and there will be again less than full repayment the next period because $v'_H(v_r) < v_r$. I interpret this as a unique protracted default episode.[12] The borrower is out of the default region the next period only after he draws θ_L ($v'_L(v) \geq \tilde{v} > v_r$ for all v).

Given the repayment policy, bond prices are uniquely pinned down by the lenders' optimality conditions. The price for short-term debt is given by:

$$\bar{q}_S(v) = \begin{cases} q & \text{if } v \in (v_r, \bar{v}] \\ q\bar{R}(v) & \text{if } v \in [v_a, v_r] \end{cases} \tag{47}$$

where $\bar{R}(v)$ is the expected recovery rate in the default region:

$$\bar{R}(v) = \zeta(v)\frac{\mu(\theta_L)r_{aL}}{1 - q\mu(\theta_H)} + [1 - \zeta(v)]\left[\mu(\theta_L)r_{rL} + \mu(\theta_H)r_{rH}\right] \tag{48}$$

The price for long-term debt can be written recursively as:

$$\bar{q}_L(v) = \begin{cases} q\sum_{i=L,H}\mu(\theta_i)\left[1 + \bar{q}_L(v'_i(v))\right] & \text{if } v \in (v_r, \bar{v}] \\ \frac{q}{1-q}\bar{R}(v) & \text{if } v \in [v_a, v_r] \end{cases} \tag{49}$$

Outside of the default region, i.e. for all $v \in (v_r, \bar{v}]$, the price of short-term debt is equal to that of a risk-free bond. Instead, the price of long-term debt is lower than the price of a risk-free consol because there is always a positive probability that there will be a default over the relevant time horizon of the bond. The next lemma shows that $\bar{q}_{LT}$ is strictly increasing in the continuation value for the borrower.

LEMMA 5. Under the assumptions in Proposition 5, for a given $\mathbf{r} = \{1, r_{rL}, r_{aL}, r_{rH}, 0\}$ with $r_{rL}, r_{aL}, r_{rH} \in (0, 1)$, $\bar{q}_L : [v_a, \bar{v}] \to \mathbb{R}$ is the unique fixed point of the contraction mapping defined by the right hand side of (49) and is strictly increasing.

Proof. Appendix. □

[12] This is consistent with the fact that there are repeated restructurings, see Cruces and Trebesch (2012).

Given the functions for bond prices $\bar{q}_S$ and $\bar{q}_L$, in the no-default region $\bar{b}_S(v)$ and $\bar{b}_L(v)$ are chosen to match the total value of debt (lenders' value) implied by the efficient allocation after θ_L and θ_H defined in (43):

$$(50) \quad b(v,\theta_L) = \bar{b}_S(v) + \bar{b}_L(v)\left[1 + \bar{q}_L(v'_L(v))\right]$$

$$(51) \quad b(v,\theta_H) = \bar{b}_S(v) + \bar{b}_L(v)\left[1 + \bar{q}_L(v'_H(v))\right]$$

A (unique) solution to (50)-(51) is guaranteed by the fact that $\bar{q}_L$ is strictly increasing and $v'_H(v) < v'_L(v)$, see Proposition 1 part (ii). Therefore $\bar{q}_L\left(v'_H(v)\right) < \bar{q}_L\left(v'_L(v)\right)$. Thus, outside of the default region the maturity composition of debt is uniquely pinned down. Simple algebra shows that:

$$(52) \quad \bar{b}_L(v) = \frac{b(v,\theta_L) - b(v,\theta_H)}{\bar{q}_L(v'_L(v)) - \bar{q}_L(v'_H(v))}$$

$$(53) \quad \bar{b}_S(v) = b(v,\theta_L) - \bar{b}_L(v)\left[1 + \bar{q}_L(v'_L(v))\right]$$

Notice that it is guaranteed that $\bar{b}_L(v) > 0$ because, by Proposition 1 part (iii), $b(v,\theta_L) - b(v,\theta_H) > 0$ and, as shown above, $\bar{q}_L(v'_L(v)) - \bar{q}_L(v'_H(v)) > 0$. Intuitively, given the *ex-post* variation in the price of long-term debt, $\bar{q}_L(v'_L(v)) - \bar{q}_L(v'_H(v))$, the long-term debt is chosen to replicate the amount of insurance, $b(v,\theta_L) - b(v,\theta_H)$, implied by the efficient allocation. Instead the short-term debt holdings can be thought as being chosen to match the total value of debt.

When there is full repayment, the fall in the value of debt after the realization of a high taste shock is obtained by imposing a capital loss on the holders of outstanding long-term debt or, in the terminology of Chatterjee and Eyigungor (2012a, 2012b), by *diluting outstanding long-term debt*. After a high taste shock, the continuation value for the borrower decreases, the overall level of indebtedness increases, and the probability that there will be a default in the near future increases. This increase in the likelihood of a future default reduces the value of the outstanding long-term debt, resulting in a capital loss for the debt holders and a capital gain for the borrower. This capital loss on the debt holders after an adverse shock mimics the debt relief for the borrower associated with the efficient allocation[13]. Thus,

[13]This is consistent with the evidence in Berndt, Lustig and Yeltekin (2011) who document how the fall in

debt dilution is not a negative feature of the equilibrium outcome path.

In the default region, bond holdings are constant. For all $v \in [v_a, v_r]$, $\bar{b}_S(v) = \bar{b}_{Sr}$ and $\bar{b}_L(v) = \bar{b}_{Lr}$, and it must be that:

$$(54) \quad b(v_r, \theta_L) = r_{rL}\left[\bar{b}_{Sr} + \frac{\bar{b}_{Lr}}{1-q}\right]$$

$$(55) \quad b(v_r, \theta_H) = r_{rH}\left[\bar{b}_{Sr} + \frac{\bar{b}_{Lr}}{1-q}\right]$$

$$(56) \quad b(v_a, \theta_L) = r_{aL}\left[\bar{b}_{Sr} + \frac{\bar{b}_{Lr}}{1-q}\right]$$

The other possible outcome follows from (56) and $\delta(v_a, \theta_H) = 0$ because

$$(57) \quad b(v_a, \theta_H) = q\left[\mu(\theta_L)b(v_a, \theta_L) + \mu(\theta_H)b(v_a, \theta_H)\right] = \frac{q\mu(\theta_L)}{1 - q\mu(\theta_H)}b(v_a, \theta_L)$$

Thus, fixing any $r_{rL} \in (0, 1)$, it must be that

$$(58) \quad r_{rH} = \frac{b(v_r, \theta_H)}{b(v_r, \theta_L)}r_{rL} \text{ and } r_{aL} = \frac{b(v_a, \theta_L)}{b(v_r, \theta_L)}r_{rL}$$

If the restrictions in (58) are satisfied, I can choose any $\left(\bar{b}_{Sr}, \bar{b}_{Lr}\right)$ that satisfies (54). Consequently (55)-(57) will also be satisfied. The split between long and short term debt is indeterminate because the two are perfect substitutes if there is default for sure in the next period. I resolve this indeterminacy by assuming that $\bar{b}_{Sr}/\bar{b}_{Lr} = \lim_{v \to v_r} \bar{b}_S(v)/\bar{b}_L(v)$. The recovery rate $r_{rL} \in (0, 1)$ is a free-parameter. It can be chosen sufficiently low that $\bar{b}_S$ is strictly positive in the default region and for v close to v_r so that a non-full repayment has a natural interpretation.

Finally, I construct the on-path tariff rates and prices for the intermediate good, $\bar{\tau}, \bar{p}$, as follows:

$$(59) \quad f'(m(v)) = \frac{1}{1 - \bar{\tau}(v)} = \bar{p}(v)$$

Then the outcome path $\mathbf{y}$ constructed from the efficient allocation $\mathbf{x}$ using (44), (46), (47),

the value of long-term debt provides fiscal insurance to the U.S. government.

(49), (52), (53), (54) and (59) satisfies the optimality conditions (35) and (36), and the equilibrium bond pricing equations (37) and (39). Moreover, it supports the level of consumption implied by the efficient allocation.

Deterring Deviation To complete the proof of Proposition 5, I need to verify that for any on-path history, the government does not have a strict incentive to deviate. A simple way to proceed is to consider a trigger strategy that reverts to autarky after any deviation by the government. If the government does not follow the prescription of $\bar{\delta}$, $\bar{b}$, and $\bar{\tau}$, it faces future bond prices equal to zero, cannot save, and receives no foreign intermediate goods. Zero intermediates and a price equal to zero for both short-term and long-term-debt can be supported as part of a sustainable equilibrium because if foreign lenders expect a tariff equal to 100 percent (full expropriation) and full default ($\delta = 0$) in any subsequent periods irrespective of the action chosen today by the government, then the government has no incentive to choose something different than $\tau = 1$ and $\delta = 0$, confirming the lenders' beliefs. The fact that the sovereign borrower cannot save after a deviation follows from an assumption. Therefore, the value of any deviation is equal to the value of the static deviation plus the continuation value associated with autarky. This value is bounded from above by $\theta U(f(m)) + \beta v_a$. Since the efficient allocation satisfies the sustainability constraint (SUST), the government has no strict incentive to deviate. This concludes the proof of Proposition 5.

Using reversion to autarky after any deviation to support the efficient allocation as an equilibrium outcome is *not* necessary. In particular, it is possible to find a history dependent pricing function $\mathbf{q}_t(h_\sigma^t, \cdot)$ that is continuous in new debt issuance $\mathbf{b}'$ for all (δ, τ). However, to support the efficient allocation, some degree of history dependence is needed. The efficient allocation cannot be supported by a Markovian equilibrium of the kind typically considered in the quantitative sovereign default literature. The pricing functions for bonds need to depend on history, not only on bonds issued today.

Proposition 5 can be generalized to the case with $|\Theta| = N \geq 2$, allowing for a richer maturity structure. For instance, I can use N types of the perpetuity introduced in Hatchondo and Martinez (2009) that pay a coupon that decays exponentially at rate $\alpha_n \in [0, 1]$. The one-period bond and the consol are special cases of this class of securities for α equal to 1

and 0, respectively. Provided that the return matrix satisfies a full-rank condition[14] (which is authomatically satisfied when $N = 2$), the statement in Proposition 5 generalizes to the case with $N > 2$.

The proposed implementation works for environments other than the one considered here. For instance, if $\beta < q$ and the realization of θ_t is public information[15] - as in the economy considered in Aguiar, Amador, and Gopinath (2009) - or if θ_t is persistent, the same logic can be applied.

C. Characterization of Equilibrium Outcome Path: Debt Holdings and Spread

The bond holdings and the prices that support the efficient allocation are qualitatively consistent with two features of the data documented in Broner, Lorenzoni, and Schmukler (2010) and Arellano and Ramanarayanan (2012) for emerging markets. First, when interest rate spreads are low, long term spreads are generally higher than short term spreads. During debt crises, the gap between long and short-term spreads tends to narrow and the term spread curve flattens or even inverts. Second, during emerging market debt crises, the debt maturity shortens. An equilibrium outcome path that supports an efficient allocation shares these features of the data.

Implications for Interest Rate Spreads Define the *short-term spread* as the difference between the interest rate implied by q_S and the risk-free international interest rate: $s_S \equiv 1/q_S - 1/q$. The *long-term spread* is defined as the difference between the consol's yield to

[14] For $\alpha_i \in \{\alpha_1 = 0, \alpha_2, ..., \alpha_N = 1\}$, define $\bar{q}_{\alpha_i}$ in a similar way as in (49):

$$\bar{q}_{\alpha_i}(v') = \begin{cases} q \sum_\theta \mu(\theta) \left[1 + (1-\alpha_i)\bar{q}_{\alpha_i}(v'(v,\theta))\right] & \text{if } v \in (v_r, \bar{v}] \\ q \frac{\bar{R}(v)}{1-(1-\alpha_i)q} & \text{if } v \in [v_a, v_r] \end{cases}$$

Then, if the return matrix

$$\underset{N\times N}{\bar{Q}(v)} \equiv \begin{bmatrix} 1 + q_{\alpha_1}(v'(v,\theta_1)) & ... & 1 + q_{\alpha_N}(v'(v,\theta_1)) \\ 1 + q_{\alpha_1}(v'(v,\theta_2)) & ... & 1 + q_{\alpha_N}(v'(v,\theta_2)) \\ ... & ... & ... \\ 1 + q_{\alpha_1}(v'(v,\theta_N)) & ... & 1 + q_{\alpha_N}(v'(v,\theta_N)) \end{bmatrix}$$

is invertible, then there exists a $\bar{b}(v) = [\bar{b}_{\alpha_1}, ..., \bar{b}_{\alpha_N}]^T$ that solves the analogue of (50)-(51) given $\bar{Q}$.

[15] In this case, defaults will not be associated with *ex-post* inefficiencies.

maturity[16] and the risk-free interest rate: $s_L \equiv [1+q_L]/q_L - 1/q$. The *term premium* is the difference between the long and the short term spreads: $s_T \equiv s_L - s_S$.

Outside the default region the short term debt is risk-free, see (47). Thus, $q_{S,t} = q$ and $s_{S,t} = 0$. Instead, from Lemma 5 it follows that $q_{L,t} < \frac{q}{1-q}$. Therefore the short term spread is zero while the long-term spread is positive. Consequently, the term spread s_T is positive. When the borrower's continuation value is in the default region, the term spread is given by:

$$(60) \quad \begin{aligned} s_T(v) &= \left(1 + \frac{1}{\bar{q}_L(v)}\right) - \frac{1}{\bar{q}_S(v)} = \left(1 + \frac{1-q}{q\bar{R}(v)}\right) - \frac{1}{q\bar{R}(v)} \\ &= \frac{1 - q(1-\bar{R}(v))}{q\bar{R}(v)} - \frac{1}{q\bar{R}(v)} < 0 \end{aligned}$$

Thus the spread for the short-term debt is higher than the long-term spread and the term structure is inverted. This behavior for the term spread is consistent with the evidence. The next proposition summarizes the argument above:

PROPOSITION 6. *Outside the default region s_L is higher than s_S. In the default region, the term structure is inverted: $s_L < s_S$.*

Maturity Shortens as Indebtedness Increases I now turn to the implications for the optimal maturity composition of debt. The main finding is that the maturity of outstanding debt issued by the sovereign gets shorter as its indebtedness increases. In particular, the amount of long-term debt, $b_{L,t}$, decreases while the amount of short-term debt, $b_{S,t}$, increases for all v in the efficient region. This result is illustrated in Figure 6. I cannot state a proposition for this result, but the findings are consistent in all of my numerical simulations.

To understand this result, notice that outside of the default region, the amount of long-term debt held by the borrower is determined by (52), reported here for convenience:

$$\bar{b}_L(v) = \frac{b(v,\theta_L) - b(v,\theta_H)}{\bar{q}_L(v'_L(v)) - \bar{q}_L(v'_H(v))}$$

The long-term debt holdings are constructed to match the debt relief implied by the optimal

[16] That is, the implicit constant interest rate at which the discounted value of the bond's coupons equals its price. Define $q_{YM,L}$ as $q_L = \frac{q_{YM,L}}{1-q_{YM,L}}$. The consol's yield to maturity is $1/q_{YM,L} = \frac{q_L}{1+q_L}$.

contract after the realization of θ_H, $b(v,\theta_L) - b(v,\theta_H)$, given the *ex-post* variation in the price of the consol, $\bar{q}_L(v'_L(v)) - \bar{q}_L(v'_H(v))$. As is shown in Figure 7, the level of cross-subsidization is approximately constant for all v over the efficient region. The *ex-post* variation in the price of the consol instead is larger the closer the borrower is to the default region. This is because as the borrower's continuation value approaches the default threshold from above, it is more likely that a realization of θ_H will push the economy into default in the near future. Hence, the long-term debt price is more sensitive to the realization of a taste shock. Therefore, a lower holding of long term debt is needed in order to replicate the same amount of insurance, i.e. the same debt relief after a high taste shock. Since the overall level of indebtedness is increasing, it must be that $\bar{b}_S$ is increasing as the borrower's continuation value approaches $\tilde{v}$ because $\bar{b}_L$ is falling at the same time. Therefore, in the efficient region, as the level of indebtedness increases, the maturity composition of debt gets shorter.

In the region with *ex-post* inefficiencies, $[v_a, \tilde{v}]$, the ratio of short-term debt to long-term debt is not always decreasing in the borrower's value under all parameterizations. This is because the *ex-post* variation in the price of long-term debt is high, but also the amount of insurance, $b(v,\theta_L) - b(v,\theta_H)$, increases a lot in this region (see Figure 7). Despite not necessarily being monotonically decreasing in this region, the maturity composition of debt is more tilted toward short-term debt than it is for continuation values associated with lower default probabilities.

To summarize, in this section I showed that an efficient allocation can be implemented with only non-contingent defaultable debt of multiple maturities. Defaults are infrequent events, are associated with *ex-post* inefficiencies, and happen on the equilibrium outcome path only when the borrower's value is minimal (or close to minimal). When there is no default, capital gains or losses on outstanding long-term debt replicate the state contingent returns implied by the efficient allocation. Moreover, the maturity of outstanding debt gets shorter as the level of indebtedness increases.

D. Discussion

[Compare to: (i) Alvarez and Jermann (2000), (ii) Angeletos, Buera-Nicolini, and (iii) DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007) and (iv) debt dilution

literature]

6. Illustrative Numerical Example

In this section, I show that an efficient equilibrium outcome path leading to a default is qualitatively consistent with the four key aspects of the data that were mentioned in the introduction. I consider the following functional forms and parameterization. Let $U(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma = 2$ and $F(m, \ell) = \left(\omega m^{1-1/\eta} + (1-\omega)\ell^{1-1/\eta}\right)^{\frac{\eta}{\eta-1}}$ with $\eta = 1.5$ and $\omega = .3$. The other parameters in the model are $\beta = .95$, $q = .96$, $\Theta = \{.9, 1.4\}$ and $\mu_L = .8$. The recovery rate r_{rL} is set to .6, r_{rH} and r_{aL} are set according to (58). This example is representative of several simulations that I perform.

Randomization is optimal under this parameterization. Consistently with Assumption 3, there is a $v_r \in (v_a, \tilde{v})$ such that B is linear over $[v_a, v_r]$. For any v in this region it is optimal to randomize between v_a and v_r. Figure 3 displays the law of motion for the borrower's continuation utility which has already been discussed. In Figure 4, I show the policy functions associated with $(\hat{\mathrm{P}})$ for intermediate imports, output, consumption, and the net transfers to the foreign lenders, $y^* - m$, as a function of the borrower's value.

The first two panels illustrate the result in Proposition 1 part (i). For low borrower values, $v \in [v_a, v^*)$, imported intermediates and output are depressed relative to the statically efficient level. The lower is the borrower's value, the higher are the distortions: imported intermediates and output are strictly increasing over the interval $[v_a, v^*)$. For borrower values sufficiently high - higher than v^*- the statically efficient level of intermediates can be sustained. The same is true for output. Then, when the economy is in the region with *ex-post* inefficiencies or the *default region*, output and intermediate imports are very low relative to "normal" times.

In the third panel, I show the decision rules for consumption after θ_L and θ_H. After a high taste shock consumption is higher than it is after a low taste shock. Notice how $c_H(v)$ is not monotone. For continuation values sufficiently high ($v \geq v^*$), it is increasing in the borrower's value, as one would expect. In the region in which the sustainability constraint is binding ($v \leq v^*$), $c_H(v)$ may have a decreasing portion. This is because providing more consumption after a high taste shock relaxes the current sustainability constraint. This is

more valuable for low borrower values.

Finally, the last panel shows the dynamics for the trade balance, $y^* - m$. The sovereign borrower experiences larger outflows after a low taste shock than it does after a high taste shock because after a high taste shock more resources are devoted to domestic consumption. In particular, when the borrower's continuation value is autarky, after θ_H there are no trade flows: $m(v_a) = 0$ and $y^*_H(v_a) = 0$ (as shown in Proposition 1 part (i) and Lemma 3 respectively). Instead, after θ_L there are positive outflows: $m(v_a) = 0$ and $y^*_L(v_a) > 0$ (as shown in Proposition 1 part (i) and Lemma 3 respectively). The set of borrower values illustrated in the picture is restricted to those that are in the support of the unique limiting distribution. The borrower experiences net outflows because it has accumulated a stock of debt. This is due to the fact that the borrower is impatient relative to the international interest rate.

Figure 5 illustrates a sample path leading to a sovereign default for output, consumption, imported intermediates, and the short-term to long-term debt ratio as well as the realizations of the taste (productivity of the non-tradable sector, $z_t = 1/\theta_t$ with $\gamma = 2$) shock. As illustrated in the first panel, the economy is hit by a sequence of high taste (low productivity) shocks that pushes the economy into the region with *ex-post* inefficiencies. In particular, at time $\tilde{t}$, the borrower's value enters the region with *ex-post* inefficiencies. At t_d, it reaches the value of autarky and there is a default. The country is stuck in autarky until the economy draws a low taste (high productivity) shock at time t_r. At t_r there is a partial repayment to the creditors and the continuation value for the sovereign borrower starts to recover.

The outcome path is qualitatively consistent with the four key facts from sovereign default episodes mentioned in the introduction. First, in the model, as in the data, defaults are associated with output and consumption losses for the sovereign borrower (see panels 3 and 4). Second, they are associated with a drop in imports of intermediate goods (see panel 2). Third, in both the model and the data, eventual recoveries are accompanied by large trade surpluses. This can be seen from the fact that consumption after the partial repayment (at t_r) is below output. Fourth, the model generates shortening of the maturity composition of sovereign debt as default approaches (see panel 5). This mirrors the observed maturity composition of sovereign debt as the debtor country approaches default.

7. Final Remarks

In this paper, I show that key aspects of sovereign debt crises can be rationalized as part of the efficient risk-sharing arrangement between a sovereign borrower and foreign lenders in an economy with informational and commitment frictions. Along the outcome path that supports an efficient allocation, sovereign default episodes happen because of the need to provide incentives, despite being *ex-post* inefficient. In this economy, intervention by a supranational authority aimed at reducing the inefficiencies in a sovereign default episode is not beneficial from an *ex-ante* perspective. Moreover, the increasing share of short term debt when a sovereign country accumulates external debt is optimal when only non-contingent defaultable debt is available. My analysis suggests that interventions that penalize the issuance of short term debt might negatively affect welfare.

It is worth noting that the implementation I propose is applicable to environments other than the one considered here. Thus the implications for the optimal maturity composition of debt may have a more general applicability. I plan to consider possible generalizations in future research.

The simple model developed in this paper is consistent with broad patterns of the data. In future work, I plan to extend the current environment along two dimensions to be able to quantitatively evaluate the performance of the model. First, I plan to add capital accumulation, bringing the simple production economy considered here closer to a standard international business cycle model used in quantitative work. Second, in this paper I assumed that shocks were *iid* only for tractability. Introducing persistence in the shock process is an interesting avenue for future research. In particular, it can help to account for the fact that debt restructuring is a lengthy process. Benjamin and Wright (2009) document that on average, the renegotiation process lasts 8 years. In my model, the sovereign borrower is out of the default region the first time he draws a high productivity (low taste) shock. The combination of *iid* shocks and the sufficient conditions in Assumption 4 imply that default episodes are resolved quickly. Introducing persistence in the shock process will help along this dimension.

Finally, while the efficient allocation can be implemented as a sustainable equilibrium outcome of the game that I proposed, the converse is not true. There is a *continuum* of

equilibria and generically they are not efficient. Thus, despite the fact that agents are able to achieve the efficient outcome in a market setting, regulation by a supranational authority may indeed be helpful in avoiding inefficient equilibria and achieving *unique implementation*. I am planning to work on this in the future, introducing a strategic supranational authority (with and without commitment) into this framework.

8. References

Aguiar, M., M. Amador and G. Gopinath (2009) "Investment Cycles and Sovereign Debt Overhang," *Review of Economic Studies*, 76, 1–31.

Aguiar, M. and G. Gopinath (2006) "Defaultable Debt, Interest Rates, and the Current Account," *Journal of International Economics*, 69(1), 64-83.

Albuquerque, R. and H. Hopenhayn (2004) "Optimal Lending Contracts and Firm Dynamics," *Review of Economic Studies*, 71(2), 285–315.

Ales, L., P. Maziero and P. Yared (2012) "A Theory of Political and Economic Cycles," NBER Working Paper n. 18354.

Alvarez, F. and U. J. Jermann (2000) "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica*, 68(4), 775–798.

Angeletos, G.-M. (2004) "Fiscal Policy with Noncontingent Debt and the Optimal Maturity Structure," *Quarterly Journal of Economics*, 117, 1105-31.

Arellano, C. and A. Ramanarayanan (2012), "Default and the Maturity Structure in Sovereign Bonds," *Journal of Political Economy*, 120:2, 187-232

Arellano, C. (2008) "Default Risk and Income Fluctuations in Emerging Economies," *American Economic Review*, 98(3), 690-712.

Arteta, C. and G. Hale (2007) "Sovereign Debt Crises and Credit to the Private Sector," *Journal of International Economics*, 74, 53-69.

Atkeson, A. and R. Lucas (1995) "Efficiency and Equality in a Simple Model of Efficient Unemployment Insurance," *Journal of Economic Theory*, 66, 64-88.

Atkeson, A. and R. Lucas (1992) "On Efficient Distribution with Private Information," *Review of Economic Studies*, 59:3, 427-453.

Atkeson, A. (1991) "International Lending with Moral Hazard and Risk of Repudiation,"

Econometrica, 59(4), 1069-89.

Benjamin, D. and M. L.J. Wright (2009) "Recovery Before Redemption: A Theory of Delays in Sovereign Debt Renegotiations," Unpublished manuscript, SUNY Buffalo and UCLA.

Berndt, A., H. Lustig and S. Yeltekin (2011) "How Does the U.S. Government Finance Fiscal Shocks?" *American Economic Journal: Macroeconomics*, forthcoming.

Borensztein, E. and U. Panizza (2008) "The Costs of Sovereign Default," *IMF Staff Papers*.

Broner, F., G. Lorenzoni and S. Schmukler (2010) "Why Do Emerging Economies Borrow Short Term?" Unpublished manuscript, Universitat Pompeu Fabra.

Buera, F. and J. P. Nicolini (2004) "Optimal Maturity of Government Debt without State Contingent Bonds," *Journal of Monetary Economics*, 51, 531–54.

Chari, V.V. and P. Kehoe (1990) "Sustainable Plans," *Journal of Political Economy*, 98, 783-802.

Chatterjee, S. and B. Eyigungor (2012a) "Maturity, Indebtedness, and Default Risk," *American Economic Review*, 102(6), 2674–2699.

Chatterjee, S. and B. Eyigungor (2012b) "Debt Dilution and Seniority in a Model of Defaultable Sovereign Debt," Working Paper N. 12-14, Federal Reserve Bank of Philadelphia.

Cruces, J. and C. Trebesch (2012) "Sovereign Defaults: The Price of Haircuts," Unpublished manuscript, Universidad Torcuato Di Tella and University of Munich.

Clementi, G. and H. Hopenhayn (2006) "A Theory of Financing Constraints and Firm Dynamics," *Quarterly Journal of Economics*, 121, 229–65.

Cole, H. and T. Kehoe (2000) "Self-fulfilling Debt Crises," *Review of Economic Studies*, 67, 91-116.

Eaton, J. and M. Gersovitz (1981) "Debt with Potential Repudiation: Theoretical and Empirical Analysis," *Review of Economic Studies*, 48, 289-309.

DeMarzo, P.M. and M.J. Fishman (2007) "Optimal Long-Term Financial Contracting," *The Review of Financial Studies*, 20(5), 2080-2128.

DeMarzo, P.M. and Y. Sannikov (2006) "Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model," *Journal of Finance*, 61(6), 2681-2724.

Feenstra, R., R. Lipsey, H. Deng, A. Ma, and H. Mo (2005) "World Trade Flows: 1962-2000," NBER Working Paper no. 11040, 2004.

Fuentes, M., and D. Saravia (2006) "Sovereign Defaulters: Do International Capital Markets Punish Them?" Pontificia Universidad Católica de Chile Instituto de Economía Documentos de Trabajo 314.

Gopinath, G. and B. Neiman (2012) "Trade Adjustment and Productivity in Large Crises," Unpublished Manuscript, Harvard University.

Green, E. (1987) "Lending and the Smoothing of Uninsurable Income," in E. Prescott and N. Wallace, (Eds.), Contractual Arrangements for Intertemporal Trade (Minneapolis: University of Minnesota Press).

Green, E. J. and R. H. Porter (1984) "Noncooperative Collusion under Imperfect Price Information", *Econometrica*, 52, 87-100.

Grossman, H. and J. Van Huyck (1988) "Sovereign Debt as a Contingent Claim: Excusable Default, Repudiation, and Reputation," *American Economic Review*, 78, 1088-1097.

Hatchondo, J. C. and L. Martinez (2009) "Long-Duration Bonds and Sovereign Defaults," *Journal of International Economics*, 79, 117–25.

Hopenhayn, H. and I. Werning (2008) "Equilibrium Default," Unpublished Manuscript, UCLA and MIT.

Kehoe, P. and F. Perri (2004) "Competitive Equilibria with Limited Enforcement," *Journal of Economic Theory*, 119 (1), 184–206.

Kehoe, P. and F. Perri (2002) "International Business Cycles with Endogenous Incomplete Markets," *Econometrica*, 70 (3), 907-928.

Kehoe, T. and D. Levine (1993) "Debt-Constrained Asset Markets," *Review of Economic Studies*, 60, 865–888.

Kocherlakota, N. (1996) "Implications of Efficient Risk Sharing without Commitment," *Review of Economic Studies*, 63(4), 595–609.

Kreps, D. (1982) "Multiperiod Securities and the Efficient Allocation of Risk: A Comment on the Black-Scholes Option Pricing Model," The Economics of Information and Uncertainty. J. McCall, editor.

Lustig, H., C. Sleet, and S. Yeltekin (2008) "Fiscal Hedging with Nominal Assets," *Journal*

of Monetary Economics, 55, 710-727.

Mendoza, E. and V. Yue (2012) "A General Equilibrium Model of Sovereign Default and Business Cycles," *Quarterly Journal of Economics*, 127, 889–946.

Panizza, U., F. Sturzenegger and J. Zettelmeyer (2009) "The Economics and Law of Sovereign Debt and Default," Journal of Economic Literature, 47:3, 651-698.

Phelan, C. (2004), "On the Irrelevance of the Maturity Structure of Government Debt without Commitment," *Games and Economic Behavior*, 46, 115–128.

Phelan, C. and R. Townsend (1991) "Computing Multi-Period, Information-Constrained Optima," *Review of Economic Studies*, 58, 853-881

Quadrini, V. (2004) "Investment and Liquidation in Renegotiation-Proof Contracts with Moral Hazard," *Journal of Monetary Economics*, 51(4):713-751.

Rodrik, D. and A. Velasco (1999) "Short-Term Capital Flows," NBER Working Paper N. 7364.

Rogoff, K. (2011) "Sovereign Debt in the Second Great Contraction: Is This Time Different?" *The 2011 Martin Feldstein Lecture, NBER Reporter*.

Rose, A. (2005) "One Reason Countries Pay their Debts: Renegotiation and International Trade," *Journal of Development Economics*, 77, 189-206.

Sleet, C. and S. Yeltekin (2006) "Optimal Taxation with Endogenously Incomplete Debt Markets," *Journal of Economic Theory*, 127, 36 – 73.

Sleet, C. (2004) "Optimal Taxation with Private Government Information," *Review of Economic Studies*, 71, 1217-1239.

Stokey, N., R. Lucas, and E. Prescott (1989), Recursive Methods in Economic Dynamics, (Cambridge, MA: Harvard University Press).

Thomas, J. and T. Worrall (1994), "Foreign Direct Investment and the Risk of Expropriation," *Review of Economic Studies*, 61 (1), 81–108.

Thomas, J. and T. Worrall (1990), "Income Fluctuations and Asymmetric Information: An Example of a Repeated Principal-Agent Problem," *Journal of Economic Theory*, 51, 367-390.

Tomz, M. and M. Wright (2007) "Do Countries Default in Bad Times?" *Journal of the European Economic Association*, 5(2–3), 352–360.

Yared, P. (2010) "A Dynamic Theory of War and Peace," *Journal of Economic Theory*, 145, 1921-1950.

Yue, V. (2010) "Sovereign Default and Debt Renegotiation," *Journal of International Economics*, 80, 176-187.

9. Appendix
A. Data and Facts

In this Appendix, I document the behavior of GDP, consumption, and imports of intermediate goods around sovereign default episodes. Moreover, I discuss an extensive literature which also documents these and other key aspects of the data.

Data Description I consider the same 23 default events as Mendoza and Yue (2012): Argentina (1982, 2002), Chile (1983), Croatia (1992), Dominican Republic (1993), Ecuador (1999), Indonesia (1998), Mexico (1982), Moldova (2002), Nigeria (1983, 1986), Pakistan (1998), Peru (1983), Philippines (1983), Russia (1998), South Africa (1985, 1993), Thailand (1998), Ukraine (1998), Uruguay (1990), Uruguay (2003), Venezuela (1995), Venezuela (1998).

Annual data for GDP and consumption are gathered from the World Development Indicators (WDI). These are measured in real US dollars. As in Mendoza and Yue (2012), imported intermediates are the sum of categories for intermediate goods based on the Broad Economic Category (BEC) classification. The categories for intermediate goods are: (111) Food and beverages, primary, mainly for industry, (121) Food and beverages, processed, mainly for industry, (21) Industrial supplies not elsewhere specified, primary, (22) Industrial and lubricants, processed, (other than motor spirit), (42) Parts and accessories of capital goods (except transport equipment), (53) Part and accessories of transport equipment. For years 1962 through 2000, data is available from Feenstra et al. (2005) but is classified using the Standard International Trade Classification, revision 4 (SITC4). I use UN concordances to map SITC4 into BEC codes. For years 1976 through 2010, data is available through the World Bank's World Integrated Trade Solution (WITS) database, which has information from the UN's Comtrade database. This database provides the series for the above BEC codes when available. When I have data from both sources, I use the WITS data, which does not rely on the concordances. For years in which both sources provide data, I have cross referenced the values. Although the levels are not exactly the same, deviations from the trend (my variable of interest) are very similar across the two sources. I deflate the intermediate import data using the US producer price index (PPI) from the Bureau of Economic Analysis (BEA). Each annual series is logged and HP-filtered with a smoothing parameter of 100.

Sovereign debt crises are associated with severe output and consumption losses for the debtor country The first two panels of Figure 1 illustrate the dynamics for GDP and consumption around a sovereign default episode. On average, in the 23 default episodes considered, output is 4.5% and 5.2% below trend in the year of a default and the year after, respectively. On average, consumption is 3.1% and 3.6% below trend in the year of a default and the year after, respectively. The same is true if I instead consider the median. This confirms the findings in Mendoza and Yue (2012). This pattern has been documented in several studies. See the references in the survey by Panizza, Sturzenegger, and Zettelmeyer (2009). Tomz and Wright (2007) is a notable exception. They find only a weak association between default episodes and output being below trend.

Sovereign debt crises are associated with trade disruptions A large literature documents that sovereign default episodes are accompanied by large drops in trade. For instance, see Rose (2005) and the references in the survey by Panizza, Sturzenegger, and Zettelmeyer (2009). Also, Borensztein and Panizza (2008) document that a default has a negative impact on trade credit. Arteta and Hale (2008) show that foreign credit to the private sector collapses in the aftermath of a default. Fuentes and Saravia (2006) show that defaults lead to a fall in FDI flows into the country.

As noted in Mendoza and Yue (2012), the drop in imports of intermediate goods is very large: it drops on average from 4.4% above trend the year before a default to about 15.5% below trend the year of a default and the year after that. See the third panel in Figure 1. This drop is larger than in recessions of similar magnitudes. To document this fact, I regress imported intermediates at time t on a constant, GDP at time t, and dummy variables that take value of one if there is a default in the country at time t, $t-1$, $t-2$ and $t-3$ from 1962 to 2010 for the 18 countries in the sample for which I have data on intermediate imports. The result for this simple regression are reported in the table below. The drop in intermediate inputs in the year of and the year following a sovereign default is more than 10 percent larger than what one would expect from a drop in output of the same magnitude, absent default. This drop can have a non-trivial impact on the economy. Gopinath and Neiman (2012) present a model calibrated to replicate the crisis in Argentina in 2002 and

show that the decline in imports of intermediate goods can account for up to a 5 percentage point decline in the welfare relevant measure of productivity.

OLS Regression: Intermediate Imports at time t

Variable	Coefficient Estimate	Standard Error
Constant	0.007	0.960
GDP at t	1.810	0.145
Default at t	-0.119	0.044
Default at $t-1$	-0.108	0.044
Default at $t-2$	-0.040	0.044
Default at $t-3$	-0.005	0.043

R^2=0.225; Number of observations = 714

Intermediate imports and GDP are logged and HP-filtered.

Recoveries are accompanied by trade surpluses Periods following a sovereign default are associated with sustained trade surpluses as the economy recovers, see Mendoza and Yue (2012), among others. Moreover, typically as the economy recovers from the recession associated with the default episode, there is a partial repayment of the defaulted debt, after which the country regains access to international credit markets. Benjamin and Wright (2009) document that settlements tend to occur when output has returned to trend.

Maturity of debt shortens when a default is more likely, as measured by interest rate spreads Broner, Lorenzoni, and Schmukler (2010) use data from 11 emerging economies from 1990 to 2009 to document that during emerging market debt crises (when spreads are high), the maturity of debt issued shortens. Moreover, when the spreads are low, long term spreads are generally higher than short term spreads. During debt crises, the gap between long and short-term spreads tends to narrow and the term spread curve flattens or even inverts. Arellano and Ramanarayanan (2012) confirm these findings.

B. Proofs

Preliminaries

To characterize the efficient allocation, I use the equivalent formulation for (P) given by (P')-($\hat{\text{P}}$). Denote the decision rules associated with ($\hat{\text{P}}$) as $m(v) : [v_a, \bar{v}] \to \mathbb{R}$ and $c(v,\theta), v'(v,\theta) : [v_a, \bar{v}] \times \Theta \to \mathbb{R}$. Moreover, define $\omega(v,\theta) \equiv \theta U(c(v,\theta)) + \beta v'(v,\theta)$. Let $V_{nr} \subset [v_a, \bar{v}]$ be the (non-empty) set of promised utility values for which randomization in the current period is *not* optimal, i.e. $B(v) = \hat{B}(v)$. Let $V_r \subset [v_a, \bar{v}]$ be the *randomization region*. That is, the (possibly empty) set of promised utility values for which it is optimal to randomize, $B(v) > \hat{B}(v)$. Without loss of generality, the randomization is between two values in V_{nr} and it solves (P').

The next lemma establishes that B is concave and that the region over which $\hat{B}$ is strictly concave.

LEMMA 6. Under Assumption 1, B is concave. If Assumption 2 holds then B is strictly concave, $B = \hat{B}$, and $V_{nr} = [v_a, v_r]$. If Assumption 2 does not hold then I can only establish that $\hat{B}$ is strictly concave over $[v^*, \bar{v}]$.

Proof. Concavity of B follows from randomization. Rewrite ($\hat{\text{P}}$) using a change of variable: instead of $(m, c(\theta), v'(\theta))$, consider choosing $(\underline{u}, u(\theta), v'(\theta))$ where $\underline{u} = U(f(m))$ and $u = U(c)$. With this change of variable, ($\hat{\text{P}}$) can be written as

$$(\hat{\text{P}}') \quad \hat{B}(v) = \max_{\underline{u}, u(\theta), v'(\theta)} H(\underline{u}) + \sum_{\theta \in \Theta} \mu(\theta) \left[-C(u(\theta)) + qB(v'(\theta)) \right]$$

subject to

$$\begin{aligned}
\sum_{\theta \in \Theta} \mu(\theta) \left[\theta u(\theta) + \beta v'(\theta) \right] &= v \\
\theta u(\theta) + \beta v'(\theta) &\geq \theta u(\theta') + \beta v'(\theta') \quad \forall \theta, \theta' \\
\theta u(\theta) + \beta v'(\theta) &\geq \theta \underline{u} + \beta v_a \quad \forall \theta \\
v'(\theta) &\geq v_a \quad \forall \theta
\end{aligned}$$

where $C : [U(0), U(\infty)] \to \mathbb{R}$ is $C = U^{-1}$ and $H(\underline{u}) \equiv f \circ \kappa(\underline{u}) - \kappa(\underline{u}) = C(\underline{u}) - \kappa(\underline{u})$ with $\kappa : [U(f(0)), U(f(m^*))] \to [0, m^*]$ is $\kappa = f^{-1} \circ C$ so that $\underline{u} = U(f(\kappa(\underline{u})))$. The constraint set

is linear in the choice variables, $(\underline{u}, u(\theta), v'(\theta))$. Under Assumption 2, H is concave. Therefore by standard arguments $\hat{B}$ is strictly concave. Then clearly $B(v) = \hat{B}(v)$ for all $v \in [v_a, \bar{v}]$. Thus $V_r = \emptyset$ and $V_{nr} = [v_a, \bar{v}]$.

In general, H is not globally concave because both C and κ are convex. Hence, randomization over v can provide a higher value. I now claim that $\hat{B}$ is strictly concave over $[v^*, \bar{v}]$. Suppose for contradiction that is not. Then there exist $v_1, v_2 \in [v^*, \bar{v}]$ and $\alpha \in (0,1)$ such that $\hat{B}(\alpha v_1 + (1-\alpha)v_2) \leq \alpha \hat{B}(v_1) + (1-\alpha)\hat{B}(v_2)$. This cannot be the case. In fact, let $x(v) = (\underline{u}(v), u(\theta, v), v'(\theta, v))$. Because $x = \alpha x(v_1) + (1-\alpha)x(v_2)$ is attainable for $\alpha v_1 + (1-\alpha)v_2$ and it attains a higher value than $\alpha \hat{B}(v_1) + (1-\alpha)\hat{B}(v_2)$. This is true by strict concavity of C, weak concavity of B, and the fact that $\underline{u}(v_1) = \underline{u}(v_2) = U(f(m^*))$ by part (i) of Proposition 1. Then $\hat{B}$ is strictly concave over $[v^*, \bar{v}]$.□

The next Lemma establishes that B is differentiable.

LEMMA 7. Under Assumption 1, $B : [v_a, \bar{v}] \to \mathbb{R}$ is differentiable.

Proof. To see that B is differentiable, notice that for $v \in V_r$, B is linear and, therefore, is differentiable. For $v \in V_{nr}$, differentiability can be established by applying the Benveniste and Scheinkman theorem, see Theorem 4.10 in Stokey, Lucas, and Prescott (SLP henceforth). For any $v_0 \in V_{nr} \cap (v_a, \bar{v})$, let $x = (m, u(\theta), v'(\theta))$ be the solution that attains $B(v_0) = \hat{B}(v_0)$. First notice that $m > 0$ and $u(\theta) > U(0)$. The next Lemma establishes this result.

LEMMA 8. Under Assumption 1, for all $v \in (v_a, \bar{v})$, if $(m, u(\theta), v'(\theta))$ is the solution to $(\hat{\text{P}})$ then $m > 0$ and $u(\theta) > U(0)$.

Proof. [Sketch] Consider m first. Suppose for contradiction that $m = 0$. By Lemma 1 part (ii), the relevant sustainability constraint is for type θ_H. There are two cases. If $\omega(\theta_H) > \theta_H U(f(0)) + \beta v_a$, then it is possible to increase m without violating the sustainability constraint and increasing the lenders' value, thus arriving at a contradiction. If instead $\omega(\theta_H) = \theta_H U(f(0)) + \beta v_a$, there are two cases. Consider for simplicity the case with $N = 2$. If $c(\theta_H) = f(0)$ then it follows that

$$\omega(\theta_L) > \theta_L U(f(0)) + \beta v_a = \theta_L U(c(\theta_H)) + \beta v_a$$

then the relevant incentive compatibility constraint is slack. It is possible to increase consumption after θ_H and decrease it after θ_L. This increases $\omega(\theta_H)$ above $\theta_H U(f(0)) + \beta v_a$ and allows m to be greater than zero without violating incentive compatibility. This variation increases the lenders' value because of the Inada condition on f. If $c(\theta_H) < f(0)$ it must be that $v'(\theta_H) > v_a$. Then it is possible to decrease $v'(\theta_H)$, increase $c(\theta_H)$, and decrease either $c(\theta_L)$ or $v'(\theta_L)$ in a way that is consistent with the incentive compatibility and the sustainability constraints. This variation increases $\omega(\theta_H)$ above $\theta_H U(f(0)) + \beta v_a$, allowing m to be greater than zero. Moreover, it increases the lenders' value because of the Inada condition on f.

Now, consider $u(\theta)$. Notice that if $u(\theta) = U(0)$ for some θ, then it must be that $u(\theta_L) = U(0)$. Given the assumption that $\theta_L U(0) + \beta v^{**} < \theta_L U(f(0)) + \beta v_a$, if $u(\theta_L) = U(0)$ then it must be that $v'(\theta_L) \geq v^{**} > \tilde{v}$ (where $\tilde{v}$ is defined in Proposition 2; notice that this proposition does not rely on the differentiability of B). Thus, $v'(\theta_L)$ is on the downward sloping portion of B. Then consider decreasing $v'(\theta_L)$ and increasing $u(\theta_L)$, leaving utility unchanged. This is incentive compatible as the incentive compatibility constraint for type $\theta > \theta_L$ claiming to be θ_L is slack at the optimal solution. This has a positive effect on the objective because of the Inada condition on U. $\square$

Now, consider a neighborhood of v_0, $D(v_0, \varepsilon) = (v_0 - \varepsilon, v_0 + \varepsilon)$ for some small $\varepsilon > 0$. Given the interiority of m and $u(\theta)$, define $\hat{x}(v) = (\hat{m}, \hat{u}(v, \theta), \hat{v}'(v, \theta))$ for any $v \in D(v_0, \varepsilon)$ as follows:

$$\hat{m}(v) = m + \frac{v - v_0}{U'\left(f(m)\right) f'(m)}, \quad \hat{u}(v, \theta) = u(\theta) + v - v_0, \quad \hat{v}'(v, \theta) = v'(\theta)$$

so that, by construction, $\hat{x}(v)$ is feasible in $(\hat{\mathrm{P}})$ for all $v \in D(v_0, \varepsilon)$ for $\varepsilon > 0$ sufficiently small. The fact that $\hat{x}(v)$ satisfies promise keeping and incentive compatibility is obvious. For the

sustainability constraint, notice that:

$$\begin{aligned}\theta U\left(f\left(\hat{m}(v)\right)\right)+\beta v_a &= \theta U\left(f(m)\right)+\int_0^{\frac{v-v_0}{U'(f(m))f'(m)}}\theta U'(f(m+x))f'(m+x)dx+\beta v_a\\ &\leq \theta U\left(f(m)\right)+\beta v_a+\frac{\theta U'\left(f(m)\right)f'(m)}{U'\left(f(m)\right)f'(m)}\left(v-v_0\right)\\ &= \theta U\left(f(m)\right)+\beta v_a+\theta\left(v-v_0\right)=\theta\hat{u}(\theta)+\beta\hat{v}'(\theta)\end{aligned}$$

where I use the fact that $U'(f(x))f'(x)$ is decreasing in x. Then $\hat{x}(v)$ is feasible in $(\hat{\mathrm{P}})$ for all $v\in D(v_0,\varepsilon)$ for $\varepsilon>0$ sufficiently small. Then, define $\underline{B}:D(v_0,\varepsilon)\to\mathbb{R}$ as

$$\underline{B}(v)=f\left(\hat{m}(v)\right)-\hat{m}(v)+\sum_{\theta\in\Theta}\mu(\theta)\left[-C(\hat{u}(\theta))+qB(v'(\theta))\right]$$

$\underline{B}(v)$ is concave and differentiable in v, for all $v\in D(v_0,\varepsilon)$, $\underline{B}(v)\leq\hat{B}(v)\leq B(v)$ because $\hat{x}$ is feasible at v and $\underline{B}(v_0)=\hat{B}(v_0)=B(v_0)$. Thus, the Benveniste and Scheinkman theorem applies: B is differentiable at v_0 and

$$\begin{aligned}(61)\quad B'(v_0) &= \underline{B}'(v_0)\\ &= \frac{f'(m)-1}{U'\left(f(m)\right)f'(m)}-\sum_{\theta\in\Theta}\mu(\theta)C'\left(u(\theta)\right)=\frac{f'(m)-1}{U'\left(f(m)\right)f'(m)}-\sum_{\theta\in\Theta}\frac{\mu(\theta)}{U'\left(c(\theta)\right)}\end{aligned}$$

This concludes the proof. □

The next Lemma establishes the continuity of the policy functions in $(\hat{\mathrm{P}})$.

LEMMA 9. Under Assumption 1, the set of maximizers $m,v'(\theta),c(\theta):[v_a,\bar{v}]\rightrightarrows\mathbb{R}$ is a compact-valued upper hemicontinuous (UHC) correspondence. If, in addition, Assumption 2 holds, then the correspondence is single valued and $m,v'(\theta),c(\theta):[v_a,\bar{v}]\to\mathbb{R}$ are continuous in v.

Proof. The first part follows from the Theorem of the Maximum. For the second part, under Assumption 2 the objective function in $(\hat{\mathrm{P}})$ is strictly concave and, therefore, it admits a unique solution. This and the first part of the lemma imply the continuity of the decision rule. □

When for some $v\in[v_a,\bar{v}]$ the solution in $(\hat{\mathrm{P}})$ is not unique, I consider an efficient

allocation obtained from a selection from the UHC correspondence $m, v'(\theta), c(\theta) : [v_a, \bar{v}] \rightrightarrows \mathbb{R}$. I assume that the selection is continuous.

Proof of Lemma 1

For part (i) see Lemma 4 part (i) in Thomas and Worrall (1990). For part (ii), consider choosing $(\underline{u}, u(\theta), v'(\theta))$ instead of $(m, c(\theta), v'(\theta))$, where $u(\theta) = U(c(\theta))$. Let $(m, u(\theta), v(\theta))$ be incentive compatible and such that $v(\theta) \geq v_a$ for all θ. Furthermore, let it be such that it satisfies the sustainability constraint for θ_H:

$$\omega(\theta_H) \geq \theta_H U(f(m)) + \beta v_a$$

where $\omega(v, \theta) \equiv \theta u(\theta) + \beta v(\theta)$. Consider two cases. First, if $U(f(m)) \geq u(\theta_H)$ then for all $\theta \in \Theta$ it follows that

$$\begin{aligned}
\omega(\theta) &\geq \omega(\theta_H) - (\theta_H - \theta)\, u(\theta_H) \\
&\geq \theta_H U(f(m)) + \beta v_a - (\theta_H - \theta)\, u(\theta_H) \\
&= \theta U(f(m)) + \beta v_a + (\theta_H - \theta)\left[U(f(m)) - u(\theta_H)\right] \\
&\geq \theta U(f(m)) + \beta v_a
\end{aligned}$$

where in the first line I use the fact that $(m, u(\theta), v(\theta))$ is incentive compatible; in the second line, the sustainability at θ_H; in the third line I add and subtract $\theta U(f(m))$; and finally in the fourth line, I use the fact that $U(f(m)) \geq u(\theta_H)$. Then the sustainability constraint holds for all $\theta \in \Theta$. Now suppose that $U(f(m)) < u(\theta_H)$. In this case, the sustainability constraint is slack at θ_H, $\omega(\theta_H) > \theta_H U(f(m)) + \beta v_a$. Suppose for contradiction that there exists $\theta \in \Theta$ such that

$$\omega(\theta) \leq \theta U(f(m)) + \beta v_a$$

Then, notice that

$$\begin{aligned}\omega(\theta_H) &\leq \omega(\theta) + (\theta_H - \theta)u(\theta_H) \\ &\leq [\theta U(f(m)) + \beta v_a] + (\theta_H - \theta)u(\theta_H) = \theta_H u(\theta_H) + \beta v_a + \theta\left[U(f(m)) - u(\theta_H)\right] \\ &< \theta_H u(\theta_H) + \beta v_a \leq \omega(\theta_H)\end{aligned}$$

where the first line follows from incentive compatibility; in the second line, I use the contradiction hypothesis; in the third line, I used the fact that, by assumption, $U(f(m)) < u(\theta_H)$; and finally that $v(\theta_H) \geq v_a$. This results in a contradiction and the result follows.

Proof of Proposition 1

Part (i). First, let $v = v_a$. Combining the the sustainability constraint and the promise keeping constraint, it follows that

$$\begin{aligned}v_a &= v = \sum_{\theta\in\Theta} \mu(\theta)\left[\theta U\left(c(\theta)\right) + \beta v'(\theta)\right] \geq \sum_{\theta\in\Theta} \mu(\theta)\left[\theta U\left(f(m)\right) + \beta v_a\right] \\ &> \sum_{\theta\in\Theta} \mu(\theta)\left[\theta U\left(f(0)\right) + \beta v_a\right] = v_a \quad \text{if } m > 0\end{aligned}$$

Then it must be that $m(v_a) = 0$. For $v \in (v_a, \bar{v}]$, at an interior solution, the optimality condition for m can be written as

$$f'(m) - 1 = \lambda_{sust}\theta_H U'\left(f(m)\right) f'(m) \geq 0$$

where $\lambda_{sust} \geq 0$ is the Lagrange multiplier on the sustainability constraint (for θ_H, the relevant one by Lemma 1). Hence $m \leq m^*$ and $m = m^*$ iff $\lambda_{sust} = 0$. First, I argue that if $\lambda_{sust}(v_1), \lambda_{sust}(v_2) > 0$ for $v_1 < v_2$ then it must be that $m(v_1) < m(v_2)$. To do so, I need to use Assumption 2. That is, H is concave. This implies that λ_{sust} is monotone decreasing in m. Suppose for contradiction that $\lambda_{sust}(v_1), \lambda_{sust}(v_2) > 0$ and $m(v_1) \geq m(v_2)$. Then it must be that:

$$(62)\quad \omega_H(v_1) \geq \omega_H(v_2), \quad \lambda_{sust}(v_1) \leq \lambda_{sust}(v_2), \quad \lambda_{pkc}(v_1) < \lambda_{pkc}(v_2)$$

where $\lambda_{pkc}(v)$ is the Lagrange multiplier associated with the promise keeping constraint. By the envelope condition, this is equal to

$$\lambda_{pkc}(v) = -B'(v)$$

and it is strictly increasing in v by strict concavity of B under Assumption 2. Consider the case $N = 2$ for simplicity. An interior optimum must also satisfy the following necessary conditions:

$$(63) \quad \frac{1}{\theta_H U'(c_H(v))} = \lambda_{pkc}(v) + \frac{\lambda_{sust}(v)}{\mu_H} - \frac{\lambda_{ic}(v)}{\mu_H}\frac{\theta_L}{\theta_H} = \Lambda(v) - \frac{\lambda_{ic}(v)}{\mu_H}\frac{\theta_L}{\theta_H}$$

$$(64) \quad -\frac{q}{\beta}B'(v'_H(v)) = \lambda_{pkc}(v) + \frac{\lambda_{sust}(v)}{\mu_H} - \frac{\lambda_{ic}(v)}{\mu_H} = \Lambda(v) - \frac{\lambda_{ic}(v)}{\mu_H}$$

where $\lambda_{ic}(v)$ is the Lagrange multiplier on the relevant incentive compatibility constraint (type θ_L reporting θ_H, see again Lemma 1) and $\Lambda(v) \equiv \lambda_{pkc}(v) + \lambda_{sust}(v)/\mu_H$. By (62) it follows that $\Lambda(v_2) > \Lambda(v_1)$. Now consider two cases. First, if

$$(65) \quad \Lambda(v_1) - \Lambda(v_2) \geq \frac{\theta_L}{\theta_H}\left[\frac{\lambda_{ic}(v_1) - \lambda_{ic}(v_2)}{\mu_H}\right]$$

then it follows that $c_H(v_1) \geq c_H(v_2)$. Moreover, $\theta_L/\theta_H \in (0,1)$ and the term in square brackets is negative. It follows that

$$\Lambda(v_1) - \Lambda(v_2) \geq \frac{\theta_L}{\theta_H}\left[\frac{\lambda_{ic}(v_1) - \lambda_{ic}(v_2)}{\mu_H}\right] > \frac{\lambda_{ic}(v_1) - \lambda_{ic}(v_2)}{\mu_H}$$

Hence $v'_H(v_1) > v'_H(v_2)$. Notice that at an optimal solution, the relevant incentive compatibility constraint must hold with equality (see the discussion of part (ii) below for a proof). Therefore, if the relevant incentive compatibility constraint binds for v_1, then the incentive compatibility constraint must not bind at the solution for v_2. In fact:

$$\begin{aligned} \omega_L(v_2) &> \omega_L(v_1) = \theta_L U\left(c_H(v_1)\right) + \beta v'_H(v_1) \\ &> \theta_L U\left(c_H(v_2)\right) + \beta v'_H(v_2) \end{aligned}$$

where the first inequality follows from the fact that $\omega_H(v_1) > \omega_H(v_2)$ and $v_2 > v_1$; the second follows from a binding incentive compatibility constraint for v_1; and the last follows from $c_H(v_1) \geq c_H(v_2)$ and $_H(v_1) \geq \omega_H(v_2)$. This is a contradiction. Consider now the case in which (62) does not hold. This implies, together with (62) and (63), that $c_H(v_1) < c_H(v_2)$. Notice that a binding incentive compatibility constraint implies that

$$\omega_L(v) = \omega_H(v) - (\theta_H - \theta_L)U\left(c_H(v)\right)$$

Using this equality in the promise keeping constraint, I can write:

$$\begin{aligned} v \quad &= \quad \mu_H\omega_H(v) + \mu_L\omega_L(v) = \omega_H(v) - \mu_L(\theta_H - \theta_L)U\left(c_H(v)\right) \\ &\iff \quad \omega_H(v) = v + \mu_L(\theta_H - \theta_L)U\left(c_H(v)\right) \end{aligned}$$

Hence, the fact that $c_H(v_1) < c_H(v_2)$ implies that $\omega_H(v_1) < \omega_H(v_2)$, a contradiction. For $N > 2$ an induction argument extends this logic to the general case.

I now turn to showing that there exists a v^* such that for all $v \geq v^*$ it must be that $m(v) = m^*$. Consider a relaxed version of ($\hat{\text{P}}$) in which the sustainability constraint is dropped. In this relaxed problem, it can be shown that $\omega_H(v) \geq \theta_H(1-\beta)v + \beta v$. Hence, if $v \geq v^{**} \equiv \left[\theta_H U(f(m^*)) + \beta v_a\right] / \left[\theta_H(1-\beta) + \beta\right]$, the solution of this relaxed problem is a solution to the original problem. Thus, $m(v) = m^*$ for all $v \geq v^{**}$. If in addition Assumption 2 holds, combining this with the fact that $m(v)$ is strictly increasing (and continuous) when λ_{sust} is binding, it follows that there must exist some $v^* \in (v_a, v^{**})$ for which $m(v) < m^*$ for all $v \in [v_a, v^*)$ and $m(v) = m^*$ for all $v \geq v^*$.

Part (ii). Consider $N = 2$ to simplify notation. First notice that the relevant incentive compatibility constraint must bind at an optimal solution. In fact, suppose for contradiction that it is slack. Then the optimality conditions imply that $v'_H \geq v'_L$ and $c_H > c_L$ or, equivalently, that $u_L > u_H$ using the change of variables in ($\hat{\text{P}}$'). Clearly this is not incentive compatible. Suppose for contradiction that $u_L \geq u_H$. For the relevant incentive compatibility constraint to be binding, it must be that $v'_H \geq v'_L$. By Lemma 3 it follows that $v'_L > v_a$. Hence the solution is interior. Thus, I can combine the first order necessary conditions with

respect to v'_H and v'_L, (69) and (70) below, to get

$$(66)\quad \frac{\lambda_{ic}}{\mu_L} \leq \frac{\lambda_{sust} - \lambda_{ic}}{\mu_H} < \frac{\lambda_{sust} - \lambda_{ic}\frac{\theta_L}{\theta_H}}{\mu_H}$$

where λ_{ic} and λ_{sust} are the Lagrange multiplier on the incentive compatibility constraint and the sustainability constraint respectively, and the last strict inequality follows from the fact that $\theta_L/\theta_H \in (0,1)$. Combining (66) with the first order conditions for u_H and u_L, (67) and (68) below, implies that

$$\frac{C'(u_H)}{\theta_H} - \frac{C'(u_L)}{\theta_L} = \frac{\lambda_{sust} - \lambda_{ic}\frac{\theta_L}{\theta_H}}{\mu_H} - \frac{\lambda_{ic}}{\mu_L} > 0 \Rightarrow u_H > u_L$$

which is a contradiction. Hence, for all v it must be that $u_H(v) > u_L(v) \iff c_H(v) > c_L(v)$. Consequently, incentive compatibility requires that $v'_L(v) > v'_H(v)$ for all v, as wanted.

Part (iii). For the cross-subsidization part, as in Thomas and Worrall (1990) Lemma 4 part (ii), suppose for contradiction that $b(v,\theta_L) < b(v,\theta_H)$ for some v. Then, consider offering the pooling allocation: $\hat{c}(v,\theta_L) = \hat{c}(v,\theta_H) = c(v,\theta_H)$ and $\hat{v}'(v,\theta_L) = \hat{v}'(v,\theta_H) = v'(v,\theta_H)$. Because the incentive compatibility constraint is binding at the optimal allocation, it follows that

$$\hat{\omega}(v,\theta_L) = \theta_L U(\hat{c}(v,\theta_L)) + \beta v'(v,\theta_L) = \theta_L u(c(v,\theta_H)) + \beta v'(v,\theta_H) = \omega(v,\theta_L)$$

Hence, the promise keeping constraint is satisfied at the proposed solution. Incentive compatibility and sustainability are also trivially satisfied. Therefore, the proposed alternative pooling solution is feasible for v and is such that

$$\mu(\theta_L)\hat{b}(v,\theta_L) + \mu(\theta_H)\hat{b}(v,\theta_H) = b(v,\theta_H) > \hat{B}(v) = \mu(\theta_L)b(v,\theta_L) + \mu(\theta_H)b(v,\theta_H)$$

This is a contradiction. So, it must be that $b(v,\theta_L) \geq b(v,\theta_H)$. Suppose now that $b(v,\theta_L) = b(v,\theta_H)$. Then it must be that the pooling allocation is a solution to $(\hat{\mathrm{P}})$. By part (ii) the allocation is dynamic, $c_H(v) > c_L(v)$ and $v'_L(v) > v'_H(v)$, hence the pooling allocation cannot be a solution.

Proof of Proposition 2

Suppose for contradiction that B is (weakly) decreasing over $[v_a, \bar{v}]$ and so $v_a \in \arg\max_{v \in V} B(v)$. I am now going to show that a level of indebtedness strictly higher than $B(v_a)$ can be supported by delivering $v > v_a$, contradicting the fact that B is decreasing over its entire domain. Denote by x_a the allocation that attains $B(v_a)$. Consider the following variation for some $\varepsilon > 0$ sufficiently small:

$$m = \varepsilon > 0, \quad c(\theta) = c_a(\theta) + \varepsilon_c(\theta), \quad v'(\theta) = v'_a(\theta) \quad \forall \theta$$

where $\varepsilon_c(\theta) > 0$ is such that for all θ

$$U\left(c_a(\theta) - \varepsilon_c(\theta)\right) - U\left(c_a(\theta)\right) = \varepsilon_u \equiv U'\left(f(\varepsilon)\right) f'(\varepsilon)\varepsilon$$

Then, by construction, the proposed variation satisfies the incentive compatibility and the sustainability constraints. This variation attains a value for the borrower equal to $v_a + \varepsilon_u > v_a$. Thus, I am left to show that it increases the lenders' value too. The change in the lenders' value can be written as

$$\frac{\Delta B}{\varepsilon} \approx -\sum_{\theta \in \Theta} \mu(\theta) \left(\frac{U'\left(f(\varepsilon)\right)}{U'\left(c_a(\theta)\right)} \right) f'(\varepsilon) + \left[f'(\varepsilon) - 1\right] = f'(\varepsilon)\left[1 - \phi\right] - 1$$

where

$$\phi \equiv \sum_{\theta \in \Theta} \mu(\theta) \left(\frac{U'\left(f(\varepsilon)\right)}{U'\left(c_a(\theta)\right)} \right) < 1$$

because from Lemma 3, it follows that $c_a(\theta) \leq f(0) < f(\varepsilon)$ and, in particular, $c_a(\theta_L) < f(0)$. Thus, $\varepsilon > 0$ can be chosen to be sufficiently small that, by the Inada condition on f, $\Delta B/\varepsilon > 0$. Therefore it must be that $B(v_a + \varepsilon_u) \geq B(v_a) + \Delta B\varepsilon > B(v_a)$. Hence, B is not strictly decreasing, a contradiction. B is increasing in a neighborhood of v_a. Moreover, B is strictly decreasing over $[v^*, \bar{v}]$. In fact, $\lambda_{sust}(v) = 0$ for $v \geq v^*$ and therefore $m(v) = m^*$. Then, from

(61) it follows that for all $v \geq v^*$

$$B'(v) = -\sum_{\theta \in \Theta} \mu(\theta) C'\left(u(\theta, v)\right) = -\sum_{\theta \in \Theta} \frac{\mu(\theta)}{U'\left(c(\theta, v)\right)} < 0$$

Thus, B is strictly decreasing over $[v^*, \bar{v}]$. The continuity and concavity of B imply that there exists $\tilde{v} \in (v_a\ , v^*)$ such that B is increasing for all $v \in [v_a, \tilde{v})$ and B is strictly decreasing over $[\tilde{v}, \bar{v}]$.

Proof of Lemma 2

Consider any v in the efficient region for which there is no randomization, $v \in [\tilde{v}, \bar{v}] \cap V_{nr}$. Letting λ_{ic}, λ_{sust}, and λ_{pkc} be the Lagrange multipliers on the incentive compatibility, sustainability, and promise keeping constraints, an interior solution must satisfy the following first order necessary conditions (fonc):

$$(67) \quad c_L \ : \quad -\lambda_{pkc} = -\frac{1}{\theta_L U'(c_L)} + \frac{\lambda_{ic}}{\mu_L}$$

$$(68) \quad c_H \ : \quad -\lambda_{pkc} = -\frac{1}{\theta_H U'(c_H)} - \frac{\lambda_{ic}}{\mu_H}\frac{\theta_L}{\theta_H} + \frac{\lambda_{sust}}{\mu_H}$$

$$(69) \quad v'_L \ : \quad -\lambda_{pkc} = \frac{q}{\beta} B'(v'_L) + \frac{\lambda_{ic}}{\mu_L}$$

$$(70) \quad v'_H \ : \quad -\lambda_{pkc} = \frac{q}{\beta} B'(v'_H) - \frac{\lambda_{ic} - \lambda_{sust}}{\mu_H}$$

and the envelope condition $B'(v) = -\lambda_{pkc}$. Combining the envelope condition with the fonc for v'_H, I can write:

$$B'(v) \leq \frac{\beta}{q} B'(v) = B'(v'_H) - \frac{\beta}{q}\left(\frac{\lambda_{ic} - \lambda_{sust}}{\mu_H}\right)$$

where I used the fact that $B'(v) \leq 0$ for all $v \in [\tilde{v}, \bar{v}]$ and that, by Assumption 1, $\beta/q < 1$. If $\lambda_{ic} > \lambda_{sust}$, I can rewrite the above inequality as

$$B'(v) \leq \frac{\beta}{q} B'(v) = B'(v'_H) - \frac{\beta}{q}\left(\frac{\lambda_{ic} - \lambda_{sust}}{\mu_H}\right) < B'(v'_H)$$

By concavity of B, it follows that $v'_H(v) < v \ \forall v \in [\tilde{v}, \bar{v}]$. Hence, it is sufficient to show that $\lambda_{ic} > \lambda_{sust}$.

Consider first the case in which there is partial insurance, i.e. $\theta_H U'(c_H) \geq \theta_L U'(c_L)$. Combine the foncs with respect to c_L and c_H to get

$$0 \geq \frac{1}{\theta_H U'(c_H)} - \frac{1}{\theta_L U'(c_L)} = \frac{1}{\mu(\theta_H)} \left[\lambda_{sust} - \lambda_{ic} \frac{\theta_L}{\theta_H} \right] - \frac{1}{\mu(\theta_L)} \lambda_{ic}$$

Rearranging terms, I obtain

$$\lambda_{sust} \leq \lambda_{ic} \left(\frac{\mu(\theta_H)}{\mu(\theta_L)} + \frac{\theta_L}{\theta_H} \right) = \lambda_{ic} \left(\frac{\mathbb{E}(\theta)}{\mu_L \theta_H} \right) \leq \lambda_{ic}$$

where in the last step I use the assumption that $\mu_L \theta_H \geq \mathbb{E}(\theta)$.

Consider now the case with $\theta_H U'(c_H) < \theta_L U'(c_L)$.[17] I contend that the following conditions cannot be jointly satisfied at a solution: (i) $v \in [\tilde{v}, v^*)$, (ii) $\theta_H U'(c_H) < \theta_L U'(c_L)$ and (iii) $v'_H(v) \geq v$. Suppose for contradiction that (i)-(iii) hold. From (i) it follows that $B(v) \leq 0$; thus, it must be that

$$(71) \quad \lambda_{sust} \theta_H = \frac{f'(m) - 1}{U'(f(m)) f'(m)} \leq \mu_L C'(u_L) + \mu_H C'(u_H) = \mathbb{E}(C'(u(\theta)))$$

From (ii) it follows that

$$(72) \quad \frac{C'(u_H)}{\theta_H} = \left[\mu_L C'(u_H) \frac{\theta_L}{\theta_H} + \mu_H C'(u_H) \right] > \mu_L C'(u_L) + \mu_H C'(u_H)$$

Furthermore, notice that the incentive compatibility constraint and the promise keeping constraint imply that

$$\begin{aligned} v &= \mu_H [\theta_H u_H + \beta v'_H] + \mu_L [\theta_L u_L + \beta v'_L] = \mu_H [\theta_H u_H + \beta v'_H] + \mu_L [\theta_L u_H + \beta v'_H] \\ &= \mathbb{E}(\theta) u_H(v) + \beta v'_H(v) \Rightarrow v'_H(v) = \frac{v - \mathbb{E}(\theta) u_H(v)}{\beta} \end{aligned}$$

[17] This case never arises in any of my numerical simulation.

Combining this with (iii) implies (using the normalization $\mathbb{E}(\theta) = 1$) that

$$(73) \quad u_H \leq v(1-\beta)$$

Thus, combining (71),(72) and (73), and using the fact that $C = U^{-1}$ is convex, I obtain:

$$(74) \quad \frac{C'(v(1-\beta))}{\theta_H} \geq \frac{C'(u_H)}{\theta_H} > \mathbb{E}(C'(u(\theta))) \geq \frac{f'(m)-1}{U'(f(m))f'(m)} = \lambda_{sust}\theta_H$$

Further notice that it must be that $m \leq \underline{m}$, defined as

$$(75) \quad \theta_H v(1-\beta) + \beta v = \theta_H U(f(\underline{m})) + \beta v_a$$

Equivalently, using a change of variable $\underline{u} = U(f(\underline{m}))$, I can write

$$(76) \quad \underline{u} = v(1-\beta) + \frac{\beta(v-v_a)}{\theta_H}$$

Then, assuming that H' is decreasing (which is always true under Assumption 2), it must be that

$$(77) \quad \lambda_{sust}\theta_H = \frac{f'(m)-1}{U'(f(m))f'(m)} \geq \frac{f'(\underline{m})-1}{U'(f(\underline{m}))f'(\underline{m})} = H'\left(v(1-\beta) + \frac{\beta(v-v_a)}{\theta_H}\right)$$

Then, I have a contradiction of (74) if the following condition is satisfied for all v in the relevant region:

$$(78) \quad H'\left(v(1-\beta) + \frac{\beta(v-v_a)}{\theta_H}\right) \geq \frac{C'(v(1-\beta))}{\theta_H}$$

There exists a $\hat{v} \in (v_a, v^*)$ such that (78) holds for all $v \in [v_a, \hat{v}]$. Moreover, the larger is θ_H, the closer is $\hat{v}$ to v^*. If θ_H is sufficiently large then it follows that (78) holds for all v in the relevant region, obtaining a contradiction. Therefore it must be that $v'_H(v) < v$ for all $v \in [\tilde{v}, \bar{v}] \cap V_{nr}$ as desired.

Sufficient Conditions for Proposition 3 for $|\Theta| > 2$

Consider the general case with $|\Theta| = N \geq 2$. Letting λ_{sust} and λ_{pkc} be the Lagrange multipliers on the sustainability and promise keeping constraints and λ_n be the Lagrange multiplier associated with the incentive constraint for type θ_n (reporting θ_{n+1}), the foncs for an interior optimum can be written as:

$$\begin{aligned}
c_1 \quad &: \quad \frac{1}{\theta_1 U'(c_1)} = \lambda_{pkc} + \frac{1}{\mu(\theta_1)}\lambda_1 \\
c_n \quad &: \quad \frac{1}{\theta_n U'(c_n)} = \lambda_{pkc} + \frac{1}{\mu(\theta_n)}\left[\lambda_n - \lambda_{n-1}\frac{\theta_{n-1}}{\theta_n}\right], \qquad n = 2, ..., N-1 \\
c_N \quad &: \quad \frac{1}{\theta_N U'(c_N)} = \lambda_{pkc} + \frac{1}{\mu(\theta_N)}\left[\lambda_{sust} - \lambda_{N-1}\frac{\theta_{N-1}}{\theta_N}\right] \\
v_1' \quad &: \quad -\frac{q}{\beta}B'(v_1') = \lambda_{pkc} + \frac{1}{\mu(\theta_1)}\lambda_1 \\
v_n' \quad &: \quad -\frac{q}{\beta}B'(v_n') = \lambda_{pkc} + \frac{1}{\mu(\theta_n)}\left[\lambda_n - \lambda_{n-1}\right], \qquad n = 2, ..., N-1 \\
v_N' \quad &: \quad -\frac{q}{\beta}B'(v_N') = \lambda_{pkc} + \frac{1}{\mu(\theta_N)}\left[\lambda_{sust} - \lambda_{N-1}\right]
\end{aligned}$$

To show that $v_N'(v) < v$, it suffices to prove that $\lambda_{sust} - \lambda_{N-1} < 0$. In the relevant case with $\theta_{n+1}U'(c_{n+1}) \geq \theta_n U'(c_n)$, combining the foncs with respect to c_N and c_{N-1}, I obtain:

$$\begin{aligned}
0 \quad &\geq \quad \frac{1}{\theta_N U'(c_N)} - \frac{1}{\theta_{N-1}U'(c_{N-1})} \\
&= \quad \frac{1}{\mu(\theta_N)}\left[\lambda_{sust} - \lambda_{N-1}\frac{\theta_{N-1}}{\theta_N}\right] - \frac{1}{\mu(\theta_{N-1})}\left[\lambda_{N-1} - \lambda_{N-2}\frac{\theta_{N-2}}{\theta_{N-1}}\right]
\end{aligned}$$

which can be rearranged as

$$\begin{aligned}
\lambda_{sust} \quad &\leq \quad \frac{\mu(\theta_N)}{\mu(\theta_{N-1})}\left[\lambda_{N-1} - \lambda_{N-2}\frac{\theta_{N-2}}{\theta_{N-1}}\right] + \lambda_{N-1}\frac{\theta_{N-1}}{\theta_N} \\
&= \quad \lambda_{N-1}\left(\frac{\mu(\theta_N)}{\mu(\theta_{N-1})} + \frac{\theta_{N-1}}{\theta_N}\right) - \lambda_{N-2}\frac{\mu(\theta_N)}{\mu(\theta_{N-1})}\frac{\theta_{N-2}}{\theta_{N-1}} < \lambda_{N-1}\left(\frac{\mu(\theta_N)}{\mu(\theta_{N-1})} + \frac{\theta_{N-1}}{\theta_N}\right)
\end{aligned}$$

Then, a sufficient condition, albeit very stringent and by no means necessary, is that

$$(79) \quad \frac{\mu(\theta_N)}{\mu(\theta_{N-1})} + \frac{\theta_{N-1}}{\theta_N} \leq 1 \iff \frac{\theta_{N-1}}{\theta_N} \leq \frac{\mu_{N-1} - \mu_N}{\mu_{N-1}}$$

which is necessarily met if (i) θ_N is sufficiently large or (ii) $\mu(\theta_N)$ is sufficiently small (as $\mu(\theta_N) \downarrow 0$, the right hand side converges to 1 and consequently the condition is satisfied).

Proof of Lemma 3

Let $v = v_a$. For all $\theta \in \Theta \backslash \{\theta_L\}$ it must be that $c(v_a, \theta) = f(0)$ and $v'(v_a, \theta) = v_a$. In fact, to deliver the value of autarky in a sustainable way, it must be that for all $\theta \in \Theta$:

$$(80) \quad \omega(v_a, \theta) = \theta U\left(c(v_a, \theta)\right) + \beta v'(v_a, \theta) = \theta U\left(f(0)\right) + \beta v_a$$

Moreover, for all θ', θ such that $\theta' > \theta$, the sustainability and the incentive compatibility constraints imply that

$$\begin{aligned} (81) \quad \omega(v_a, \theta) &\geq \omega(v_a, \theta') - (\theta' - \theta) U\left(c(v_a, \theta')\right) \\ &= \theta' U\left(f(0)\right) + \beta v_a - (\theta' - \theta) U\left(c(v_a, \theta')\right) \end{aligned}$$

Combining (80) and (81), it follows that

$$\begin{aligned} \theta U\left(f(0)\right) + \beta v_a \quad &\geq \quad \theta' U\left(f(0)\right) + \beta v_a - (\theta' - \theta) U\left(c(v_a, \theta')\right) \\ &\iff \quad (\theta' - \theta) U\left(c(v_a, \theta')\right) \geq (\theta' - \theta) U\left(f(0)\right) \end{aligned}$$

which implies that $c(v_a, \theta') \geq f(0)$. (80) and the fact that $v'(\theta) \geq v_a$ imply that $c(v_a, \theta') = f(0)$, as desired. Then, it is only possible that $c(v_a, \theta_L) < f(0)$ for $\theta = \theta_L$.

For θ_L, there are two possibilities: (i) $c(v_a, \theta_L) < f(0)$ and $v'(v_a, \theta_L) > v_a$ or (ii) $c(v_a, \theta_L) = f(0)$ and $v'(v_a, \theta_L) = v_a$. If (ii) is true, v_a is an absorbing state and $B(v_a) = 0$. Suppose for contradiction that we are in case (ii). Consider a two-period variation that decreases current consumption after θ_L by ε and it increases it by ε/K in any states in the next period. That is:

$$\begin{aligned} c_H &= f(0), \quad v'_H = v_a \\ c_L &= f(0) - \varepsilon, \quad v'_L = \sum_{\theta \in \Theta} \mu(\theta) \theta U\left(f(0) + \varepsilon/K\right) + \beta v_a \end{aligned}$$

for some $\varepsilon > 0$ and $K > 0$ such that the variation satisfies the promise keeping constraint at

v_a:

$$(82)\quad \theta_L U\left(f(0)\right)+\beta v_a = U\left(f(0)-\varepsilon\right)+\beta\sum_{\theta\in\Theta}\mu(\theta)\theta U\left(f(0)+\varepsilon/K\right)+\beta^2 v_a$$
$$\iff \left[U\left(f(0)\right)-U\left(f(0)-\varepsilon\right)\right]=\beta\frac{\mathbb{E}\left(\theta\right)}{\theta_L}\left[U\left(f(0)+\varepsilon/K\right)-U\left(f(0)\right)\right]$$

To get a contradiction, it suffices to show that there exist $\varepsilon>0$ and $K>0$ such that (82) holds and

$$(83)\quad B(v_a)=0<\mu(\theta_L)\left[\varepsilon-q\varepsilon/K\right]=\mu(\theta_L)\left[1-q/K\right]\varepsilon \iff K>q$$

Rewrite (82) as

$$\int_0^{\varepsilon} U'\left(f(0)-e\right)de=\frac{\beta}{K}\frac{\mathbb{E}\left(\theta\right)}{\theta_L}\int_0^{\varepsilon} U'\left(f(0)+e/K\right)de$$

which implies, for $\varepsilon>0$ sufficiently close to zero, that

$$\begin{aligned} K &= \beta\frac{\mathbb{E}\left(\theta\right)}{\theta_L}\left[\frac{\int_0^{\varepsilon}U'\left(f(0)+e/K\right)de}{\int_0^{\varepsilon}U'\left(f(0)-e\right)de}\right]\approx\beta\frac{\mathbb{E}\left(\theta\right)}{\theta_L}\left[\frac{U'\left(f(0)\right)\varepsilon}{U'\left(f(0)\right)\varepsilon}\right]=\beta\frac{\mathbb{E}\left(\theta\right)}{\theta_L}\\ &> \left(q\frac{\theta_L}{\mathbb{E}\left(\theta\right)}\right)\frac{\mathbb{E}\left(\theta\right)}{\theta_L}=q\end{aligned}$$

where in the last step I use the fact that from Assumption 1 $\beta/q\in\left(\theta_L/\mathbb{E}\left(\theta\right),1\right]$. Then (82) and (83) hold. This is a contradiction. Therefore it must be that $c(v_a,\theta_L)<f(0)$ and $v'(v_a,\theta_L)>v_a$.

Proof of Lemma 4

Consider first borrower values in the region with *ex-post* inefficiencies. Let $v\in[v_a,\tilde{v})$. In this interval, it must be that $v'_L(v)\geq\tilde{v}$. Suppose, to the contrary, that $v'_L(v)<\tilde{v}$. By Lemma 8 we know that $c_L>0$. Consider then the following variation: decrease c_L by ε_c and increase v'_L by ε_v for some $\varepsilon_v>0$ sufficiently small and $\varepsilon_c(\varepsilon_v)>0$, defined as the unique solution to

$$\theta'_L U\left(c_L-\varepsilon_c(\varepsilon_v)\right)+\beta v'_L+\varepsilon_v=\theta_L U\left(c_L\right)+\beta v'_L$$

This variation is feasible for v in $(\hat{\mathrm{P}})$[18] and has a positive effect on the objective function:

$$\Delta B(v) = \mu_L \left[qB\left(v'_L + \varepsilon_v\right) - qB\left(v'_L\right) + \varepsilon_c(\varepsilon_v)\right] > 0$$

because it decreases the cost of providing consumption today and it also increases the value of future transfers if $\varepsilon_v > 0$ is sufficiently small. This is a contradiction. Hence, it must be that $v'_L(v) \geq \tilde{v}$ for all $v \in [v_a, \tilde{v})$. Notice how this argument applies for all v. Hence, $v'_L(v) \geq v$ for all v.

Consider now $v \in [\tilde{v}, \bar{v}]$. Let λ_{ic} be the Lagrange multiplier on the incentive compatibility constraint. Combining the necessary fonc with respect to v_L and the envelope condition, the intertemporal condition for v'_L can be written as:

$$B'(v) = \frac{q}{\beta} B(v'_L) + \frac{\lambda_{ic}}{\mu_L} \tag{84}$$

Consider first $v = \tilde{v}$. In this case, (84) can be written as:

$$0 = \frac{\beta}{q} B'(\tilde{v}) = B(v'_L(\tilde{v})) + \frac{\beta}{q} \frac{\lambda_{ic}}{\mu_L} > B(v'_L(\tilde{v}))$$

Then it must be that $v'_L(\tilde{v}) > \tilde{v}$.

Consider now borrower values in $(\tilde{v}, \bar{v}]$. I have to consider two cases, $\beta = q$ and $\beta < q$. For $\beta = q$, (84) specializes to

$$B'(v) = B(v'_L) + \frac{\lambda_{ic}}{\mu_L} > B'(v'_L)$$

Then, by concavity of B, it follows that $v'_L(v) > v$ for all $v \in (\tilde{v}, \bar{v}]$.

For $\beta < q$, rewrite (84) as:

$$B'(v) = \frac{q}{\beta} B(v'_L) + \frac{\lambda_{ic}}{\mu_L} = B'(v'_L(v)) + \frac{q - \beta}{\beta} B'(v'_L(v)) + \frac{\lambda_{ic}(v)}{\mu_L}$$

[18] Note that the same variation is not feasible for θ_H because it would violate the incentive compatibility constraint. For θ_L instead the proposed variation is actually relaxing a non-binding incentive constraint (type θ_H not reporting θ_L).

Then $v'_L(v) < v$ if and only if

$$(85) \quad -\frac{q-\beta}{\beta}B'(v'_L(v)) > \frac{\lambda_{ic}(v)}{\mu_L}$$

Suppose for contradiction that $v'_L(v) \geq v$ for all $v \in [v_a, \bar{v}]$. Then it must be that for all v condition (85) does not hold and $B'(v) \geq B'(v'_L(v))$. Therefore it follows that

$$-\frac{q-\beta}{\beta}B'(v) \leq -\frac{q-\beta}{\beta}B'(v'_L(v)) < \frac{\lambda_{ic}(v)}{\mu_L}$$

Since $\lambda_{ic}(v)$ is bounded from above, $B'(v)$ is strictly decreasing for all $v \geq \tilde{v}$, and $\lim_{v\to\bar{v}} B'(v) = \lim_{c\to\infty} -1/U'(c) = -\infty$, for v sufficiently large it must be that $-\frac{q-\beta}{\beta}B'(v) > \frac{\lambda_{ic}(v)}{\mu_L}$. This is a contradiction. Then, for v sufficiently high, condition (85) is met. Denote by $\bar{v}_q \in (\tilde{v}, \bar{v})$ the smallest value of promised utility such that (85) holds for all $v > \bar{v}_q$.[19] By the above argument, such a $\bar{v}_q$ exists.

Proof of Proposition 4

In light of Lemma 4, I can restrict attention to the compact set $[v_a, \bar{v}_q] \subset [v_a, \bar{v}]$. In fact, starting from any $v \in (\bar{v}_q, \bar{v}]$ the continuation utility is transiting to $[v_a, \bar{v}_q]$ in a finite number of periods because $v > v'_H(v) > v'_L(v)$ for all $v \in (\bar{v}_q, \bar{v}]$. To show that there exists a unique stationary distribution, I will show that the conditions in Theorem 12.12 in SLP are satisfied. In particular, I need to show that Assumption 12.1 in SLP is satisfied. To this end, define the transition $Q : [v_a, \bar{v}_q] \times \mathcal{B}([v_a, \bar{v}_q]) \to \mathbb{R}$ as

$$Q(v, A) = \sum_{\theta\in\Theta} \mu(\theta) \int_0^1 \mathbb{I}\{v'(\theta, v, \xi) \in A\}\, d\xi$$

I need to show that there exists a mixing point $v \in [v_a, \bar{v}_q]$, $K \geq 1$, and $\varepsilon > 0$ such that $Q^K(v_a, [v, \bar{v}_q]) \geq \varepsilon$ and $Q^K(\bar{v}, [v_a, v]) \geq \varepsilon$. Consider $\tilde{v}$ as the mixing point. Because $v'_H(v) < v$ for all $v \geq \tilde{v}$, it follows that starting at $\bar{v}_q$ after a sufficiently long (but finite) string of

[19] Notice that I only show that it exists a $\bar{v}_q$ such that $v'_L(v) > v$ for $v \in [\bar{v}_q, \bar{v})$ and $v'_L(\bar{v}_q) = \bar{v}_q$. I haven't shown that for all $v < \bar{v}_q$ it must be that $v'_L(v) > v$. It is however possible to define $v_q \leq \bar{v}_q$ as the largest v such that for all $v \leq v_q$ we have that $v'_L(v) > v$. The support of the limiting distribution (see the next proposition) is a subset of $[v_a, v_q] \subset [v_a, \bar{v}_q]$. In all of my numerical simulations I find that $v_q = \bar{v}_q$.

realizations of θ_H, the continuation utility transits to the region with *ex-post* inefficiencies. Thus for some finite K, $Q^K(\bar{v}, [v_a, \tilde{v}]) \geq \mu_H^K > 0$. Furthermore, by Lemma 4, $v'_L(v) \geq \tilde{v}$ for any v. Hence, starting from any v_a after drawing K realizations of θ_L, the continuation value is in the efficient region. Therefore $Q^K(v_a, [\tilde{v}, \bar{v}_q]) \geq \mu_L^K > 0$. Then just let $\varepsilon = \min\{\mu_L^K, \mu_H^K\}$. This shows that $\tilde{v}$ is a mixing point. Therefore, Theorem 12.12 in SLP applies and there exists a unique stationary distribution Ψ^* to which any efficient allocation converges. The fact that Ψ^* is non-degenerate follows from Lemmas 2, 3 and 4.

Proof of Lemma 5

Let Q_L be the space of bounded functions $q_L : [v_a, \bar{v}] \to [0, q/(1-q)]$ and let $T : Q_L \to Q_L$ be defined by the right hand side of (49). That is:

$$(Tq_L)(v) = \begin{cases} q \sum_{i=L,H} \mu(\theta_i) \left[1 + q_L(v'_i(v))\right] & \text{if } v \in (v_r, \bar{v}] \\ \frac{q}{1-q} \bar{R}(v) & \text{if } v \in [v_a, v_r] \end{cases}$$

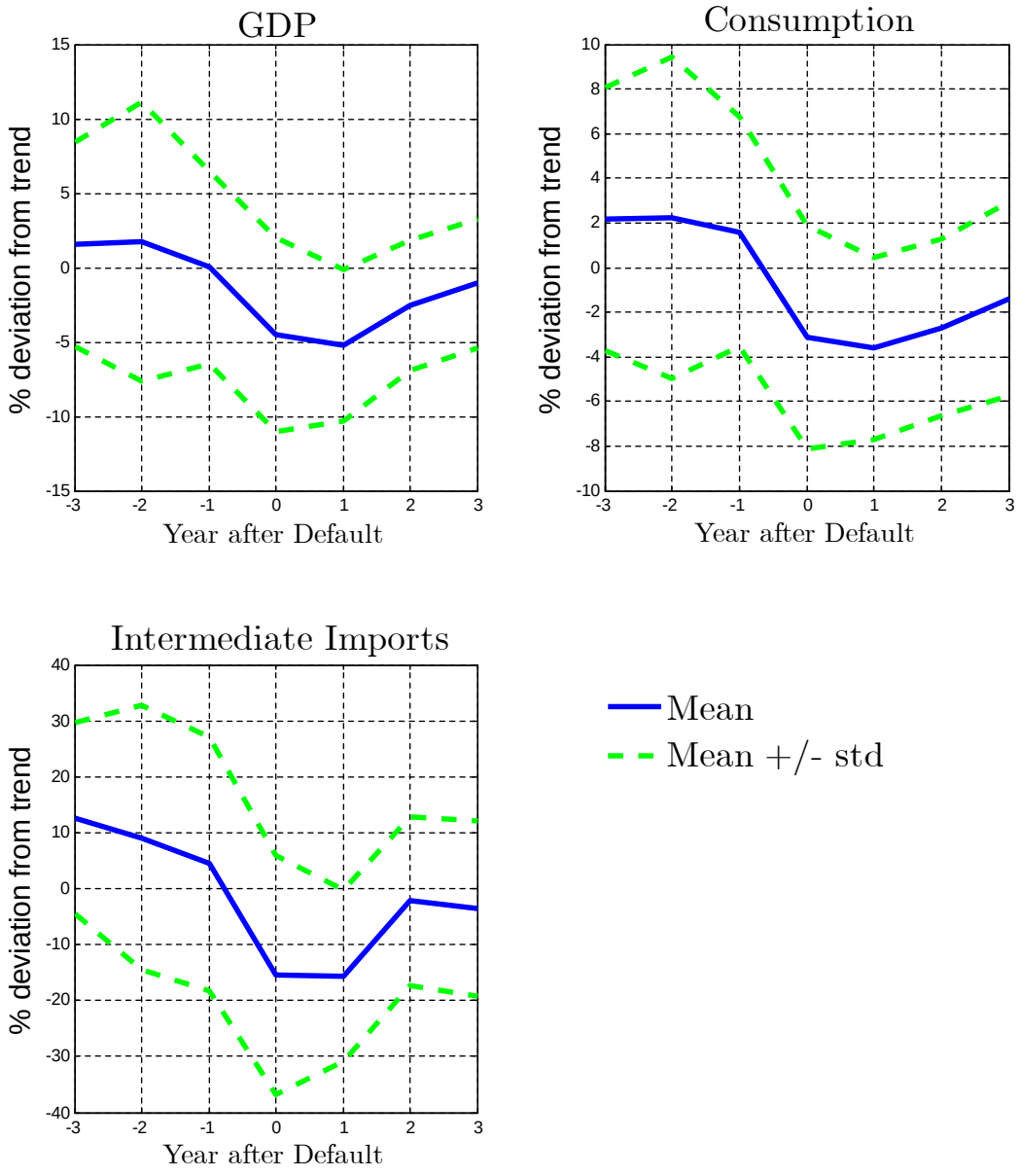
T satisfies the Blackwell's sufficient condition for a contraction mapping, see Theorem 3.3 in SLP. Then, by the contraction mapping theorem, there exists a unique fixed point of T, $\bar{q}_L$. To see that $\bar{q}_L$ is strictly increasing, first notice that $\bar{q}_L$ must be (weakly) increasing. T maps increasing functions into increasing functions. Then, by a corollary of the contraction mapping theorem (see Corollary 3.1 in SLP) it must be that $\bar{q}_L$ is increasing. To see that $\bar{q}_L$ is strictly increasing, first notice that, by definition, $\bar{q}_L$ is strictly increasing over $[v_a, v_r]$. Second, suppose for contradiction that $\bar{q}_L$ is constant over some interval. Let $[v_1, v_3] \subset [v_r, \bar{v}]$ be the first of such intervals so that for all $v \in [v_a, v_1)$, $\bar{q}_L$ is strictly increasing. For all $v_2 \in (v_1, v_3]$ we have that

$$(86) \quad \bar{q}_L(v_1) = q \sum_{i=L,H} \mu(\theta_i) \left[1 + \bar{q}_L(v'_i(v_1))\right] < q \sum_{i=L,H} \mu(\theta_i) \left[1 + \bar{q}_L(v'_i(v_2))\right]$$

To see why (86) holds, first notice that $v'_L(v_2) > v'_L(v_1)$ and $\bar{q}_L$ is weakly increasing; it follows that $\bar{q}_L(v'_L(v_2)) \geq \bar{q}_L(v'_L(v_1))$. Second, because $v'_H(v_2) > v'_H(v_1)$ and $v'_H(v_1) < v_1$, it must be that $v'_H(v_1) \in [v_a, v_1)$. Since $\bar{q}_L$ is strictly increasing in that region, $\bar{q}_L(v'_H(v_2)) > \bar{q}_L(v'_H(v_1))$. Hence (86) holds. This is a contradiction. Then $\bar{q}_L$ is strictly increasing.

C. Figures

Figure 1: Real Variables Around Sovereign Default Episodes



Data for 23 default events over the 1977- 2009 period. Same sample as in Mendoza and Yue (2012). See Data Appendix for a description of each variable.

Figure 2: Pareto and Utility Possibility Frontiers

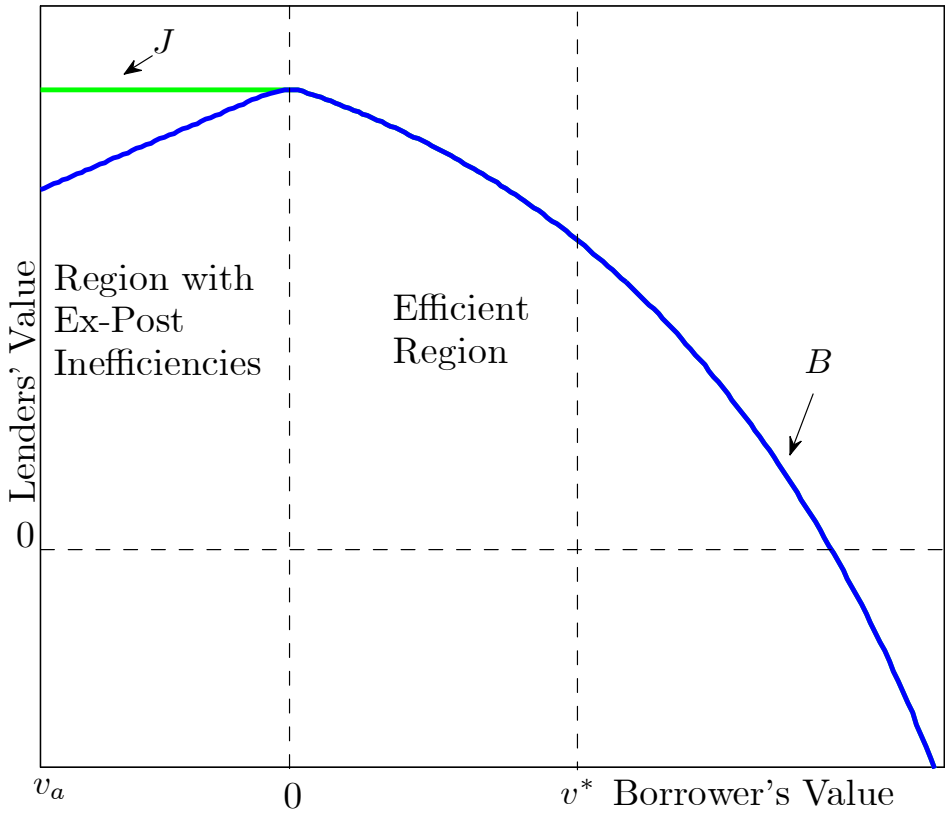


Figure 3: Law of Motion for Borrower's Value

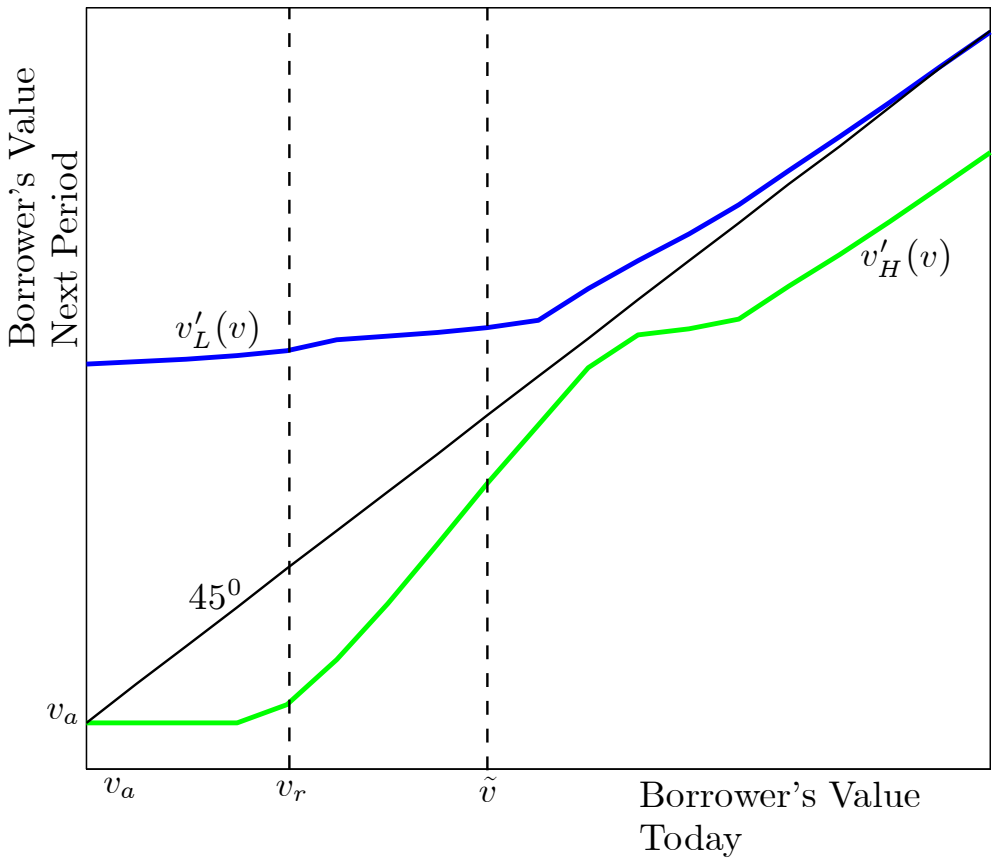


Figure 4: Policy Functions: $m(v), Y(v,\theta), c(v,\theta)$ **and** $y^*(v,\theta) - m(v)$**.**

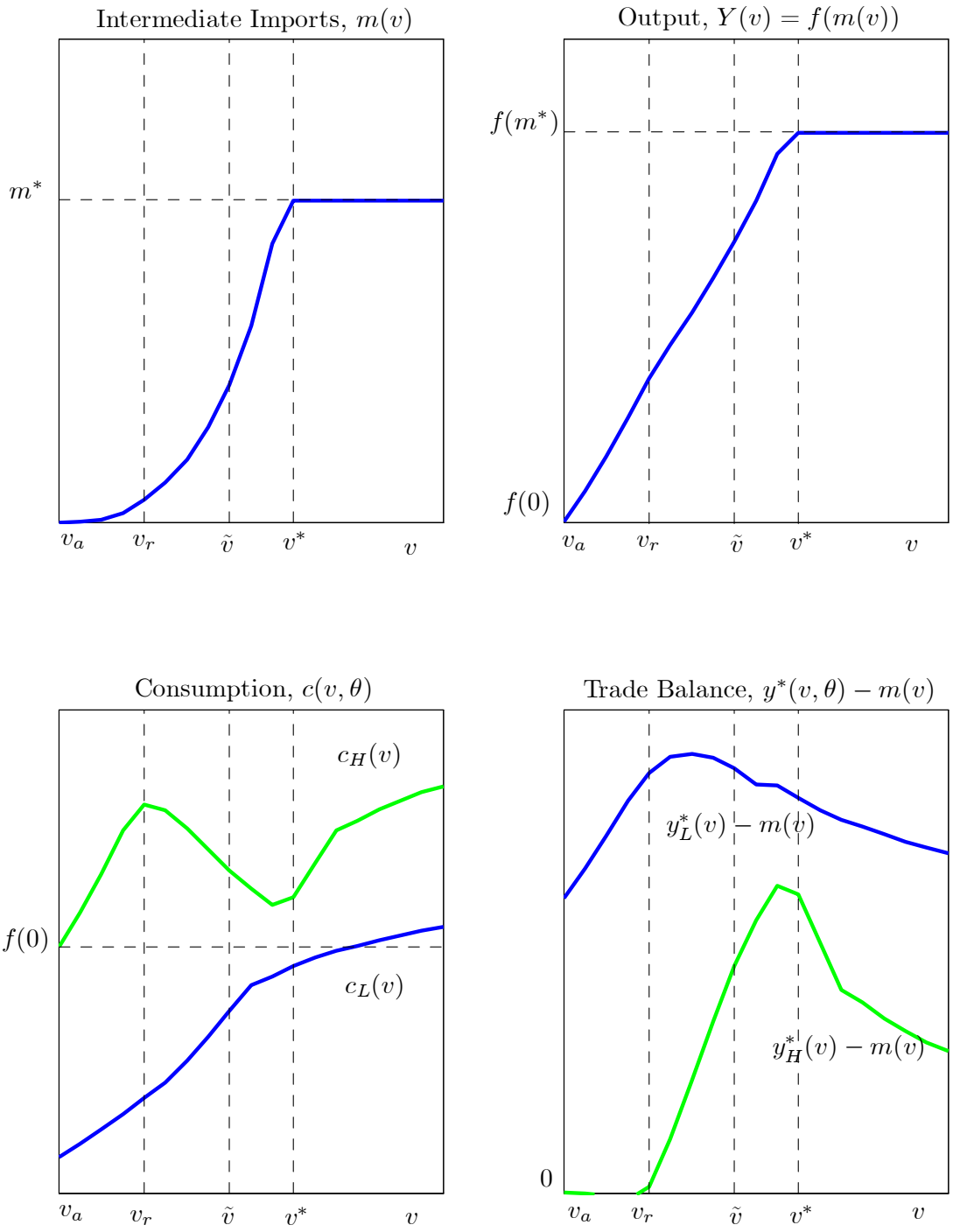


Figure 5: Outcome Path with Example of a Crisis

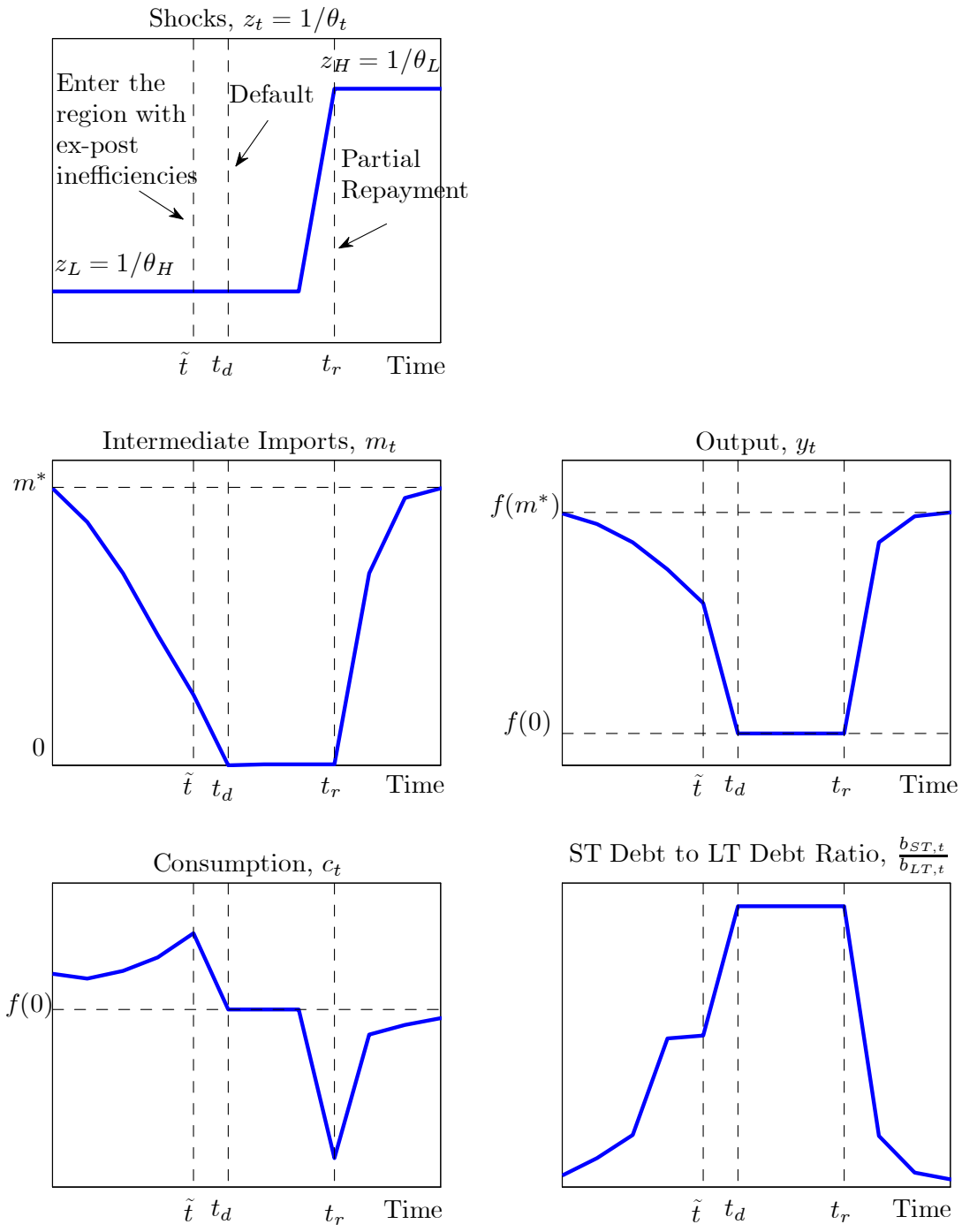


Figure 6: Bond Prices and Bond Holdings

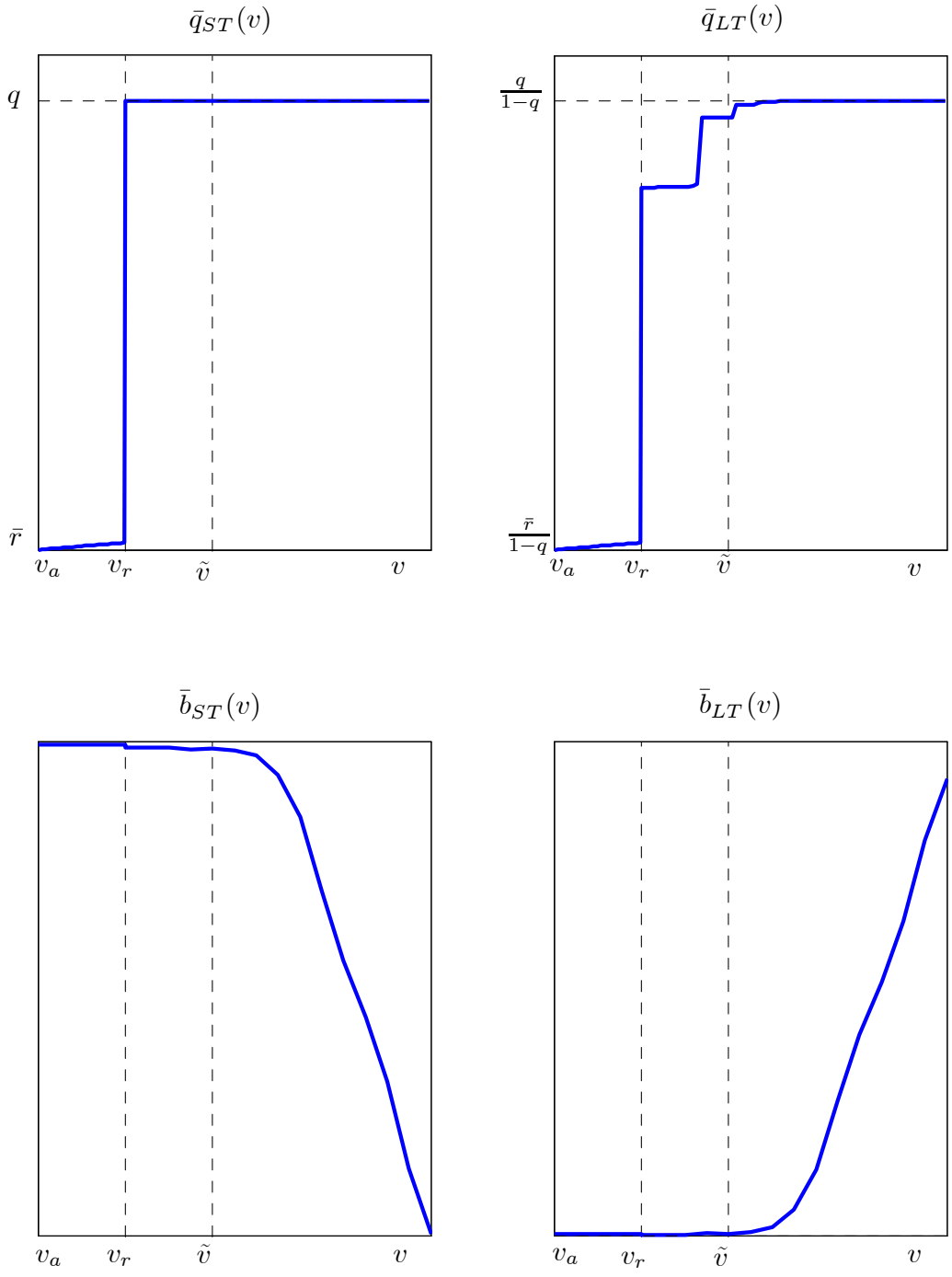


Figure 7: Ex-Post Variation in LT-Debt Price and Insurance