Endogenous Market Incompleteness Without Market Frictions

Dynamic Suboptimality of Competitive Equilibrium in Multiperiod Overlapping Generations Economies

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Abstract

In this paper, we show that within the set of stochastic three-period-lived OLG economies with productive assets (such as land), markets are necessarily sequentially incomplete, and agents in the model do not share risk optimally. We start by characterizing perfect risk sharing and find that it requires a state-dependent consumption claims which depend only on the exogenous shock realizations. We show then that the recursive competitive equilibrium of any overlapping generations economy with weakly more than three generations is not strongly stationary. This then allows us to show directly that there are short-run Pareto improvements possible in terms of risk-sharing and hence, that the recursive competitive equilibrium is not Pareto optimal. We then show that a financial reform which eliminates the equity asset and replaces it with zero net supply insurance contracts (Arrow securities) will implement to Pareto optimal stochastic steady-state known to exist in the model. Finally, we also show via numerical simulations that a system of government taxes and transfers can lead to a Pareto improvement over the competitive equilibrium in the model.

Keywords: Stochastic overlapping generations, incomplete markets

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1 Introduction

The issue of systematic risk has been central in the recent discussion of the 2008 financial market meltdown and the subsequent steep decline in real economic activity. Indeed, a key fact of the crisis was that none of the so-called financial innovations meant to help diversity risk – the securitization of long-term debt contracts and the various hedging and insurance instruments meant to backstop these contracts – were effective in preventing the market meltdown that occurred when housing prices fell. From an economic perspective, this exposure of the economy to such systematic risk was likely the result of a fundamental market incompleteness that is not captured (and indeed, cannot be captured) in conventional real business cycle macroeconomic models, where the first welfare theorem is known to be valid.

This failing has led a number of economists to examine models in which frictions such as imperfect competition or market incompleteness lead to violations of the first welfare theorem. A key development in this respect has been the large and growing literature on endogenous market incompleteness based on informational frictions – primarily involving costly monitoring of states or problems of moral hazard or adverse selection (see Sleet [19] for a survey of these models). These models derive constrained optimal general equlibrium allocations in which the binding of incentive constraints can lead to deviations from the first-best risk-sharing that occurs in a standard RBC framework. As such, these models have provided some insights into how government interventions might improve on market allocations. The constrained optimality of the allocations that result in these models is something of a drawback, however, since it implies that there can be no policies that will improve on the allocations in the model short of actually removing the friction that leads to the market incompleteness to begin with.

In this paper, we show that endogenous market-incompleteness can result in an overlapping generations framework without any informational frictions, solely due to the structure of the financial markets, and, in particular, the nature of the assets traded. Specifically, we show that the presence of a productive asset (such as land or a Lucas tree) in positive net supply in an otherwise standard stochastic exchange environment leads to sequentially incomplete markets and inefficient risk-sharing. Based on this result, we then show that a financial reform which replaces the positive net supply asset with a set of insurance contracts in zero net supply generates a strict Pareto improvement on the *liassez-faire* competitive equilibrium allocation. Actually implementing this reform requires the somewhat drastic step of having the government confiscate and redistribute the dividends associated with the productive asset, so we also examine (via simulations) the effects of government tax and transfer schemes and show that these can also generate Pareto improvements.

The remainder of the paper is organized as follows. Section 2 reviews the literature on optimality and overlapping generations models. In Section 3 we lay out the three-period-lived OLG model and show some basic results on competitive equilibrium in the model. Section 4 derives our main theoretical results on the optimality of competitive equilibrium in the model. Section 5 provides a numerical example of these results. In Section 6 we examine the Pareto improvements associated first with financial reform, and then with respect to taxes and transfers. Section 7 provides a brief overview of the empirical implications of the model and evidence in the data of the kinds of effects the model predicts. Finally, Section 8 concludes.

1.1 Literature

Economists have been concerned with the optimality of competitive equilibrium allocations in dynamic, stochastic economies since the first discovery that dynamic economies can exhibit phenomena that lead to competitive allocations which are not Pareto optimal. The Cass criterion provides a way for determining whether an allocation is suboptimal – generally due to capital overaccumulation – in the context of the neo-classical growth model using the competitive equilibrium prices associated with the allocation (Cass [6]). Gale [12] examined the optimality properties of competitive equilibrium allocations in the overlapping generations model and found that when the intertemporal marginal rate of substitution was greater than one (which he dubbed the "classical case"), the competitive allocation was dynamically Pareto optimal, but when the MRS was less than one (which he dubbed the "Samuelsonian case"), there could exist competitive equilibria whose allocations were not optimal.

In the context of overlapping generations models, most analyses of the welfare properties of equilibrium have been undertaken in the context of the simplest version of the model in which agents live two periods, and trade a single good, generally called ”consumption”. The earliest examination of welfare properties in stochastic models were by Muench [17], Peled [18], and Aiyagari and Peled [2]. In the simple model, one can show that stochastic steady-state competitive equilibria are *strongly stationary*, in the sense that endogenously determined variables are functions of the exogenous shocks alone. Hence, for models in which the exogenous shock is taken to have finite support, the model can be analyzed using standard finite-dimensional vector space techniques. The Peled and Aiyagari and Peled papers exploit this fact to show that if the competitive equilibrium is Pareto optimal, then the dominant root of the pricing kernel (i.e. the matrix of state contingent asset prices) at the equilibrium allocation will be strictly less than one.

These results have been extended in a number of directions. Work by Abel, Mankiw, Summers and Zeckhauser [1] characterized the efficiency properties of competitive equilibrium in the benchmark model with production using the Cass criterion. Zilcha [24] provides a similar characterization. Chattopadhyay and Gottardi [7] show the optimality of competitive equilibrium in a two-period-lived agents model with more than one good traded in each period. This extension is not trivial, since Spear [21] shows that strongly stationary equilibria don't exist generically when agents trade more than a single good in each period. A similar result obtains for single commodity models if agents live more than two periods. Finally, Demange [10] provides a general characterization of various notions of optimality in stochastic OLG settings, and shows that the stationary competitive equilibrium in a model in which agents trade a single good and live more than two periods satisfies the Cass criterion if markets are sequentially complete, and hence will be Pareto optimal. Our work builds on Demange's in showing that when markets are not sequentially complete, the competitive equilibrium does not allocate risk efficiently.

The observation that the risk sharing among the individuals in the society might be better with a social security system than without, has been made by among others Ball and Mankiw [3], Bohn [5] and Smetters [20].

2 The Model

We work with an overlapping generations model in which agents become economically active at age 20, and live for three 20 year periods, which we call youth, middle-age, and retirement. While our attention will be focused primarily on steady-states of the model, our interest in examining optimality issues in the model leads us to view time as unfolding from an initial period 0 in which there is a given population of initial agents who are either middle-aged or old. We impose this modeling assumption in order to make different steady-state allocations in the model Pareto comparable in the sense that we can move from such allocation to another in ways that make all agents no worse off. In particular, we wish to exclude the possibility of reallocations that make all future generations better off at the expense of an initial generation. The work of Benveniste and Cass [4] shows that if there is no initial period, then only steady-state by steady-state optimality comparisons matter. We can encompass a bi-infinite time sequence if we interpret period 0 as an initial starting period chosen arbitrarily for the purposes of analyzing the model. For the same reasons, we do not consider the question of multiple steady-state equilibria, even though such equilibria are known to exist in OLG economies. In models in which agents live two-period lives and trade a single good per period, for example, no-trade can always be supported as a competitive equilibrium, even though there will exist additional steady-state equilibria for the model. This question has been studied by (among others) Kehoe and Levine [13] and Wang [23].

Households in the model receive a deterministic labor income when young (ω_y) and middle-aged (ω_m), but must save in order to finance consumption when retired. They have two assets available: bonds which are in zero net supply and pay one unit of consumption next period, and equity, which is in fixed supply, normalized to one. Each period a dividend δ is paid out to the equity holders. The dividend is assumed to be stochastic, and to keep the development relatively simple, we assume that the dividend can take on one of two values, $\delta \in \left\{\delta^h, \delta^l\right\}$ with $\delta^h > \delta^l$. We assume that the stochastic dividend process is i.i.d., with the probablity of δ^s denoted π^s for $s = h.l$. This assumption can easily be relaxed to allow the dividend process to be Markovian.

Agent's preferences are given by a von Neumann-Morgenstern utility function

$$\mathrm{E}\left(U\right) = u\left(c_y\right) + \beta \mathrm{E}\left[u\left(c_m\right)\right] + \beta^2 \mathrm{E}\left[u\left(c_r\right)\right]$$

where the discount factor β is such that $0 < \beta \leq 1$, and c_i denotes consumption in period $i = y, m, r$. The period utility functions $u\left(\cdot\right)$ are strictly concave, strictly increasing and satisfy Inada conditions.

The assumption that one of the assets is productive is not necessary for the results we obtain, and can be replaced by the assumption that both assets are in zero net supply, as in Citanna and Siconolfi [8]. While the results we obtain will not be changed, the interpretation of the model is different in each case. Specifically, when the asset is in positive net supply, it is productive in that the dividend adds to the total resources available to the economy. When this asset is in zero net supply, it is non-productive, or speculative, in that the dividend is simply a return paid by one agent to some other agent. For our analysis, we will focus on the positive net supply case.

2.0.1 Competitive Equilibrium

We focus now on competitive equilibrium. Each agent maximizes discounted life-time expected utility conditional on the state in which the agent is born.

$$\mathrm{E}\left(U\right) = u\left(c_y\right) + \beta \mathrm{E}\left[u\left(c_m\right)\right] + \beta^2 \mathrm{E}\left[u\left(c_r\right)\right]$$

subject to the individuals' period-by-period budget constraints

$$\begin{aligned} c_y &= \omega_y - q b_y - p e_y \\ c_m &= \omega_m + b_m + \left(p + \delta\right) e_y - q b_m - p e_m \\ c_r &= b_m + \left(p + \delta\right) e_m \end{aligned}$$

where the various prices, asset holdings and consumption allocations are as yet unspecified random variables. In terms of these random variables, market clearing conditions are

$$\begin{aligned} b_y + b_m &= 0, \\ e_y + e_m &= 1, \end{aligned}$$

and the overall resource constraint

$$c_y + c_m + c_r \leq \omega_y + \omega_m + \delta = \omega$$

where ω denotes to the total resources of the economy.

In any period, young and middle aged individuals will solve their optimization problems. The first order condition with respect to the two assets for the young agents are

$$\begin{aligned} -q u'\left(c_y\right) + \beta \mathrm{E}\left[u'\left(c_m\right)\right] &= 0 \\ -p u'\left(c_y\right) + \beta \mathrm{E}\left[\left(p' + \delta\right) u'\left(c_m\right)\right] &= 0 \\ -q u'\left(c_m\right) + \beta \mathrm{E}\left[u'\left(c_r\right)\right] &= 0 \\ -p u'\left(c_m\right) + \beta \mathrm{E}\left[\left(p' + \delta\right) u'\left(c_r\right)\right] &= 0 \end{aligned}$$

Since the results we develop below depend on the structure of the stochastic processes followed by the equilibrium prices, asset holdings and consumption allocations, we first define a *short memory* (following the terminology of Citanna and Siconolfi [8]) competitive equilibrium as one in which the equilibrium prices and allocations depend on at most finitely many realizations of the exogenous shock. We will call the case where prices and allocations only depend on only the current realization of the exogenous shock a *strongly stationary equilibrium*. A key result is the following

Lemma 1: For an open and dense of OLG economies, there is no short memory competitive rational expectations equilibrium.

Proof: See Appendix A ■

This non-existence result is important because it implies that any competitive rational expectations equilibrium for the model must include lagged, endogenous variables as state variables. This was first shown in Spear and Srivastava [22]. Duffie *et al.* [11] subsequently established general equilibrium existence results for OLG economies with lagged endogenous state variables. In practice, macroeconomists working with these kinds of models (particularly for numerical simulations) typically took the endogenous state variables to be the distribution of wealth across agents. Equilibria of this type are generally referred to as recursive equilibria. While it is not possible to prove that such equilibria always exist, recent results by Citanna and Siconolfi [9] show that the set of OLG economies for which such equilibria exist are dense in the space of OLG economies, and hence, even though the model we are working with doesn't satisfy the Citanna-Siconolfi conditions required for existence of an exact recursive equilibrium, the density result in their paper justifies our focus on this equilibrium concept, particularly for our computational simulations where the best one can hope for is an approximation of the competitive equilibrium[1]. As noted earlier, there are also issues of whether the recursive equilibrium is unique or determinate that we cannot address here, though we acknowledge their importance.

To analyze the recursive equilibrium, we assume that at any point in time, the economy is characterized by the realization of the endowment process (ω), lagged bond holdings and equity holdings by the middle aged $\left[b_{m(t-1)}, e_{m(t-1)}\right]$. (In principle, the endogenous state variables should also include the bond and equity holdings of the young agents, but these can be

[1] While a detailed explication of the Citanna-Siconolfi results is beyond the scope of this paper, our application of their results is based on the two key results of their paper. The first result shows existence of simple Markov equilibria of type examined by Duffie *et al.*. The second result then shows that there is a dense subset of economies for which the Markov equilibrium can be represented as a recursive equilibrium. Since our model lies in the larger space for which existence of Markov equilibria is guaranteed, Citanna-Siconolfi's second result implies that we can find a sequence of economies having exact recursive equilibria which converges to our economy. For the numerical work we pursue to show that the competitive equilibria in our model are not Pareto optimal, this is sufficient. We note also that this approach is consistent with the Kubler and Schmedders [14] interpretation of recursive solutions in numerical simulations as approximations via "nearby" economies for which such equilibria exist.

eliminated via the market-clearing conditions.) The vector

$$\sigma = \left[b_{m(t-1)}, e_{m(t-1)}, s_t\right] \in \Sigma \subset \mathbb{R}^3$$

is the state of the economy. Hence, an equilibrium is a sequence of allocations $\{b_m(\sigma), e_m(\sigma)\}$ and a sequence of prices $\{q(\sigma), p(\sigma)\}$ such that

1. Each individual solves her/his optimization problem subject to budget constraint;
2. The bond and equity markets clear and the aggregate resource constraint holds.

Finally, and most importantly, we note that in the recursive equilibrium, markets are **not** sequentially complete. Sequential market completeness requires that if we fix the state variables at any time t, there exist sufficiently many financial instruments for agents to transfer wealth between states of the world in period $t+1$. For the recursive formulation of the model, if we fix the state variables at time t at their equilibrium values, we take the realizations of past bond and asset holdings and current endowment realizations as fixed. The states at time $(t+1)$ now consist of the current bond and asset holdings of the middle aged, together with the two possible time $(t+1)$ endowment realizations. Because we know there is no equilibrium in which time $(t+1)$ prices don't depend on the lagged state variables, there are necessarily more than two future states. But agents at time t have only the two financial instruments with which to transfer wealth across states, so the markets are necessarily sequentially incomplete. This result is obviously crucial to our results, since Demange [2002] has shown that when markets are sequentially complete, the competitive equilibrium allocation is Pareto optimal. We can also show that this market incompleteness result extends necessarily to any Markovian equilibrium in which prices depend on lagged endogenous variables. This will follow from the fact that if markets are sequentially complete, then it is possible to transform the sequential budget constraints that agents face into a single life-cycle constraint. If we let $p(z_{t-1}, s_t)$ denote the price of the equity asset when the resource shock is s_t and the lagged endogenous variables are given by z_{t-1}, and $q(z_{t-1}, s_t)$ denote the price of the bond, then the sequential budget constraints are given by

$$\begin{aligned}
c_y &= \omega_y - q(z_{t-1}, s_t)\, b_y - p(z_{t-1}, s_t)\, e_y \\
c_m &= \omega_m + b_y + \left[p(z_t, s_{t+1}) + \delta^{s_{t+1}}\right] e_y - q(z_t, s_{t+1})\, b_m - p(z_t, s_{t+1})\, e_m \\
c_r &= b_m + \left[p(z_{t+1}, s_{t+2}) + \delta^{s_{t+2}}\right] e_m
\end{aligned}$$

To convert this to a single budget constraint, we multiply c_m by $q(z_{t-1}, s_t)$ and c_r by $q(z_{t-1}, s_t)\, q(z_t, s_{t+1})$ and add the three constraints. This yields (after some manipulation)

$$\begin{aligned}
& c_y + q(z_{t-1}, s_t)\, c_m + q(z_t, s_{t+1})\, q(z_{t-1}, s_t)\, c_r = \omega_y + q(z_{t-1}, s_t)\, \omega_m + \\
& + \left[q(z_{t-1}, s_t) \left[p(z_t, s_{t+1}) + \delta^{s_{t+1}} \right] - p(z_{t-1}, s_t) \right] e_y + \\
& + \left[q(z_t, s_{t+1}) \left[p(z_{t+1}, s_{t+2}) + \delta^{s_{t+2}} \right] - p(z_t, s_{t+1}) \right] q(z_{t-1}, s_t)\, e_m
\end{aligned}$$

Now, for this to reduce to a single life-cycle budget constraint, we would require that the following no-arbitrage conditions hold

$$q(z_{t-1}, s_t) \left[p(z_t, s_{t+1}) + \delta^{s_{t+1}} \right] = p(z_{t-1}, s_t)$$

and

$$q(z_t, s_{t+1}) \left[p(z_{t+1}, s_{t+2}) + \delta^{s_{t+2}} \right] = p(z_t, s_{t+1})$$

Now, fix z_{t-1} and $s_t = h$ (say). These conditions would then require that

$$p(z_t, h) + \delta^h = p\left(z'_t, h\right) + \delta^h$$

for any z'_t which requires strong stationarity. We turn next to a discussion of optimality in the model.

3 Optimality in the Stochastic Economy

We turn now to the main question of the paper, and examine the competitive equilibrium for a model with aggregate shocks to total resources laid out in Section 2, and examine the risk-sharing properties and related optimality issues of this equilibrium. Because the model we are considering is stationary and we are focusing on stationary competitive equilibrium allocations, we will limit our notion of optimality to the case of stationary allocations. In order to compare the results we obtain for the incomplete markets environment of the model with those obtained elsewhere in the literature, we also look at the three standard notions of optimality: *ex ante*, *ex interim*, and resource conditional optimality as defined, for example, in Demange [10].

Our first result characterizes perfect risk sharing.

The optimal stationary risk-sharing allocation will be a solution to the optimization problem

$$\max_f \sum_{i=y,m,r} \gamma^i E_i u_i(f_i)$$

subject to

$$f_y(\omega^s, z) + f_m(\omega^s, z) + f_r(\omega^s, z) = \omega^s$$

where each agent's allocation is assumed to be a function of current resources ω^s, and lagged endogenous variables, which we denote by z. The γ^i's are social welfare weights the planner assigns to each agent. We index the expectation operator by each agent's type to allow for different notions of optimality. The different notions we consider are *ex ante* optimality across all agents, resource state conditional *ex interim* optimality (in which the planner considers takes the expectation for young agents over lagged endogenous variables, but not over the resource state in which the young were born), and conditional optimality (in which the planner takes the current resource state and realized lagged endogenous variables of the young as given). Letting μ be the (assumed known) invariant distribution of lagged endogenous variables, the expected utility for each middle-aged or retired agent in the optimization problem for any optimality concept is given by

$$\int \left[\pi^l u_i\left(f_i\left[\omega^l, z\right]\right) + \pi^h u_i\left(f_i\left[\omega^h, z\right]\right)\right] d\mu\left(z\right) \text{ for } i = m, r.$$

For an *ex ante* notion of optimality, this will also be the expected utility of the young. In the resource conditional *ex interim case*, the expected utility of the young will be

$$\int u_y\left(f_y\left[\omega^s, z\right]\right) d\mu(z)$$

while in the conditional case, the objective function for the young will simply be $u_y\left(f_y\left[\omega^s, z\right]\right)$. With these definitions, we can now state our result.

Theorem 1: Perfect risk sharing implies a strongly stationary consumption sequence.

Proof: For the *ex ante* and resource conditional *ex interim* cases, the first-order conditoins for the planner's problem are

$$\gamma^i \pi^s u_i'\left(f_i\right) g\left(z\right) - \lambda^s = 0, \text{ for } s = l, h \text{ and } i = y, m, r$$
$$f_y\left(\omega^s, z\right) + f_m\left(\omega^s, z\right) + f_r\left(\omega^s, z\right) = \omega^s$$

where $g(z)$ is the Radon-Nikodym derivative of the measure μ with respect to z. Theses conditions imply that

$$\gamma^i \pi^s u_i'\left(f_i\right) = \gamma^j \pi^s u_j'\left(f_j\right) \text{ for } s = h, l \text{ and } i \neq j.$$

Hence, we can solve for say f_y and f_m in terms of f_r. Substituting back into the resources constraints will then yield allocations which are strongly stationary. Taking ratios in the first set of first-order conditions, we get the usual equality of state contingent marginal rates of substitution condition:

$$\frac{\pi^l u_i'\left(f_i\left[\omega^l, z\right]\right)}{\pi^h u_i'\left(f_i\left[\omega^h, z\right]\right)} = \frac{\lambda^l}{\lambda^h} \text{ for } i = y, m, r$$

so that risk in this allocation is being shared optimally. Note in particular that the probability $g(z)$ for lagged endogenous state variables drops out, since it is the same across current realizations of total resources.

For the conditional optimality case, the first-order conditions for the middle-aged and retired are as above, while for the young, they become

$$\gamma^y u_y'(f_y) - \lambda^s = 0.$$

In this case, since the Lagrange multiplier associated with the resource constraint doesn't depend on lagged endogenous variables, the allocation of the young will also be independent of these variables. Since the first-order conditions of the middle-aged and retired remain as before, their allocations are also independent of lagged endogenous variables, and we again obtain the result that optimal allocations are strongly stationary.■

This result is quite intuitive. Since the exogenous uncertainty is independent of any endogenous uncertainty, the optimal allocation simply ignores endogenous fluctuations, and allocates total resources in a way that minimizes the variance associated with these fluctuations. There is one additional curious possibility that we might need to consider, however, which is that the weights in the social planner's problem might themselves be functions of lagged endogenous variables. Since the first-order conditions require that

$$\gamma_y \pi^s u_y'(f_y) = \gamma_m \pi^s u_m'(f_m) = \gamma_r \pi^s u_r'(f_r) \text{ for } s = l, h$$

this would yield allocations depending on the lagged endogenous variables. But in this case, since the planner is free to choose the weights, *ex ante* optimization would lead him to choose constant weights, since these minimize the variances of each agent's allocation. Note finally that this result can be extended readily to Markovian shock processes.

This result tells us immediately that the competitive equilibrium in a stochastic OLG economy in which agents live for more than two periods does not allocate risk optimally. Since it is only the young and middle-aged who have incentives to engage in risk-sharing trades, the inefficient risk-sharing implies that there are trades available to the young and middle-aged which will improve risk-sharing, and hence, lead to a short-run Pareto improvement for the economy. Hence, we have the following result.

Theorem 2: If agents live more than two periods, the laissez-faire competitive equilibrium allocation is not dynamically Pareto optimal.

Proof: Since the competitive equilibrium is not strongly stationary, risk is not allocated optimally. If it occurs that in some period t the young and middle-aged agents contingent allocations in the following period are such that their state-contingent marginal rates of substitution at these allocations are not equal then these agents can generate a bilateral *ex ante* Pareto improvement by contracting between themselves to have the agent with the smaller MRS move along her indifference curve toward the diagonal (reducing variance while keeping expected utility constant), while the agent with the larger MRS takes the opposite side of this swap. This makes the agent with the higher MRS strictly better off. Since this improvement makes no other agent in the economy worse off, it constitutes a short-run Pareto improvement, and hence, the competitive equilibrium is not short-run optimal. Since short-run optimality is necessary for long-run optimality, it follows that the competitive equilibrium is not Pareto optimal.■

As a corollary, the same argument establishes that the competitive allocations will not be Pareto optimal for any equilibrium which depend non-trivally on lagged endogenous state variables. These results can also be extended to allow for finitely many heterogenous types of agents and life cycles of more than three periods.

We turn next to a set of numerical simulations that illustrate the results shown here.

4 Numerical simulations

The numerical simulations are meant to illustrate our theoretical results, to give an indication of the quantitative importance of the suboptimality of the C.E., and to give an estimate for the time necessary to reach the stationary, optimal equilibrium. For the numerical work, we assume that utility funcitons are CRR, of the form

$$u(c_i) = \frac{c_i^{1-a}}{1-a} \text{ for } i = y, m, r$$

where a is the agent's coefficient of relative risk aversion. The numerical solution technique uses the parameterized expectations approach to find the rational expectations equilibrium of the model. The details about the computational approach are in Appendix B.

4.1 Parametrization

As a benchmark, we work with a parametrization of the model in which both the dividend and labor income are stochastic. Specifically, we assume that labor's share of the total endowment is $\frac{2}{3}$, and the ratio of labor income when young to labor income when middle-aged is $\frac{3}{5}$. So, for any given total endowment ω, we have

$$\begin{aligned}\delta &= \frac{1}{3}\omega, \\ w_y &= \frac{3}{8}\frac{2}{3}\omega = \frac{1}{4}\omega, \\ w_m &= \frac{5}{8}\frac{2}{3}\omega = \frac{5}{12}\omega.\end{aligned}$$

The endowment-process follows a two-stage Markov process. For the benchmark simulations, the endowment process is assumed to be i.i.d. with realizations $\{\omega^l, \omega^h\} = \{0.95, 1.05\}$.

The time-preference parameter $\beta = 1$, and the risk aversion coefficient $a = 2$.

4.2 Numerical results

For these parameters, the socially optimal allocation gives each agent 0.35 units of consumption in the high state, and 0.317 units in the low state. The expected utility for the social optimum is $\mathrm{E}u = -9.02$. The standard deviation of the social optimum is 0.0233.

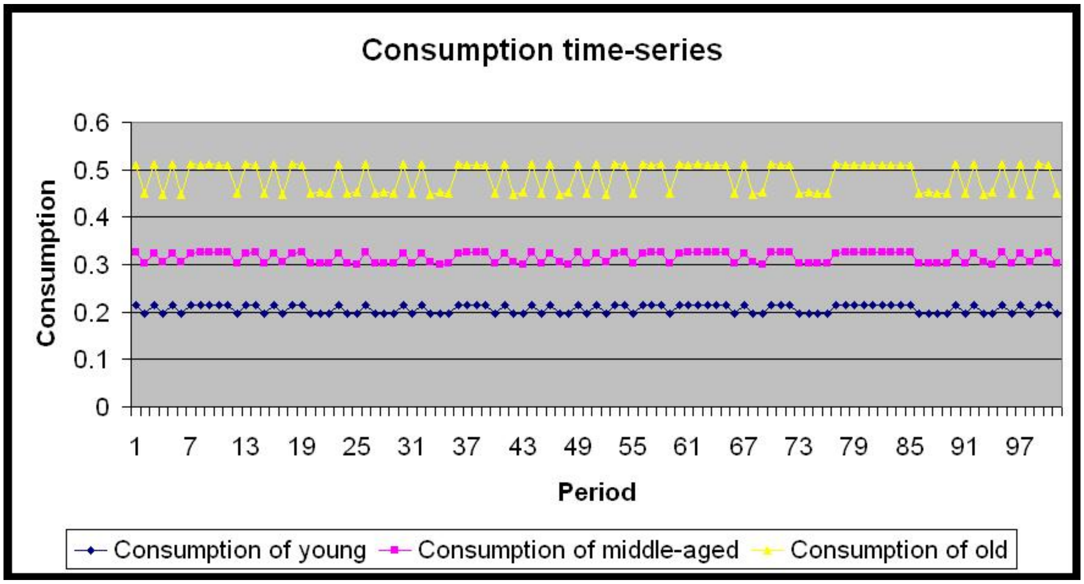
At the competitive allocation, Table 1 shows the average consumptions and standard deviations of consumption for the various types of agents in the high and low states

	c_y	c_m	c_r
Low	0.19719	0.30268	0.45013
(SD)	0.00018	0.00099	0.00117
High	0.21388	0.32490	0.51122
(SD)	0.00020	0.00112	0.00131

Table 1: Summary statistics, benchmark

The chart below shows the time-series of consumption for a typical equilibrium simulation of the model. The time-series clearly illustrate that the variance of consumption for the old is significantly larger than for the middle-aged, and an order of magnitude higher than that of the young, which strongly suggests that there may be room for improved risk-sharing.

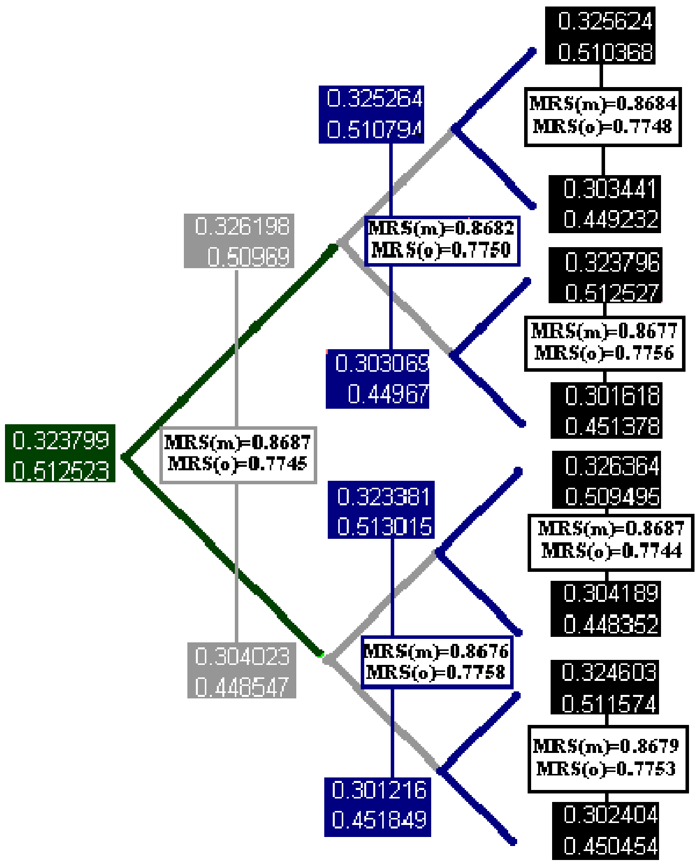
The expected utility for a typical agent can be estimated from the simulation data using the fact that the equilibrium allocations follow an ergodic stochastic process, so that time-series and cross-sectional averages will be the same. From this data, we find that $\mathrm{E}u = -10.14$. Hence, the socially



optimal allocation improves overall expected utility for this calibration of the model by roughly 11%. The overall average standard deviations of consumption for each type of agent are $\sigma_y = 0.008$, $\sigma_m = 0.011$, and $\sigma_r = 0.031$. Clearly, the old bear far more risk in the competitive equilibrium than they do at the social optimum.

We illustrate the results implied by Theorem 2 in our numerical simulation by first showing directly that the kind of improved risk-sharing described in the theorem is indeed possible in the simulated economy. The diagram below shows the tree of realized consumptions for middle-aged and old agents over a span of four periods. In the tree, branches going up indicate realizations of the high resource state, while branches going down indicate a low resource state realization. For each allocation at each node of the tree, we calculate the marginal rate of substitution between low and high state allocations for both middle-aged and old, and, as is apparent from the diagram, the middle-aged have uniformly higher MRS's than do the old, indicating that there is always a risk-improving reallocation of the type indicated above that is possible.

We can generalize this direct demonstration to show that such improvements are always possible using the fact that the distribution of allocations generated by the competitive equilibrium is ergodic, and hence time averages and cross-sectional averages will be the same. Consider a middle-aged



and a retired agent at any time t. For the parametrization of the model, the state-contingent MRS for the middle-aged agent will be

$$MRS_m^{hl} = \left(\frac{c_m^l}{c_m^h}\right)^2$$

while that of the old agent will be

$$MRS_r^{kl} = \left(\frac{c_r^l}{c_r^h}\right)^2$$

where c_j^i is the $j = H, L$ state allocation of agent $i = m, r$. Using the resource constraint, we can write the old agent's MRS as

$$
\begin{aligned}
MRS_r^{hl} &= \left(\frac{r^l - c_y^l - c_m^l}{r^h - c_y^h - c_m^h} \right)^2 \\
&= \left(\frac{\hat{r}^l - c_m^l}{\hat{r}^h - c_m^h} \right)^2
\end{aligned}
$$

where $\hat{r}^s$ is the total state s resources net of the allocation of the young. Now, consider the middle-aged agent's share of net resources in the low state

$$
\frac{c_m^l}{\hat{r}^l}
$$

and suppose that this agent gets the same share of net resources when the state is high. In this case, her MRS will be

$$
MRS_m^{lh} = \left(\frac{c_m^l}{\left(\frac{c_m^l}{\hat{r}^l} \hat{r}^h \right)} \right)^2 = \left(\frac{\hat{r}^l}{\hat{r}^h} \right)^2 .
$$

Since the old agent's share in the high state in this case is $1 - \frac{c_m^l}{\hat{r}^l} = \frac{c_r^l}{\hat{r}^l}$, it follows that the old agents state-contingent MRS will be

$$
MRS_r^{lh} = \left(\frac{c_r^l}{\left(\frac{c_r^l}{\hat{r}^l} \hat{r}^h \right)} \right)^2 = \left(\frac{\hat{r}^l}{\hat{r}^h} \right)^2
$$

and hence, since the MRS's are equal, we would have optimal risk-sharing. Since we know that the CE does not allocate risk optimally, it must be the case that one agent's share of the net resources in, say, the high state is greater than the other agent's. So, suppose the middle-aged agent gets a smaller share in the high state than she does in the low state. Let $0 < \varepsilon < 1$ and assume that her high state share is

$$
(1 - \varepsilon) \frac{c_m^l}{\hat{r}^l} .
$$

Then the old agent's high state share will be

$$
(1 + \varepsilon) \frac{c_r^l}{\hat{r}^l} .
$$

Plugging these into the MRS formulas, we have

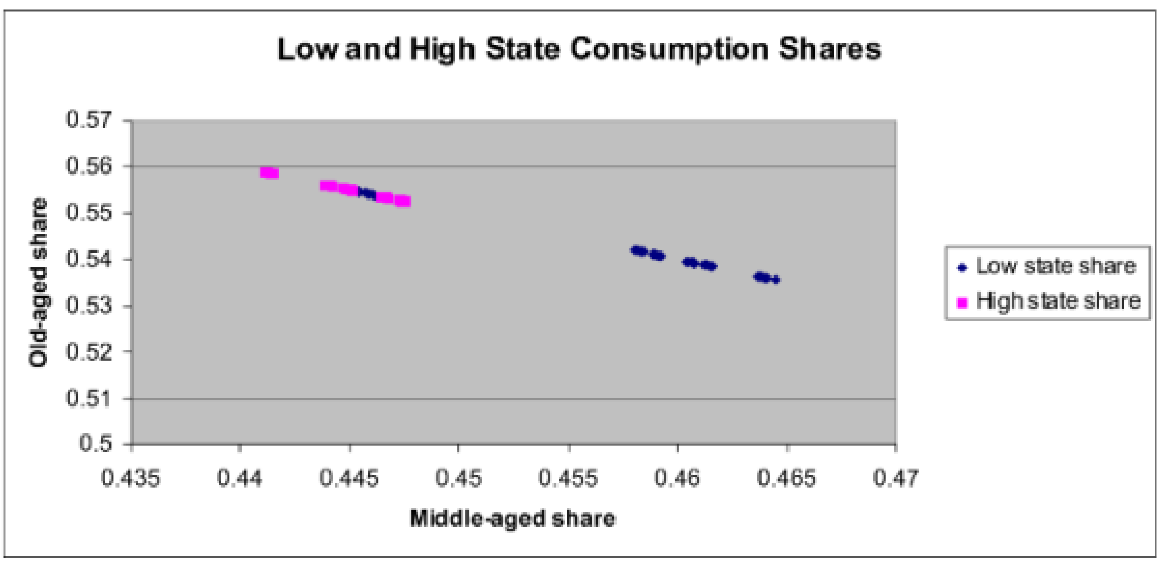
$$MRS_m^{lh} = \left(\frac{c_m^l}{(1-\varepsilon) \frac{c_m^l}{\hat{r}^l} \hat{r}^h} \right)^2 = \left(\frac{\hat{r}^l}{(1-\varepsilon) \hat{r}^h} \right)^2$$

and

$$MRS_r^{lh} = \left(\frac{c_r^l}{(1+\varepsilon) \frac{c_r^l}{\hat{r}^l} \hat{r}^h} \right)^2 = \left(\frac{\hat{r}^l}{(1+\varepsilon) \hat{r}^h} \right)^2$$

and the old agent will have a smaller state-contingent MRS than the middle-aged.

The simulated data show that these kinds of opportunities for improved risk-sharing always exist. The chart below shows the shares for old and middle-aged in each of the resource states for a simulated time-series (after convergence of the numerical algorithm used to compute the equilibrium) consisting of 3000 periods. Clearly, the old get a uniformly higher share of net resources in the high state, while the middle-aged get a higher share in the bad state. Hence, via the argument above, there are always opportunities for improved risk-sharing.



5 Decentralized optimality

Since the socially optimal allocation for the economy is a Pareto optimal steady-state, the second welfare theorem implies that it can be supported as a competitive equilibrium after some reallocation of resources. Since we know that when the productive asset pays any positive dividend, the resulting competitive equilibrium will not be strongly stationary and hence not optimal, it follows that to implement the optimal steady-state, the central planner (which we will refer to hereafter as the government) must completely tax away the dividend, and then redistribute it back to the agents as lump sum transfers. One way the government could implement the optimal steady-state would be to give agents exactly the right shares of total resources so as to implement the optimal allocation as a no-trade equilibrium. While this is easy enough to do in our simple three-period setting, in more complicated economies, it would require the government to hit an allocation target having measure zero in the space of all possible state contingent allocations. So, we inquire instead whether, having taxed away the dividend and then rebated it arbitrarily, agents can then trade via competitive markets to the optimal allocation. The answer to this question is yes, if we impose a kind of financial reform and introduce a set of Arrow securities to allow agents to insure themselves against the intertemporal effects of the resource shocks.

5.1 Financial Reform

To show this result, we denote the post-transfer endowments of agents as $\tilde{\omega}_i^s$, for $i = y, m, r$ and $s = h, l$. We assume that agents can (as before) trade one period bonds in zero net supply in order to allocate income intertemporally. We also introduce a set of Arrow securities, denoting the holdings of a type i agent born in resource state s which pays off in one unit of consumption in state s' by $a^{ss'}$. We will denote the bond prices by q^s, as before, and let p^s be the (nominal) price of the Arrow security that pays off in state s. With these modifications, the budget constraints for the model (under the assumption that we are at a strongly stationary allocation) take the form

$$\begin{aligned}
c_y^s &= \tilde{\omega}_y^s - q^s b_y^s \ \text{ for } s = h, l \\
\sum_{s'} p^{s'} a_y^{ss'} &= 0 \ \text{ for } s = h, l \\
c_m^{ss'} &= \tilde{\omega}_m^{s'} + b_y^s + a_y^{ss'} - q^{s'} b_m^{s'} \ \text{ for } \left(s, s'\right) \in \{h, l\}^2 \\
\sum_{s''} p^{s''} a_m^{s's''} &= 0 \ \text{ for } s' = h, l \\
c_r^{s's''} &= b_m^{s'} + a_m^{s's''} \ \text{ for } \left(s', s''\right) \in \{h, l\}^2 .
\end{aligned}$$

The constraint on the Arrow securities requires that they be self-financing in the sense of allowing transfers only between different states of nature and not over time.

The first-order conditions for the budget constrained utility maximizations of young and middle-aged agents are

$$\begin{aligned}
-q^s u'\left(c_y^s\right) + \sum_{s'} \pi^{s'} u'(c_m^{ss'}) &= 0 \quad \text{for } s = h, l \\
u'\left(c_m^{ss'}\right) - \lambda^s p^{s'} &= 0 \ \text{ for } \left(s, s'\right) \in \{h, l\}^2 \\
-q^{s'} u'\left(c_m^{ss'}\right) + \sum_{s''} \pi^{s''} u'(c_r^{s's''}) &= 0 \ \text{ for } \left(s', s''\right) \in \{h, l\}^2 \\
u'\left(c_r^{s's''}\right) - \lambda^{s'} p^{s''} &= 0 \ \text{ for } \left(s', s''\right) \in \{h, l\}^2
\end{aligned}$$

together with the budget constraints above. Here, λ^s is the Lagrange multiplier associated with the self-financing condition on the Arrow securities. Market clearing for the model requires that

$$\begin{aligned}
c_y^s + c_m^{s's} + c_r^{s's} &= \tilde{\omega}_y^s + \tilde{\omega}_m^s + \tilde{\omega}_r^s \ \text{ for } s = h, l \\
b_y^s + b_m^s &= 0 \ \text{ for } s = h, l \\
a_y^{s's} + a_m^{s's} &= 0 \ \text{ for } \left(s, s'\right) \in \{h, l\}^2 .
\end{aligned}$$

Now, the first-order condition for the middle-aged agent implies that consumption of the middle-aged at equilibrium can't depend on the agent's birth state, since the right-hand expected utility of consumption when old for the agent doesn't depend on the birth state. Together with the resource constraints, this implies that consumption when old only depends on the current resource state. The first-order conditions for the Arrow securities then imply that the Lagrange multipliers are independent of the lagged shock. Note also that the consumption market-clearing conditions are redundant given asset market-clearing and the budget constraints. Finally, the first-order conditions for the Arrow securities implies that

$$\frac{u'\left(c_m^h\right)}{u'\left(c_m^l\right)} = \frac{p^h}{p^l}$$

while the first-order condition with respect to the middle-aged agent's bond holdings implies that

$$\frac{u'\left(c_m^h\right)}{u'\left(c_m^l\right)} = \frac{q^l}{q^h}$$

so that the bond and Arrow security prices are not independent.

Hence, we are left then with a system of 14 equations in 14 variables: the 2 bond prices, 4 bond holdings, 8 Arrow security holdings. These equations can be solved under standard conditions using standard techniques.

To show that the competitive equilibrium allocation allocates risk optimally, consider the first-order conditions for the Arrow security holdings for a middle aged agent. Fixing s and taking $s' = h, l$, and taking ratios we get

$$\frac{u'\left(c_m^h\right)}{u'\left(c_m^l\right)} = \frac{p^h}{p^l}.$$

Doing the same thing for the old agent, we find

$$\frac{u'\left(c_r^h\right)}{u'\left(c_r^l\right)} = \frac{p^h}{p^l}.$$

Hence, the middle-aged and old have their state-contingent marginal rates of substitution equalized and are sharing risk optimally. The first-order conditions with respect to bond holdings, together with the price dependence between bond and Arrow security prices show that the state-contingent MRS of the young is also equal to p^h/p^l.

To show finally that the socially optimal allocation will be attained in this setting, we need to make one more assumption: the government transfer to the old is not so large that the endowment allocation is already optimal. We need this assumption since it's possible that if the transfer to the old is large, the endowment allocation will be in the classical region. Given this assumption, we can determine the competitive equilibrium prices which support the social optimum by plugging in the consumptions at the social optimum in the equilibrium equations above, and backing out the prices.

There are a couple of important observations about this result that we should make. First, the result shows in particularly stark form the importance of providing some form of intergenerational insurance if we are interested in agents' overall welfare. It is clear from the budget constraints, particularly for the old agents, that the Arrow securities allow the agent to smooth consumption across shock realizations in ways that the bond holdings alone do not. This suggests an obvious role for government insurance programs in the absence of private provision of assets of this form.

The second observation concerns the fact that we can find a strongly stationary equilibrium with Arrow securities, but not with privately held productive assets. While it might be tempting to think that this is due to the fact that the productive asset is in positive net supply, this is not the reason, since we can show the same inoptimality result if the equity asset is assumed to be in zero net supply. (Indeed, none of our optimality analysis relies at all on the fact the the equity asset is in positive net supply.) Rather, the reason we can find a strongly stationary equilibrium with the Arrow securities stems from the self-financing constraint. Because the insurance actions of the Arrow securities are required to be self-financing, there is no change in wealth associated with realizations of states of nature, and hence no active changes in what would otherwise be the endogenous state variables in the model. In the presence of a productive asset, or a non-productive speculative asset which pays a state contingent dividend, there are real welath effects generated by the resolution of uncertainty. These wealth effects create subsidy relationships between agents that break the possibility of agents facing a single, life-time budget constraint, and thus generate the incomplete markets phenomenon.

While we have demonstrated how the second welfare theorem can be applied, the requirement that the government essentially confiscate the full return on the assets is obviously problematic. To the extent that asset returns provide incentives in more complicated economic environments in which agents make investments in productive activities, removing this incentive by taxing it completely away will cause obvious problems. So, we look next at an alternative to the full application of the second welfare theorem, by considering a tax on dividends that is significantly less than confiscatory, coupled with a lump sum rebate of the tax which moves the economy in the "right direction" toward the social optimum, and ask whether this will implement a set of Pareto improving allocations.

5.1.1 Stochastic steady state with taxes and transfers

Kubler and Kruger [14] showed via a numerical example that a pay-as-you-go social security system can lead to Pareto improvements after bad state realizations in a model similar to ours, so there is reason to think this will also be so here. The model with taxes and transfers is the same as our benchmark model, except that we modify the budget constraints to reflect the fact that the government now imposes a proportional tax of t on the dividend realization, and uses the proceeds from this tax to give lump sum transfers τ^i, $i = y, m, r$ to the households. Hence, the sequential budget constraints become

$$
\begin{aligned}
c_y &\leq \omega_y - qb_y - pe_y + \tau_y \\
c_m &\leq \omega_m + b_y + e_y \left[p' + (1-t)\,\delta\right] - q'b_m - p'e_m + \tau_m \\
c_r &\leq b_m + e_m \left[p" + (1-t)\,\delta\right] + \tau_r.
\end{aligned}
$$

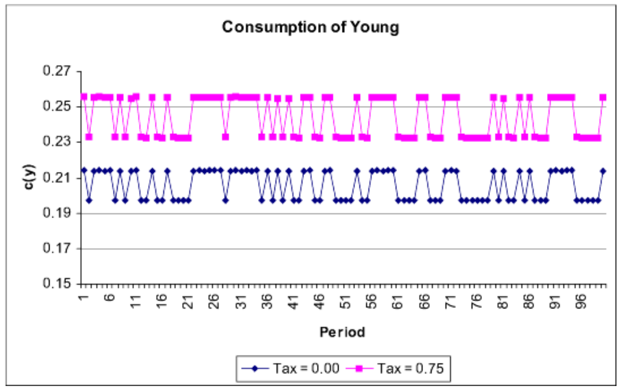
The government's budget constraint requires that in each period

$$\tau_y + \tau_m + \tau_r = t\delta.$$

We compute the recursive competitive equilibrium of the tax-transfer model for tax rates of 25%, 50% and 75%, under the assumption that all of the lump sum transfer goes to the old. The results for each tax level clearly improve risk-sharing and raise expected utility. The table below shows the variance of consumption for the old, the overall expected utility, and minimum consumption for the young at each tax rate.

Tax rate	$var(c_r)$	min (c_y)	EU
0.00	0.000934	0.19692	-10.14377318
0.25	0.000859	0.2053	-9.963402732
0.50	0.000770	0.21629	-9.75796138
0.75	0.000645	0.23241	-9.51129027

The chart below plots the consumptions of the young in the zero tax case and the 75% tax case over a typical run of 100 periods. From the diagram, it is clear that the young will be getting strictly more consumption under the tax-transfer equilibrium than in the zero tax case. Combining this observation with the expected utility improvement, we infer that the tax-transfer equilibrium will in fact Pareto dominate the zero tax equilibrium.



6 Empirical Implications

While the current model – particularly given its abstracting away of production – is far to coarse to provide detailed (even calibrated) empirical predictions, it is not without empirical content. First, the model predicts that given the standard equity-structured asset market, a purely *laissez-faire* market system misallocates risk, and, in particular, imposes much more risk on retired agents than it does on young or middle-aged agents. This result is intuitive: young and middle-aged agents can actively balance or rebalance their asset portfolios to offset market risk. The old cannot. In the model, the optimal response to this is for the old to save in excess of what they would if they didn't face the degree of risk imposed in the *laissez-faire* competitive equilibrium. Indeed, when we move from the equity-based model of assets to one based purely on decentralized social insurance, the average consumption of the old falls along with the variance of their consumption, and that of the young rises significantly.

From an empirical perspective, then, one obvious implication of the model is that insuring the consumption flows of old agents, who cannot hedge against systemic risk leads to improved risk-sharing. This in turn suggests an important role for social insurance programs like the U.S. Social Security system and other similar old-age insurance systems elsewhere in the world. But the model also predicts that a second Pareto improvement associated with reducing the risk face by old agents is a concurrent increase in the consumption for the young. Hence, we need to ask whether we in fact see such reallocative transfers occurring in the data. We would argue that we do. In addition to Social Security transfers that help reduce consumption variance in old age, we can identify another substantial set of income transfers at work that actually benefit younger households, which are paid for through taxes levied on older, wealthier households. These benefits for the young and middle-aged households include the deductibility of home mortgage interest, the deductibility of property taxes (which largely support public education), exclusion of employer contributions for health care and insurance, direct federal support of public education at both K-12 and post-secondary levels, together with various education-related tax deductions, standard deductions for child support, and direct child care credits. In 2004, these expenditures amounted to about \$315 billion. Add to this another \$170 billion that the States spend to support their public university systems, and we get a total benefit of \$485 billion that society provides to younger families as they get started. This is on a par with the \$500 billion spent in 2004 on Social Security payments. These transfers are typically justified in terms of the benefit principle: young households that benefit from the various cost reductions associated with the transfers should pay for them, though delaying repayment until later in life provides a net benefit to the individual beneficiaries. Our model suggests that there is a deeper *quid pro quo* at work in the form of a social contract that leads to improved risk-sharing over the life-cycle.

7 Conclusion

The analysis we have presented demonstrates unambiguously that the *laissez-faire* competitive equilibrium in a multi-period OLG economy with productive assets will be Pareto suboptimal because of imperfect risk-sharing. The deviation from the first welfare theorem arises because of the restricted market participation imposed on unborn agents by the finite lifetimes assumption underlying the OLG environment, and the endogenous market-incompleteness generated by the weak stationarity of the competitive equilibrium in the multi-period setting.

On a very fundamental level, these results also have clear and obvious policy implications for the ongoing debate over whether governments should provide social insurance. Compared with a situation where the government has no role in redistributing income across generations, our exercise shows that government intervention can improve upon the risk sharing between the individuals and therefore the welfare of everybody. The exact extent of government intervention is harder to quantify. Since the two factors of production are supplied inelastically in our model economy, we are also ignoring any potential tax-induced distortions which might reduce welfare.

In a world where governments can not solely rely on lump-sum taxation, there will exist trade-offs between risk sharing and efficiency in production. We leave this open for further research.

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8 Appendix A

Proof of Lemma 1.

We will outline the proof for the non-existence of a strongly stationary equilibrium. The extension to short memory equilibria is straightforward, and details can be found in Citanna and Siconolfi (2007). So, assume, for the moment, that there is a strongly stationary equilibrium. Then the assumption that the exogenous dividend shock $s_t \in \{h, l\}$ and the shocks are i.i.d., allows us to write the budget constraints and first-order conditions as

$$\begin{aligned} c_y^s &= \omega_y - q^s b_y^s - p^s e_y^s \\ c_m^{ss'} &= \omega_m + b_y^s + \left(p^{s'} + \delta^{s'}\right) e_y^s - q^{s'} b_m^{s'} - p^{s'} e_m^{s'} \\ c_3^{s's''} &= b_2^{s'} + \left(p^{s''} + \delta^{s''}\right) e_2^{s'} \end{aligned}$$

and

$$\begin{aligned} -u'\left(c_y^h\right) q^h + \pi^h u'\left(c_m^{hh}\right) + \pi^l u'\left(c_m^{hl}\right) &= 0 \\ -u'\left(c_y^h\right) p^h + \pi^h u'\left(c_m^{hh}\right)\left[p^h + \delta^h\right] + \pi^l u'\left(c_m^{hl}\right)\left[p^l + \delta^l\right] &= 0 \\ -u'\left(c_y^l\right) q^l + \pi^h u'\left(c_m^{lh}\right) + \pi^l u'\left(c_m^{ll}\right) &= 0 \\ -u'\left(c_y^l\right) p^l + \pi^h u'\left(c_m^{lh}\right)\left[p^h + \delta^h\right] + \pi^l u'\left(c_m^{ll}\right)\left[p^l + \delta^l\right] &= 0 \end{aligned}$$

$$\begin{aligned} -u'\left(c_m^{sh}\right) q^h + \pi^h u'\left(c_r^{hh}\right) + \pi^l u'\left(c_r^{hl}\right) &= 0,\ s = h, l \\ -u'\left(c_m^{sh}\right) p^h + \pi^h u'\left(c_r^{hh}\right)\left[p^h + \delta^h\right] + \pi^l u'\left(c_r^{hl}\right)\left[p^l + \delta^l\right] &= 0,\ s = h, l \\ -u'\left(c_m^{sl}\right) q^l + \pi^h u'\left(c_r^{lh}\right) + \pi^l u'\left(c_r^{ll}\right) &= 0,\ s = h, l \\ -u'\left(c_m^{sl}\right) p^l + \pi^h u'\left(c_r^{lh}\right)\left[p^h + \delta^h\right] + \pi^l u'\left(c_r^{ll}\right)\left[p^l + \delta^l\right] &= 0,\ s = h, l \end{aligned}$$

Here, we index second and third period consumptions with both the current and lagged shock realizations because agents' holdings of the equity asset will generally depend on the state in which the asset was purchased. Market clearing requires that

$$b_y^s + b_m^s = 0, \text{ for } s = h, l$$

and

$$e_y^s + e_m^s = 1, \text{ for } s = h, l.$$

These equations have several implications. Note first that since the expected marginal utility expressions in each of the first-order conditions of the middle-aged is independent of the lagged state, this implies that $c_m^{hl} = c_m^{ll} \equiv c_m^l$, and $c_m^{hh} = c_m^{lh} \equiv c_m^h$. Since first period consumptions only depend on the current state, the resource constraint and the fact that the endowment process is i.i.d. then implies that $c_r^{hs} = c_r^{ls} \equiv c_r^s$. Via the budget constraints above, we can show explicitly that the bond and equity holdings in the model must be state independent. To see this, consider

$$c_m^{hh} = \omega_m + b_y^h + \left(p^h + \delta^a\right) e_y^h - q^h b_m^h - p^h e_m^h$$

and

$$c_m^{lh} = \omega_m + b_y^l + \left(p^h + \delta^a\right) e_y^l - q^h b_m^h - p^h e_m^h.$$

Since $c_m^{hh} = c_m^{lh}$, this implies that

$$b_y^h + \left(p^h + \delta^h\right) e_y^h = b_y^l + \left(p^h + \delta^h\right) e_y^l.$$

Similarly, since $c_m^{hl} = c_m^{ll}$,

$$b_y^h + \left(p^l + \delta^l\right) e_y^h = b_y^l + \left(p^l + \delta^l\right) e_y^l.$$

Subtracting the second equation from the first, we get

$$e_y^h \left[p^h + \delta^h - p^l - \delta^l\right] = e_y^l \left[p^h + \delta^h - p^l - \delta^l\right].$$

From this expression, either $e_y^h = e_y^l$ or $p^h + \delta^h = p^l + \delta^l$. In the first case, we also infer that $b_y^h = b_y^l$. In this case, then, we are left with a system of 10 equations in the eight variables q^h, q^l, p^h, p^l, b_y, b_m, e_y, and e_m. In the latter case, we have two functional dependencies between the prices of the asset, and hence, we will have a system of 12 equations in the 11 variables

q^h, q^l, p^h, b^h_y, b^l_y, b^h_m, b^l_m e^h_y, e^l_y, e^h_m, and e^l_m. In both cases, it is possible to show that generically there cannot be an equilibrium using techniques similar to those developed by Citanna and Siconolfi (2007).[2].

Extending this result to all for short memory equilibria requires noting that the only way allocations could depend on two or more lagged exogenous shocks would be if the prices depended on these lagged shocks. In this case, if consumptions depend on n lagged shock realizations, then a middle-aged agent looking forward one period will integrate out the next period shock, so that the expected marginal utility component of the first-order condition will be independent of the first lagged shock in the middle-aged agent's consumption, so that this consumption must, in fact, not depend on this shock. From this observation, we can unravel the shock dependence back to the strongly stationary case.

9 Appendix B

We solve the model by approximating the decision rules of the individuals of the economy by a smooth function. The parameters of the function are then revealed by imposing identifying restrictions dictated by our model economy.

A general representation of the problem is to approximate the solution to

$$\mathrm{E}_t F(y_{t+1}, x_{t+1}, z_{t+1}, y_t, x_t, z_t) = 0$$

where $F : \mathbb{R}^{n_y} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_y+n_x}$ describes the decision problem of the individual. x is the set of state variables that are endogenous to the model economy, y is the set of individual decision variables, and z is the set of exogneous state variables. The solution of the model is a set of decision rules for the control variables

$$y_t = g\left(x_t, z_t\right)$$

Next period's exogenous state is defined by $z_{t+1} = f(z_t, \varepsilon_{t+1})$ and next period endogeous state variables as

$$x_{t+1} \equiv h\left(x_t, y_t, z_t\right)$$

[2] This result was first shown by Spear for OLG economies in which agents live two periods but trade multiple goods. To the best of our knowledge, the corresponding result for single good economies in which agents live more than two periods was first shown by Aiyagari while he was visiting Carnegie Mellon in 1984, though the result does not appear to have been published anywhere.

hence

$$x_{t+1} = h\left(x_t, g\left(x_t, z_t\right), f(z_t)\right)$$

and

$$y_{t+1} = g\left(x_{t+1}, z_{t+1}\right) = g\left(h\left(x_t, y_t, z_t\right), f(z_t)\right) = g\left(h\left(x_t, g\left(x_t, z_t\right), z_t\right), f(z_t)\right)$$

such that the model can be rewritten

$$\mathrm{E}_t R(x_t, z_t, g) = 0$$

The idea of the minimum method is to replace the true decision rule g by a parametric approximation function, $\Phi(x_t, z_t\theta)$, of the current state variables x_t and z_t and a vector of parameters θ.

We approximate the decision rules using Chebyshev polynomials as basis functions.

$$\Phi(x_t, z_t, \theta) = \sum_{i=0}^{n} \theta_i T_i\left(\varphi(x_t, z_t)\right)$$

where $T_i(\cdot)$ is the Chebychev polynomial of order $i = 0, \ldots, n$.

These polynomials are described by the recursion

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{with} \quad T_0(x) = 1, T_1(x) = x$$

which admits as solution

$$T_n(x) = \cos(n \cos^{-1}(x)).$$

The polynomials form an orthogonal basis with respect to the weighting function $\omega(x) = (1 - x^2)^{-1/2}$ over the interval $[-1, 1]$. This interval may be generalized to $[a, b]$, by transforming the data using the formula

$$x = 2\frac{y-a}{b-a} \quad \text{for} \quad y \in [a, b]$$

Beyond the standard orthogonality property, Chebyshev polynomials exhibit a discrete orthogonality property which writes as

$$\sum_{i=1}^{n} T_i(r_k)T_j(r_k) = \begin{cases} 0 & \text{for } i \neq j \\ n & \text{for } i = j = 0 \\ \frac{n}{2} & \text{for } i = j \neq 0 \end{cases}$$

where $r_k,\ k = 1, \ldots, n$ are the roots of $T_n(x) = 0$.

Procedure

Choose an order of approximation n_ω, n_{a^y} and n_{a^m} for each dimension, compute the $n_\omega + 1$, $n_{a^y} + 1$ and $n_{a^m} + 1$ roots of the orthogonal polynomial of order $n_\omega + 1$, $n_{a^y} + 1$ and $n_{a^m} + 1$ asz

$$
\begin{aligned}
{}^{i}_{\omega} &= \cos\left(\frac{(2i-1)\,\pi}{2\,(n_\omega+1)}\right) && \text{for } i = 1, \ldots, n_\omega + 1 \\
z^i_{a^y} &= \cos\left(\frac{(2i-1)\,\pi}{2\,(n_{a^y}+1)}\right) && \text{for } i = 1, \ldots, n_{a^y} + 1 \\
z^i_{a^m} &= \cos\left(\frac{(2i-1)\,\pi}{2\,(n_{a^m}+1)}\right) && \text{for } i = 1, \ldots, n_{a^m} + 1
\end{aligned}
$$

and formulate an initial guess for θ. Compute ω_i as

$$
\begin{aligned}
{}_{i} &= \underline{\mathrm{w}} + \left(z^i_\omega + 1\right)\frac{\bar{\omega} - \underline{\mathrm{w}}}{2} && \text{for } i = 1, \ldots, n_\omega + 1 \\
a^y_i &= \underline{\mathrm{a}}^y + \left(z^i_{a^y} + 1\right)\frac{\bar{a}^y - \underline{\mathrm{a}}^y}{2} && \text{for } i = 1, \ldots, n_{a^y} + 1 \\
a^m_i &= \underline{\mathrm{a}}^m + \left(z^i_{a^m} + 1\right)\frac{\bar{a}^m - \underline{\mathrm{a}}^m}{2} && \text{for } i = 1, \ldots, n_{a^m} + 1.
\end{aligned}
$$

to map $[-1, 1]$ into $[\underline{\mathrm{a}}^m, \bar{a}^m]$. At each node $\left(\omega_i, a^y_j, a^m_k\right)$, $i = 1, \ldots, n_\omega +$ 1, $j = 1, \ldots, n_{a^y} + 1$, and $k = 1, \ldots, n_{a^m} + 1$ compute

$$
a^y_{t+1} \approx \Phi\left(\omega_i, a^y_j, a^m_k, \theta^{a^y}\right) = \sum_{j_\omega=0}^{n_\omega} \sum_{j_{a^y}=0}^{n_{a^y}} \sum_{j_{a^m}=0}^{n_{a^m}} \theta^{a^y}_{j_\omega, j_{a^y}, j_{a^m}} T_{j_\omega}(\omega_i)\, T_{j_{a^y}}\left(a^y_j\right) T_{j_{a^m}}\left(a^m_k\right)
$$

2.

$$
a^m_{t+1} \approx \Phi\left(\omega_i, a^y_j, a^m_k, \theta^{a^m}\right) = \sum_{j_\omega=0}^{n_\omega} \sum_{j_{a^y}=0}^{n_{a^y}} \sum_{j_{a^m}=0}^{n_{a^m}} \theta^{a^m}_{j_\omega, j_{a^y}, j_{a^m}} T_{j_\omega}(\omega_i)\, T_{j_{a^y}}\left(a^y_j\right) T_{j_{a^m}}\left(a^m_k\right)
$$

hence

$$
\begin{aligned}
c^y_t &= w^y - a^y_{t+1} \\
c^m_t &= w^m + a^y_t\,(1 + r - \delta) - a^m_{t+1} \\
c^r_t &= w^r + a^m_t\,(1 + r - \delta)
\end{aligned}
$$

where w^y, w^m, w^r and r all are functions of ω_t and $a^y_t + a^m_t$.

3. Then, for each node $\left(\omega_i, a^y_j, a^m_k\right)$, compute the (*ex ante*) expected consumption the following period. For $\ell = 1, \ldots, q$

$$
\begin{aligned}
a^y_{t+2,\ell} &\approx \Phi\left(\rho\omega_i + z_\ell\sqrt{2\sigma_\varepsilon}, a^y_{t+1}\left(\omega_i, a^y_j, a^m_k\right), a^m_{t+1}\left(\omega_i, a^y_j, a^m_k\right), \theta^{a^y}\right) \\
&= \sum_{j_\omega=0}^{n_\omega} \sum_{j_{a^y}=0}^{n_{a^y}} \sum_{j_{a^m}=0}^{n_{a^m}} \theta^{a^y}_{j_\omega, j_{a^y}, j_{a^m}} T_{j_\omega}\left(\rho\omega_i + z_\ell\sqrt{2\sigma_\varepsilon}\right) \\
&\qquad \cdot T_{j_{a^y}}\left(a^y_{t+1}\left(\omega_i, a^y_j, a^m_k\right)\right) T_{j_{a^m}}\left(a^m_{t+1}\left(\omega_i, a^y_j, a^m_k\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
a^m_{t+2,\ell} &\approx \Phi\left(\rho\omega_i + z_\ell\sqrt{2\sigma_\varepsilon}, a^y_{t+1}\left(\omega_i, a^y_j, a^m_k\right), a^m_{t+1}\left(\omega_i, a^y_j, a^m_k\right), \theta^{a^m}\right) \\
&= \sum_{j_\omega=0}^{n_\omega} \sum_{j_{a^y}=0}^{n_{a^y}} \sum_{j_{a^m}=0}^{n_{a^m}} \theta^{a^m}_{j_\omega, j_{a^y}, j_{a^m}} T_{j_\omega}\left(\rho\omega_i + z_\ell\sqrt{2\sigma_\varepsilon}\right) \\
&\qquad \cdot T_{j_{a^y}}\left(a^y_{t+1}\left(\omega_i, a^y_j, a^m_k\right)\right) T_{j_{a^m}}\left(a^m_{t+1}\left(\omega_i, a^y_j, a^m_k\right)\right)
\end{aligned}
$$

4. Compute the residuals of (*ex ante*) expected marginal rates of substitution

$$
\begin{aligned}
R^{a^y}\left(\omega_i, a^y_j, a^m_k, \theta\right) &= \left(c^y_t\right)^{-\sigma} - \frac{\beta}{\sqrt{\pi}} \sum_{\ell=1}^{q} \lambda_\ell \left(c^m_{t+1,\ell}\right)^{-\sigma}(1 + r - \delta) \\
R^{a^m}\left(\omega_i, a^y_j, a^m_k, \theta\right) &= \left(c^m_t\right)^{-\sigma} - \frac{\beta}{\sqrt{\pi}} \sum_{\ell=1}^{q} \lambda_\ell \left(c^r_{t+1,\ell}\right)^{-\sigma}(1 + r - \delta)
\end{aligned}
$$

5. If all residuals are close enough to zero then stop, if not update θ.