Geometric Backstepping for Strict Feedback Simple Mechanical Systems

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Abstract—We propose a geometric control design method for a class of underactuated geometric mechanical systems, which we term strict-feedback simple mechanical systems. The configuration space of these class of systems can be split into two parts: one is a Riemannian manifold with full actuation, whose dynamics we call the shape dynamics; the other is a Lie group with a certain degree of underactuation, whose dynamics we call the body dynamics. Using the idea of backstepping, we create a generalizable geometric controller that is able to realize tracking of the body dynamics by adjusting the state of shape dynamics in a coordinate-free manner. Based on a geometric Lyapunov candidate function, we also prove local exponential stability of the proposed controller. Finally, we demonstrate and validate our controller on two example systems.

I. INTRODUCTION

Most mechanical systems are governed by second order differential equations and evolve on nonlinear manifolds, [12]. Typical control techniques for these systems rely on the introduction of local coordinates so as to set up a smooth one-to-one correspondence between the actual state space and Cartesian space within an admissible range. Although this converts control on a manifold to control in Cartesian space, the issue of singularity exists for any choice of local coordinates on certain nonlinear manifolds, such as the special orthogonal group $SO(3)$.

Geometric control is developed to circumvent this dilemma of singularity in mechanical systems [3]. Contrary to controllers which are based on Cartesian spaces, geometric controllers treat the configuration spaces of mechanical systems as manifolds and develop feedback laws based on differential geometry. For instance, research in [3] proposes exponentially-stabilizing controllers for fully-actuated, simple mechanical systems. These type of mechanical systems admit their configuration spaces as Riemannian manifolds with corresponding configuration errors and transport maps to compare states. Moreover, based on general expressions of Lyapunov candidate functions, we are able to guarantee local exponential stability of such geometric controllers. Furthermore, Maithripala et al. [11] proposes a geometric controller for Lie groups using the group structure. In this case, by choosing a proper Morse function, the authors are able to design an exponentially stable feedback law. Though these controllers are quite powerful in their generality, the restriction to fully-actuated systems would make them hard to apply to systems with underactuation.

For underactuated mechanical systems, theoretical analysis about controllability and various controllers have been set up for different types of underactuated systems in [10]. The concepts of strong and weak controllability for underactuated systems whose configuration space is a principal fiber bundle is presented in [7]. Partial feedback linearization has been introduced in [15] for systems which satisfy “Strong Inertial Coupling” conditions so that certain underactuated states can be controlled through input transformation. Finally, backstepping control design methodology, as originally shown in [8], was introduced to provide an intuitive way to design controllers for strict-feedback systems which are underactuated. The backstepping methodology also enables the construction of a Lyapunov candidate function iteratively.

For unceractuated systems that evolve on nonlinear manifolds, the method of backstepping has been applied to very specific systems, such as geometric control of underactuated multi-copters [4], [5], [9], [16]. However, all these systems under study are very specific, and a general backstepping methodology for such systems has not been exhaustively explored. The main contributions of this paper with respect to prior work, such as [4], [9], [16], lie in the following aspects:

- We introduce the concept of strict-feedback simple mechanical systems, and extract out the common properties shared by such underactuated systems so as to apply the method of backstepping.
- We propose a coordinate-free geometric feedback law that can realize tracking of the underactuated
part using previous research in geometric control.

- We show local exponential stability based on iterative Lyapunov candidate construction using backstepping.

The rest of this paper is organized as follows: Sec. II focuses on the introduction of necessary mathematical concepts about Riemannian manifold and Lie group. Sec. III shows the detailed steps of our controller and theoretical results regarding its performance. Sec. IV gives some analysis on specific systems where our controller can be applied. Sec. V summarizes what we have done and draws the final conclusion.

II. Mathematical Preliminaries

This section introduces necessary concepts in geometric control and differential geometry. In particular, we will cover the elements of geometric control and its connection with Riemannian geometry, Lie group and the model of strict-feedback simple mechanical systems in an intuitive and concise way. For a more thorough and detailed introduction to differential geometry and Lie groups, we refer to [1], [2], [6], [14].

A. Fundamentals of Differential Geometry

A smooth $m$ dimensional manifold $M$ is a subset of Cartesian space where we can still define calculus in the language of differential geometry. For any point $r \in M$, an open ball in $\mathbb{R}^m$ could be smoothly deformed into a local open neighborhood of $r$ in $M$ through a mapping $\varphi$. We define the range of the Jacobian matrix $J_\varphi$ at $\varphi^{-1}(r)$ to be the tangent space at $r$, denoted as $T_r \mathcal{M}$. A cotangent vector $u : T_r \mathcal{M} \to \mathbb{R}$ is a linear functional for $T_r \mathcal{M}$. We denote the collection of cotangent vectors as $T^*_r \mathcal{M}$ and $\langle u, v \rangle = u(v)$ where $u \in T^*_r \mathcal{M}$, $v \in T_r \mathcal{M}$. A vector field is a mapping from each point $r \in M$ to a tangent vector in $T_r \mathcal{M}$. An one form is a mapping from each point $r \in M$ to a cotangent vector in $T^*_r \mathcal{M}$. For a smooth vector field $X$, we can define its integral curve as $\gamma : (0,1) \to M$ such that

$$
\frac{d\gamma}{dt}(t) = X_{\gamma(t)}, \quad \forall t \in (0,1),
$$

where $X_r$ indicates the tangent vector of $X$ at $r \in M$.

For a smooth function $f : M \to \mathbb{R}$, we could get an one form $df$ as its differential such that:

$$
\langle df, X \rangle_r = \left. \frac{df(\gamma(t))}{dt} \right|_{\gamma(t)=r}, \quad \forall r \in M,
$$

where $\gamma$ is the integral curve of $X$ passing through $r$.

It is obvious that the usage of vector field and one form is to introduce integration and differentiation on the manifold $M$. Similarly, if $f : M \to N$ is a smooth map, the differential map $df : T_r \mathcal{M} \to T_{f(r)} \mathcal{N}$ is called the pushforward for a particular vector field $X \in T \mathcal{M}$. If this function is a diffeomorphism on $N$, then we could also push forward a vector field $Y$ on $N$ using $df^{-1}$ from $N$ back to $M$, and we call this the pullback of tangent vector. Correspondingly, the pullback or pushforward of one form between different manifolds could be defined using the pushforward and pullback operations of vector fields. Further, if $f(r_1, r_2, \ldots, r_k)$ is a map from a product of manifolds $M_1, M_2, \ldots, M_k$ to a new manifold $N$, we denote its differential map with respect to the $i^{th}$ component as $d_{r_i} f(v_i)$ where $v_i \in T_{r_i} M_i$ while keeping the other variables fixed.

A Riemannian manifold is a manifold equipped with a Riemannian metric $\mathcal{M}$ and a Riemannian connection $\nabla$. The role of metric is to associate an inner product to each tangent space. Moreover, the cotangent space would inherit this inner product structure through Rietz representation theorem. In this way, we are able to define the norm of a tangent or cotangent vector as well as perform projection to subspaces in the tangent space $T_r \mathcal{M}$. A connection is a map on the tangent space which can be utilized to represent higher order derivatives on the manifolds. Correspondingly, a Riemannian connection is a specific connection which can be uniquely determined by the Riemannian metric.

B. Geometric Control Design on Riemannian Manifold

Based on the previously introduced concepts, we are able to interpret elements of mechanical systems from the perspective of differential geometry:

- The configuration space of a mechanical system is a Riemannian manifold $\mathcal{M}$ while the state space is the corresponding tangent bundle $T \mathcal{M} = \cup_r T_r \mathcal{M}$.
- The inertia of mechanical system is a Riemannian metric $\mathcal{M}$.
- Potential force is the negative of the differential of a smooth potential function $\mathcal{V} : \mathcal{M} \to \mathbb{R}$.
- External force is a collection of one forms $\{f_i\}$, whose magnitude is determined by the control input.
- A measure of distance between two points $r_1, r_2 \in M$ is a smooth function $\mathcal{V}_M : \mathcal{M} \times \mathcal{M} \to [0, \infty)$ called configuration error, and its differential $d_{r_1} \mathcal{V}_M$ is an one-form called position error denoted as:

$$e_r = d_r \mathcal{V}_M(r, r_d), \quad (1)$$
A measure of distance between two tangent vectors \( \dot{r} \in T_r M, \dot{r}_d \in T_{r_d} M \) is given based on the Riemannian metric norm of the velocity error, which is defined as:

\[
e_r = \dot{r} - \mathcal{T}_{(r,r_d)} \dot{r}_d;
\]

where the function \( \mathcal{T}_{(r,r_d)} : T_r M \to T_r M \) is called the transport map that is capable of converting vectors in different tangent spaces from each other in a way that is compatible with \( \Psi_M \) given.

**C. Basics of Lie Groups**

A Lie group \( G \) is a smooth manifold with additional algebraic structure of a group. Moreover, we require that group inverse and group addition are smooth which don’t break smoothness property. In this way, any group member \( g \) can be identified uniquely as a smooth action on the group, and thus can pushforward vector field or pullback one form through left and right translation. We denote the group identity as \( e \). Due to the fact that this pullback is an isomorphism at each point \( g \), we could characterize every tangent space by only analyzing tangent space at identity \( e \). Thus, the structure of this tangent space \( T_e G \) is very important, is called the Lie algebra, which is denoted as \( \mathfrak{g} \). We also denote its dual space as \( \mathfrak{g}^* \). In this vector space \( \mathfrak{g}^* \), we could define adjoint operation \( Ad_g : \mathfrak{g} \to \mathfrak{g} \) as \( Ad_g \xi = g \cdot (\xi \cdot g^{-1}) = g \cdot \xi \cdot g^{-1} \) and Lie bracket \([\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\) which could be interpreted as the Riemannian connection at \( e \). For vector fields on the Lie group \( G \), it is called invariant if it remains unchanged to any group action. Based on the concept of the invariant vector fields, invariance of one form and metric could also be defined. For Lie group with left invariant Riemannian metric, we could restrict our scope to the body-fixed velocity as \( \xi = g^{-1}\dot{g} \) with constant inertia metric \( \mathbb{I} : \mathfrak{g} \to \mathfrak{g}^* \). According to \([11]\), the configuration error on Lie group could be defined as \( \Psi_G = K(g_d^{-1}g) \) where \( K \) is a Morse function with \( e \) a local minimum. The corresponding position error and velocity error are given both in \( \mathfrak{g} \):

\[
e_g = (g_d^{-1}g) \cdot dK, \quad e_\xi = Ad_d(\xi - \xi_d),
\]

where \( \xi_d = g_d^{-1} \cdot \dot{g}_d \).

**D. Strict-Feedback Simple Mechanical Systems Model**

Based on the system dynamics studied in \([4, 9, 16]\), we are able to propose a model called strict-feedback simple mechanical systems. Consider a mechanical system with configuration space \( Q = G \times M \), which is the direct product of a Lie group \( G \) and a Riemannian manifold \( M \). The Lie group \( G \) is \( n_1 \)-dimensional with inertia tensor \( I : \mathfrak{g} \to \mathfrak{g}^* \) and \( n_1 > 1 \). The Riemannian manifold \( M \) is \( n_2 \)-dimensional with the inertia metric \( \left\langle \cdot, \cdot \right\rangle_M \). Suppose the overall system dynamics is given by:

\[
\dot{\xi} = \mathbb{I}^{-1}(ad^\xi(\mathbb{I}) + h_0(g, \xi) + h_1(g, r)v), \quad (4)
\]

\[
\nabla_r \dot{r} = M^{-1}(f_0(g, r) + \sum_{i=1}^{n_2} f_i(r)u_i), \quad (5)
\]

where \( h_0 : G \times \mathfrak{g}^* \to \mathfrak{g}^* \) is a Lie-algebra valued map representing dissipation and potential force in \( G, h_1 : G \times M \to \mathfrak{g}^* \) reflects the coupling between \( G \) and \( M \), \( v \in \mathbb{R} \) is the single input applied to \([4]\), \( f_0(g, r) : G \times M \to TM \) is the potential force on \( M \), and each \( f_i : M \to T^*_M \) are the external force applied to \([5] \) with the control input \( u_i \) \((i = 1, 2, \ldots, n_2)\).

**Definition 1:** (Strict-Feedback Simple Mechanical System)

We will call the overall system containing \([4] \) and \([5] \) a strict-feedback simple mechanical system, if it satisfies the conditions below:

- **P1** \( \text{span}(f_i(g, r)) = T^*_a M \) for each \( r \in M \), which means that \([5] \) is fully actuated.
- **P2** The inertia metric is left invariant for \([4] \).
- **P3** The induced norms by metric on \( T_r M \) and \( \mathfrak{g} \) are denoted as

\[
\|\dot{r}\|^2_M = \left\langle \dot{r}, \dot{r} \right\rangle_M, \quad \|\xi\|^2 = \left\langle \xi, \xi \right\rangle_B
\]

where \( B = \mathbb{I}^{-1} \mathbb{I} \). We will also use the same notation for induced norms on \( T^*_a M \) and \( \mathfrak{g}^* \). The value would be computed depending on the arguments.

- **P4** Both \( G \) and \( M \) have local quadratic configuration errors \( \Psi_G, \Psi_M \), which means that there exist \( C_1, C_2, D_1, D_2, L_1, L_2 > 0 \) such that

\[
C_2 \|e_r\|^2_M \leq \Psi_M(r, r_d) \leq C_1 \|e_r\|^2_M, \quad (6)
\]

\[
D_2 \|e_g\|^2 \leq \Psi_G(g, g_d) \leq D_1 \|e_g\|^2, \quad (7)
\]

whenever \( \Psi_M \leq L_1, \Psi_G \leq L_2 \). The position and velocity error are also well-defined as \( (e_r, e_{\dot{r}}) \) and \( (e_g, e_{\dot{g}}) \) respectively.

- **P5** The inertia matrix \( \mathbb{I} \) satisfies for any \( \zeta_1, \zeta_2 \in \mathfrak{g}^* \)

\[
\left\langle \zeta_1, \mathbb{I}^{-1} \zeta_2 \right\rangle \leq B \|\zeta_1\|_1 \cdot \|\zeta_2\|_1 \quad (8)
\]

for some constant \( B > 0 \).
Remark 1: We will call subsystem (4) body dynamics, and subsystem (5) fully-actuated shape dynamics. Also we will denote \( u = [u_1, u_2, \ldots, u_{n_2}] \) as the control vector.

III. GEOMETRIC BACKSTEPPING DESIGN OF STRICTLY FEEDBACK SYSTEM

In this section, we propose a control design method to track a smooth reference \( g_d(t) \in G \). The control idea is based on a combination of geometric control method and backstepping method: geometric control method provides a suitable Lyapunov function candidate to start with; backstepping method allows us to realize tracking of (4) through control of (5). We will illustrate the whole design process through three steps with relevant propositions.

The first step is to realize an exponentially stabilizing control for the shape dynamics (5) using geometric control theory. Suppose we have already been given a feedback law according to Prop. 1 to G to get a feedback law \( \alpha = \alpha(t, g, \xi) \). But to better employ the Lie group structure, we will select the following feedback law according to (11):

\[
\alpha(g, \xi, t) = 1(k_p Ad_{\Psi_{\alpha}^{-1}} e_g - k_d (\xi_d - \xi)) + 1(\xi_d + [\xi, \xi_d] - ad^*_\xi (1(\xi) - h_0(g, \xi)),
\]

where \( e_g^* \) is the unique vector in Lie algebra such that \( [e_g, \xi] = \langle e_g^*, \xi \rangle I \).

Also, a Lyapunov function can be chosen as:

\[
V_g = k_p K_1(g_d^{-1} g) + \| e_g \|_1 + \varepsilon \| e_g \|_1,
\]

where \( K \) is a Morse function on \( G \), and \( \varepsilon > 0 \) can be chosen to make \( V_g \) quadratic as P4).

It can be shown that with a proper Morse function and gain values, this Lyapunov function satisfies the following properties which will be utilized in further analysis:

\[
\begin{align*}
\frac{a_1(\Psi_G(g, g_d) + \| e_{\xi} \|_1)}{\| V_g \|_1} & \leq V_g \leq a_2(\Psi_G(g, g_d) + \| e_{\xi} \|_1),
\end{align*}
\]

\[
\dot{V}_g \leq -b_1(\Psi_G(g, g_d) + \| e_{\xi} \|_1),
\]

\[
\| d_\xi V_g \|_1 \leq c_1(\sqrt{\Psi_G(g, g_d)} + \| e_{\xi} \|_1),
\]

where \( a_2 > a_1 > 0, b_1, c_1 > 0 \) and \( d_\xi V_g \) is a point in the Lie algebra \( g^* \).

Based on the feedback law \( \alpha \) given for \( G \), we need to convert it to a reference for \( r \in M \). In particular, we want to determine a smooth reference trajectory \( r_c(t) \) and \( v_c(t) \) such that \( v_c h_1(g, r_c) = \alpha(t, g, \xi) \). The proposition below provides sufficient conditions when such reference exists.

Proposition 2: (Sufficient Conditions for the Existence of Smooth Reference \( r_c \))

If the following conditions are satisfied:

- H2.1) There exists a family of linear invertible maps \( \varphi_g : g^* \rightarrow g^* \) indexed by \( g \) which is smooth-
varying with respect to $g$ such that $h_1(g, r) = \varphi_g(h_1(e, r)), \varphi_g(0) = 0$.

- **H2.2** Define the unit sphere in $g^*$ as $S_0 = S^{n-1} \subset g^*$ and the map $h_1(e, r) = \hat{h}_1 : M \rightarrow g$. If there exists a compact submanifold $N \subset M$ such that the restricted map $\hat{h}_1 = \hat{h}_1|_N : N \rightarrow S_0$ is bijective and the differential map $d\hat{h}_1$ is an isomorphism for each $r \in N$.

- **H2.3** The feedback function $\alpha(g, \xi, t) \neq 0$ for any $(g, \xi, t) \in G \times g \times (0, \infty)$. Then there exists a smooth function $r_c(g, \xi, t)$ and $v(g, \xi, t)$ such that $v h_1(g, r_c) = \alpha(g, \xi, t)$ for any $(g, \xi, t) \in G \times g \times (0, \infty)$.

**Proof:** We get the reference $r_c$ in two steps: The first step is to narrow down our scope to $S_0$ and apply inverse function theorem. Consider $r \in N, s \in S_0$ with $h_1(r) = s$. By H2.2, the differential map $df : T_r N \rightarrow T_s S_0$. By Inverse Function theorem for manifold in [6], $\hat{h}_1$ is also a local diffeomorphism around $r$. This means that the inverse function $\hat{h}_1^{-1}$ is also a local diffeomorphism around $s$ and thus it is a global diffeomorphism. Denote the global inverse function as $\hat{h}_1^{-1} = \hat{s} : S_0 \rightarrow N$.

Then the next step is to expand this function to the whole space except the origin. Since we are looking for $r_c$ such that $v h_1(g, r_c) = \alpha(g, \xi, t)$. By H2.1, this means that $\varphi g(h_1(e, r)) = \alpha(g, \xi, t)$ and $\varphi g(v h_1(e, r)) = \alpha$ implies $v h_1(e, r) = \varphi g^{-1}$. Also, H2.1 implies that $\varphi g(x) \neq 0$ for all $x \neq 0$. Combining H2.1, H2.3 yields that $\varphi g^{-1}$ can never be zero.

Thus a pair of candidate functions can be given as:

$$v = \left\| \varphi g^{-1}(\alpha(g, \xi, t)) \right\|_t,$$

$$r_c(g, \xi, t) = \hat{s}(\varphi g^{-1}(\alpha)/v)$$

which is well defined and satisfy the feedback law $\alpha$.

Then due to the fact that $\hat{h}_1^{-1}$ is a global diffeomorphism, all the functions including norms are smooth, as required by the system properties. We have the reference $r_c$ is a smooth function since it is the composition of smooth functions.

Based on this smooth reference $r_c(g, \xi, t)$ and $v = \left\| \alpha(g, \xi, t) \right\|_t$, we can express its derivative explicitly using differential map. The higher derivatives of $r_c$ are given as:

$$\dot{r}_c = S_1 + \left\| \alpha(g, \xi, t) \right\|_t \cdot d\xi r_c(\Pi^{-1} h_1(g, r))$$

$$\n abla_r \dot{r}_c = S_2 + \left\| \alpha(g, \xi, t) \right\|_t \cdot \left\langle d\xi r_c, d_r h_1(\hat{r}) \right\rangle_1$$

(14)

where the terms $S_1$ is independent of $r$ in $\dot{r}_c$, and $S_2$ is independent of $\dot{r}$ in $\dot{r}_c$.

**Remark 3:** A key about $r_c$ is that its derivative up to second order only depends on the current state $r, \dot{r}$. This means that we could treat $r_c$ as a smooth command for $r$, which can be tracked using $u$. This fact is crucial for the stability proof of the next proposition.

If the conditions of Prop. 2 are satisfied, we could get a smooth reference $r_c$ based on the current state and higher order derivatives of $g_d(t)$. Let’s also assume that a Lyapunov function candidate could be given as $\{13\}$ for the feedback law $\alpha$. We propose a backstepping feedback law as:

$$u = u_{ff} + u_{pd}, \quad v = \left\| \varphi g^{-1}(\alpha(g, \xi, t)) \right\|_t,$$

(15)

$$u_{pd} = A^{-1}(\beta_1 e_r + \beta_2 \dot{e}_r),$$

$$u_{ff} = A^{-1}M(-f_0(g, r) + \nabla \epsilon (T(r, r_c) \hat{r}))$$

$$+ A^{-1}M \left( \frac{d}{dt} \right)_{\hat{r} \text{fixed}} (T(r, r_c) \hat{r})$$

where $f(g, r_c) = \alpha(g, \xi, t), e_r = d_r \Psi_M(r, r_c)$ is the position error between $r$ and $r_c$, $e_\dot{r} = \dot{r} - T(r, r_c) \dot{r}_c$ is the corresponding velocity error.

Note that the only difference [9] and [15] is that we replace $r_d$ with $r_c$, which is dependent on the current states, but rather than specified. Based on the previous arguments, it can be seen that the derivative $r_c$ can also be computed using current states and the higher order derivatives of the reference $g_d$, which makes the feedback law feasible. Regarding the stability of the corresponding closed-loop system, we have the following proposition.

**Proposition 3:** (Exponential Convergence of Geometric Backstepping Controller for [4]) Suppose the following conditions are satisfied:

- **H3.1** Conditions of Prop. 2 are satisfied.
- **H3.2** There exist positive number $\gamma$ such that $\left\| h_1(g, r) - h_1(g, s) \right\|_t \leq \gamma \sqrt{\Psi_M(g, r, s)}$ holds for any $g \in G, r, s \in M$ (Lipschitz-type inequality).
- **H3.3** There exist $\eta_1, \eta_2, \eta_3 > 0$ such that $\left\| \alpha(g, \xi, t) \right\|_t \leq \eta_1 \sqrt{\Psi_C(G, g, g_d) + \eta_2} \epsilon_\xi \left\| \epsilon \right\|_t \left( \eta_3 \right)$ (Linear bound on the feedback law).
- **H3.4** The initial error $\Psi_M(r(0), r_c(0)) < L_1$.
- **H3.5** There exist $\beta_1, \beta_2 > 0$ and $0 < \epsilon < 2 \sqrt{\beta_1 C_2}$ such that the matrix

$$\begin{bmatrix} b_1 I_2 & Q \\ Q P & P \end{bmatrix}$$

is positive definite where the matrices $P, Q$ defined in [10] is positive definite and

$$Q = -\begin{bmatrix} B \gamma_1 \beta_1 / 2 & B \gamma_1 \beta_1 / 2 \\ 0 & 0 \end{bmatrix}.$$

Then the reference $g_d$ is exponentially stable for the closed loop system with the feedback law [15].

**Proof:** According to H3.1, we have a well-defined $r_c$ and $v$. Then the proof is based on a candidate Lyapunov function as the sum of two candidates:

$$V = V_g + V_r,$$

where $V_g$ is defined in [12] and $V_r$ is defined in [11] with
First check the derivative of $V_g$ as:

$$
V_g \leq \frac{d}{dt} V_g(\|\alpha\|^{-1}\left[ h_1(g, r) - h_1(g, r_c) \right])
- b_1(\Psi_G(g, gd) + \langle e_\xi, e_\xi \rangle) \\
= \|\alpha\| \cdot \frac{d^2}{dt^2} V_g(\|\alpha\|^{-1}\left[ h_1(g, r) - h_1(g, r_c) \right])
- b_1(\Psi_G(g, gd) + \langle e_\xi, e_\xi \rangle) \quad \text{(linearity of } d^2V_g) \\
\leq \|\alpha\| \cdot \|d^2V_g\| \| h_1(g, r) - h_1(g, r_c) \| \\
- b_1(\Psi_G(g, gd) + \langle e_\xi, e_\xi \rangle), \quad \text{according to } (8)
$$

where we have also used Cauchy-Schwartz inequality in the last derivation.

Based on H3.2), H3.3) and (13), we could give an upper bound for the term below:

$$
\|d^2V_g\| \| h_1(g, r) - h_1(g, r_c) \| \|\alpha\| \leq \gamma \sqrt{\Psi_M(r, r_c)} \\
\cdot (\eta_1 \sqrt{\Psi_G(g, gd)} + \eta_2 \|e_\xi\| \|\alpha\| + \eta_3) \\
\cdot c_1(\sqrt{\Psi_G(g, gd)} + \|e_\xi\|) = x^T \begin{bmatrix} \sqrt{\Psi_M(r, r_c)} W & -Q \\ -Q & 0_{2 \times 2} \end{bmatrix} x
$$

where

$$
W = \begin{bmatrix}
\gamma c_1 \eta_1 & \gamma c_1 (\eta_1 + \eta_2)/2 \\
\gamma c_1 (\eta_1 + \eta_2)/2 & \gamma c_1 \eta_2
\end{bmatrix},
$$

and

$$
x = \begin{bmatrix} \sqrt{\Psi_G(g, gd)} \|e_\xi\| \\
\sqrt{\Psi_M(r, r_c)} \|e_\xi\| \|\alpha\| \end{bmatrix}^T.
$$

Plugging this into the expression of $\dot{V}_g$ and applying Proposition. 1 yields:

$$
\dot{V} \leq -x^T \begin{bmatrix} b_1 L_2 & Q \\ Q & P \end{bmatrix} x - \sqrt{\Psi_M(r, r_c)} \begin{bmatrix} BW & 0_{2 \times 2} \\ 0_{2 \times 2} & 0_{2 \times 2} \end{bmatrix} x
$$

Denote $\lambda_1$ as the smallest eigenvalue of $P_1$ and $\lambda_2 = \|P_2\|_2$. Then by H3.5), $0 < \lambda_1 \leq \|P_1\|_2$. Using H3.4), applying Prop. 1 yields that there exists $t_0 > 0$ such that $\Psi_M(r, r_c) \leq [\lambda_1/(2\lambda_2)]^2$ for all $t \geq t_0$.

Thus for all $t \geq t_0$, it holds that the smallest eigenvalue of $P_1 - \sqrt{\Psi_M(r, r_c)} P_2 \geq \lambda_1 - \lambda_2 \cdot [\lambda_1/(2\lambda_2)] = \lambda_1/2$, where we are using the exponential decaying of $\Psi_M(r, r_c)$ to estimate the effect of $P_2$. Hence, the condition holds that

$$
\dot{V} \leq -\frac{\lambda_1}{2} x^T x
$$

for the quadratic function $V$.

IV. Simulation Results on Some Strictly Feedback Systems

In this section, we will implement our controller for two systems, a single 3D-moving quadrotor and a modified 3D pendulum model. The simulation results of both systems are shown which are generated using Matlab ode solver.

A. Geometric Control of a Single Quadrotor UAV

Geometric controller design of a single quadrotor has been well-studied in [9] and become the basis for many relevant research topics [5], [16]. Consider a single quadrotor as shown in Fig. 1a, its dynamics can be given as:

$$
\dot{x} = v, \quad \dot{v} = m \cdot I_v = -mg \cdot e + vR e_3, \quad (16)
\dot{R} = R \hat{\Omega}, \quad \dot{\Omega} = J^{-1}((J\hat{\Omega}) \times \Omega + u), \quad (17)
$$

where $x \in \mathbb{R}^3$ is the quadrotor CoM’s position, $v \in \mathbb{R}$ represents the thrust force, $R \in SO(3)$ is the quadrotor’s orientation, $\Omega \in \mathbb{R}^3$ represents the body-fixed angular velocity and $u \in \mathbb{R}^3$ is the moment in the body-fixed frame.

Correspondingly, the body dynamics is (16) on $(\mathbb{R}^3, +)$ and the shape dynamics is (17) on $SO(3)$. Let’s check the conditions of Prop. 2. For (16), its Lie algebra is just $\mathbb{R}^3$ where $[\xi_1, \xi_2] = 0$ for any $\xi_1, \xi_2 \in \mathfrak{g}^\circ$. Any reference would be globally exponentially stable using PD control. For (17), it is also a Lie group and we could choose a candidate configuration error as follows:

$$
\Psi(R, R_d) = K(R_d^{-1} R) = \frac{1}{2} \text{trace}(I - R_d^{-1} R), \\
e_R = \frac{1}{2} (R_d^T R - R^T R_d^T)^{\circ}, \quad e_\Omega = \Omega - R^T R_d \Omega_d,
$$

where $K$ is a Morse function on $SO(3)$, the vee map $\vee: \mathfrak{so}(3) \rightarrow \mathbb{R}^3$ sets up an isomorphism between skew symmetric matrix and real vector:

$$
\begin{bmatrix}
0 & -x_3 & x_2 \\
x_3 & 0 & -x_1 \\
-x_2 & x_1 & 0
\end{bmatrix}^{\circ} = \begin{bmatrix} x_1 \\
x_2 \\
x_3
\end{bmatrix},
$$

with hat map $\hat{\cdot}: \mathbb{R}^3 \rightarrow \mathfrak{so}(3)$ its inverse.

If we use $L_2$ norm in $\mathbb{R}^3$ for $e_R$, then according to [9], it holds that

$$
\|e_R\|^2 = \Psi \cdot (2 - \Psi), \quad \|h_1(x, R) - h_1(x, R_d)\| \leq \|e_R\|_2, \quad (18)
$$

which indicates that $\|e_R\|^2 \leq (2 - c) \Psi$ for any $\Psi < c < 2$. Since every norm is equivalent on finite dimensional space, we have the fact that $\Psi$ is locally quadratic. The force function $h_1(x, R) = Re_3$ and we could just pick $\varphi_g = id$ which is the identical map. To invert $h_1$, we
choose the following submanifold $R_N = \{ R \in SO(3) : s(R) = q_{1d} \cdot Re_2 = 0 \}$ for a constant unit vector $q_{1d}$. The image of the differential $ds(R) = q_{1d} \cdot Re_2$ since we could characterize each tangent vector in $R$ as $R\eta$ for $\eta \in \mathbb{R}^3$. For any point $R \in s^{-1}(0)$, it satisfies that $q_{1d} \cdot Re_2 = 0$ which will implies that the quantities $q_{1d} \cdot Re_1$ or $q_{1d} \cdot Re_3$ should be non-zero since $q_{1d}$ lies in the plane spanned by $Re_1, Re_2$. Assume $q_{1d} \cdot Re_1$ is nonzero, then if we choose $\eta = ke_3, ds(R) = k q_{1d} \cdot Re_1$ which shows that $ds$ is a submersion at each point $s^{-1}(0)$ and 0 is a regular value of $s$. Hence, by the preimage theorem, $R_N$ is a submanifold with dimension $3 - 1 = 2$. Then, we could set up a global diffeomorphism of $h_1$ on $R_N$ with open subset of the sphere $\mathbb{S}^3 \setminus \{ q_{1d}, -q_{1d} \}$ using the following inverse map:

$$h_1^{-1}(q_3) = \left[ -\frac{1}{\| q_3 \times q_{1d} \|} q_3^2 q_{1d}, \frac{1}{\| q_3 \times q_{1d} \|} q_3 \times q_{1d} \right].$$

Here we are using the unit vector $q_{1d}$ to construct a submanifold in $SO(3)$ where we could invert $h_1$. As discussed in [13], singularity exists for this systems through this construction, where we can never reach $q_{1d}$. Using [13], we have $\| h_1(x, R) - h_1(x, R) \| \leq \sqrt{2} \Psi$ which satisfies the Lipschitz-type condition. Thus we could apply our method to [16] using PD controller with proper gains. Note that this controller coincides with the geometric controller in [9] except that the control scalar $v$ is defined as the projection $v = \| q_3, f(q, r) \|_2$. We have performed a simulation result of this quadrotor to track a reference position $x_d(t) = [\sin t, \cos t, \cos 1.5t]^T$ and show the corresponding position error and velocity error in Fig. 2 and Fig. 3. As can be seen, each component of the errors decrease to zero in a very fast manner, and this implies that the corresponding closed-loop system is stable.

**B. Geometric Backstepping of a Modified 3D Pendulum**

The system considered here is a 3D pendulum with modified model, shown in Fig. 1b. The moment at the pivot point is applied to adjust the orientation of the pendulum through an actuator at the pivot. For this system, the moment magnitude can be directly controlled but the direction of the moment has internal dynamics. Accordingly, the corresponding system dynamics is given below as:

\[ \dot{R} = R\dot{\Omega}, \quad \Omega = J_1^{-1}((J_1) \times \Omega + vq), \]
\[ \dot{q} = \omega \times q, \quad \dot{\omega} = J_2^{-1}(-q \times (q \times u)), \]

where $R \in SO(3)$ is the orientation of pendulum, $J_1 \in \mathbb{R}^{3 \times 3}$ is the pendulum inertia with respect to the pivot, $v \in \mathbb{R}$ represents the moment magnitude, $q \in \mathbb{S}^2$ represents the direction of the moment applied in the body-fixed frame, $J_2 > 0$ reflects the inertia of the actuator and $u \in \mathbb{R}^3$ is the internal torque of this actuator.

Here, the body dynamics is [19] and the shape dynamics is [20]. For the two-sphere $\mathbb{S}^2$, we use the following configuration error from [17]:

\[ \Psi(q, q_d) = 1 - q \cdot q_d, \]
\[ e_q = q_d \times q, \quad e_\omega = \omega + \dot{q} \times \omega_d, \quad \omega_d = q_d \times \dot{q}_d, \]

where we will use $L_2$ norm and $\Psi$ has been shown to be locally quadratic [3].

The force function $h_1(R, q) = q$ with $\varphi_\omega = id$. Since $M$ is a two-sphere, it can be directly embedded into the Lie algebra of $SO(3)$ which is $\mathbb{R}^3$. Hence, we will choose $N = \mathbb{S}^2 \subset so(3)$. Also, it holds obviously that

\[ \|h_1(R, q_1) - h_1(R, q_2)\|_2 = \|q_1 - q_2\|_2, \]

where the Lipschitz type inequality holds.

Given a smooth reference $R_{id}$, a virtual feedback law
Based on this, we could apply the final feedback law for
the reference for the orientation is specified as:
formed a simulation test for this 3D pendulum where
a control input from the fully-actuated shape dynamics.

A general Lyapunov function is constructed iteratively
based on the idea of backstepping
proposed to track a reference trajectory for the underac-
tuated body dynamics. V. C
ONCLUSION

We define a class of underactuated systems called
strict-feedback simple mechanical systems, which has a
fully-actuated shape dynamics and underactuated body
dynamics. A generalizable geometric control design is
proposed to track a reference trajectory for the underac-
tuated body dynamics based on the idea of backstepping
a control input from the fully-actuated shape dynamics.
A general Lyapunov function is constructed iteratively
through backstepping and is used to prove local expo-
tential stability for the proposed controller and validated
through numerical simulations on two example systems.

REFERENCES


