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An Efficient Weighting Update Method to Achieve Acceptable Consistency Deviation in Analytical Target Cascading

Weighting coefficients are used in analytical target cascading (ATC) at each element of the hierarchy to express the relative importance of (a) matching targets passed from the parent element and (b) maintaining consistency of linking variables and consistency with designs achieved by subsystem child elements. Proper selection of weight values is crucial when the top-level targets are unattainable, for example when "stretch" targets are used. In this case, strict design consistency cannot be achieved with finite weights; however, it is possible to achieve arbitrarily small inconsistencies. This article presents an iterative method for finding weighting coefficients that achieve solutions within user-specified inconsistency tolerances and demonstrates its effectiveness with several examples. The method also led to reduced computational time in the demonstration examples.
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Introduction

Analytical target cascading (ATC) is a model-based, hierarchical optimization methodology for systems design. ATC requires a set of analysis or simulation models that predict responses (the characteristics) of each system, subsystem, and component as a function of the design variables (the decisions) [1]. The analysis models are organized using design optimization models that are the elements or building blocks of the hierarchy, as shown in Fig. 1 with the standard index notation. The top level represents the overall system, and each lower level represents a subsystem or component of its parent element. In the ATC process, top-level system design targets are propagated down to lower subsystem and component level targets that are then optimized to meet the targets as closely as possible. The resulting responses are rebalanced at higher levels by iteratively adjusting targets and designs to achieve consistency.

Following Michelena et al. [2], and using the general notation introduced by Michalek and Papalambros [3], the *original design target problem* is

$$\underset{\mathbf{x}}{\text{minimize}} \|\mathbf{r}(\mathbf{x})-\mathbf{T}\|_2^2$$
$$\text{subject to}\quad \mathbf{g}(\mathbf{x})\leq\mathbf{0},\quad \mathbf{h}(\mathbf{x})=\mathbf{0} \tag{1}$$

where $\mathbf{T}$ is the vector of targets, $\mathbf{r}$ is the vector-valued response function, $\mathbf{x}$ is the complete vector of design variables, $\mathbf{g}$ and $\mathbf{h}$ are vectors of design constraint functions, and $\|\ \|_2^2$ denotes the square of the l_2 norm. Equation (1) represents the entire large-scale system, and it is solved all at once (AAO) (i.e., all variables and functions are evaluated together during search). Given that the system has an implied hierarchical structure of $N+1$ levels, as in Fig. 1, the formulation (still solved AAO) can be equivalently represented by designating response variables and linking variables, creating copies of these variables at parent and child levels, and adding constraints forcing the copies to be equal

$$\underset{\overline{\mathbf{x}}_{ij},\mathbf{y}^i_{(i+1)j}}{\text{minimize}} \|\mathbf{R}^0_{0l}-\mathbf{T}\|_2^2$$
$$\text{subject to}\quad \sum_{k\in C_{ij}}\|\mathbf{R}^i_{(i+1)k}-\mathbf{R}^{i+1}_{(i+1)k}\|_2^2=0$$
$$\sum_{k\in C_{ij}}\|\mathbf{S}_k\mathbf{y}^i_{(i+1)j}-\mathbf{y}^{(i+1)}_{(i+1)k}\|_2^2=0$$
$$\mathbf{g}_{ij}(\overline{\mathbf{x}}_{ij})\leq\mathbf{0},\quad \mathbf{h}_{ij}(\overline{\mathbf{x}}_{ij})=\mathbf{0}$$
$$\text{where}\quad \mathbf{R}^i_{ij}=\mathbf{r}_{ij}(\overline{\mathbf{x}}_{ij})$$
$$\overline{\mathbf{x}}_{ij}=[\mathbf{x}^i_{ij},\mathbf{y}^i_{ij},\mathbf{R}^i_{(i+1)k_1},\ldots,\mathbf{R}^i_{(i+1)k_{c_{ij}}}]^T$$
$$\forall j\in E_i,\ i=0,1,\ldots,N \tag{2}$$

where $\mathbf{x}^i_{ij}$ is the vector of local variables for element j at level i, $\mathbf{y}^i_{ij}$ is the vector of linking variables for element j at level i, $\mathbf{y}^{i-1}_{ij}$ is the copy of the vector of linking variables at element j level i coordinated by the parent element at level $(i-1)$, $\mathbf{S}_j$ is the selection matrix indicating which terms of the parent coordinating linking variable vector $\mathbf{y}^{i-1}_{ij}$ are relevant to the linking variable vector $\mathbf{y}^i_{ij}$ at element j, $\mathbf{R}^i_{ij}$ is the vector of responses at element j level i, $\mathbf{R}^{i-1}_{ij}$ is the vector of response targets for element j at level i that are set by the parent element at level $(i-1)$, $\mathbf{r}_{ij}$ is the vector-valued response function of element j at level i, $\mathbf{g}_{ij}$ is the vector of inequality constraints at element j level i, $\mathbf{h}_{ij}$ is the vector of equality constraints at element j level i, E_i is the set of elements at level i, C_{ij} is the set of element j's children numbered 1 through c_{ij}, and l designates the top-level element. Note that $\mathbf{y}^i_{ij}$ drops out for elements that do not have linking variables, such as element l, and $\mathbf{R}^i_{(i+1)k}$ terms drop out for leaf elements (elements that do not have children).

Following Michelena et al. [2], the formulation in Eq. (2) is relaxed by allowing deviation between linking variable and response variable copies to be within a tolerance ε and minimizing ε. Additionally, vectors of weighting coefficients $\mathbf{w}$ are introduced for linking and response variables to specify the relative importance of matching each target at each level. This yields the *relaxed AAO formulation*, which is set up to be, but has not yet been, decomposed

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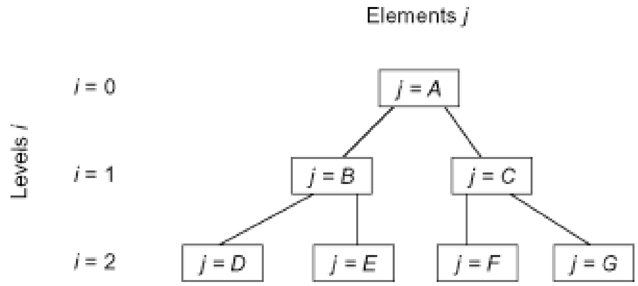

Fig. 1 Example of index notation for a hierarchically partitioned design problem

$$\underset{\bar{\mathbf{x}}_{ij},\mathbf{y}^i_{(i+1)j},\varepsilon^{\mathrm{R}}_{ij},\varepsilon^{\mathrm{y}}_{ij}}{\text{minimize}} \quad \|\mathbf{R}^0_{0l}-\mathbf{T}\|_2^2+\sum_{i=0}^{N-1}\sum_{j\in E_i}\varepsilon^{\mathrm{R}}_{ij}+\sum_{i=0}^{N-1}\sum_{j\in E_i}\varepsilon^{\mathrm{y}}_{ij}$$

subject to

$$\sum_{k\in C_{ij}}\|\mathbf{w}^{\mathrm{R}}_{(i+1)k}\circ(\mathbf{R}^i_{(i+1)k}-\mathbf{R}^{i+1}_{(i+1)k})\|_2^2\leq\varepsilon^{\mathrm{R}}_{ij}$$

$$\sum_{k\in C_{ij}}\|\mathbf{S}_k\mathbf{w}^{\mathrm{y}}_{(i+1)j}\circ(\mathbf{S}_k\mathbf{y}^i_{(i+1)j}-\mathbf{y}^{(i+1)}_{(i+1)k})\|_2^2\leq\varepsilon^{\mathrm{y}}_{ij}$$

$$\mathbf{g}_{ij}(\bar{\mathbf{x}}_{ij})\leq\mathbf{0},\quad \mathbf{h}_{ij}(\bar{\mathbf{x}}_{ij})=\mathbf{0}$$

where

$$\mathbf{R}^i_{ij}=\mathbf{r}_{ij}(\bar{\mathbf{x}}_{ij}),$$

$$\bar{\mathbf{x}}_{ij}=[\mathbf{x}^i_{ij},\mathbf{y}^i_{ij},\mathbf{R}^i_{(i+1)k_1},\ldots,\mathbf{R}^i_{(i+1)k_{c_{ij}}}]^T$$

$$\forall j\in E_i, i=0,1,\ldots,N \tag{3}$$

where $\varepsilon^{\mathrm{R}}_{ij}$ is the response deviation tolerance variable for element j level i, $\varepsilon^{\mathrm{y}}_{ij}$ is the linking deviation tolerance variable for element j level i, $\mathbf{w}^{\mathrm{R}}_{ij}$ is the response deviation weighting coefficient vector for element j at level i, $\mathbf{w}^{\mathrm{y}}_{ij}$ is the linking variable deviation weighting coefficient vector for element j at level i, and the $\circ$ symbol is used to indicate term-by-term multiplication of vectors such that $[a_1a_2\ldots a_n]^T\circ[b_1b_2\ldots b_n]^T=[ab_1ab_2\ldots ab_n]^T$.

Finally, the problem is decomposed into separate elements P_{ij}, and monotonicity analysis [4] is used to show that the ε-bound constraints of each element are active, allowing them to be solved for ε and moved into the objective function. The general notation for a *single ATC element* P_{ij} in the hierarchy is then

$$\underset{\bar{\mathbf{x}}_{ij},\mathbf{y}^i_{(i+1)j}}{\text{minimize}}\|\mathbf{w}^{\mathrm{R}}_{ij}\circ(\mathbf{R}^i_{ij}-\mathbf{R}^{i-1}_{ij})\|_2^2+\|\mathbf{S}_j\mathbf{w}^{\mathrm{y}}_{ip}\circ(\mathbf{S}_j\mathbf{y}^{i-1}_{ip}-\mathbf{y}^i_{ij})\|_2^2$$

$$+\sum_{k\in C_{ij}}\|\mathbf{w}^{\mathrm{R}}_{(i+1)k}\circ(\mathbf{R}^i_{(i+1)k}-\mathbf{R}^{i+1}_{(i+1)k})\|_2^2$$

$$+\sum_{k\in C_{ij}}\|\mathbf{S}_k\mathbf{w}^{\mathrm{y}}_{(i+1)j}\circ(\mathbf{S}_k\mathbf{y}^i_{(i+1)j}-\mathbf{y}^{(i+1)}_{(i+1)k})\|_2^2$$

subject to

$$\mathbf{g}_{ij}(\bar{\mathbf{x}}_{ij})\leq\mathbf{0},\quad \mathbf{h}_{ij}(\bar{\mathbf{x}}_{ij})=\mathbf{0}$$

where

$$\mathbf{R}^i_{ij}=\mathbf{r}_{ij}(\bar{\mathbf{x}}_{ij}),$$

$$\bar{\mathbf{x}}_{ij}=[\mathbf{x}^i_{ij},\mathbf{y}^i_{ij},\mathbf{R}^i_{(i+1)k_1},\ldots,\mathbf{R}^i_{(i+1)k_{c_{ij}}}]^T \tag{4}$$

The sequence of solving each optimization problem element P_{ij} and passing its solution to the rest of the hierarchy is called a coordination strategy. Michelena et al. [2] proved that using certain classes of coordination strategies to manage elements of the ATC formulation in Eq. (4), will result in convergence to the same solution as that of the relaxed AAO formulation in Eq. (3). Under these specific coordination strategies, managing the ATC hierarchy can be viewed as solving a series of Hierarchical Overlapping Coordination (HOC) problems, which have been shown to have nonascent, global convergence properties [5–7].

ATC has been applied to automotive applications [8–11], including the design of product families [11], as well as to the design of building systems [12]. Decomposing large-scale problems can be advantageous because it organizes and separates models and information by focus or discipline, provides communication only where necessary, and facilitates concurrent design. Moreover, ATC can solve some problems that are computationally difficult or impossible to solve all at once. Occasionally, decomposition can also result in improved computational efficiency because the formulation of each element typically has fewer degrees of freedom and fewer constraints than the AAO formulation. However, computational efficiency of ATC is not yet well understood, and empirical evidence shows that it can vary dramatically, depending on the choice of weighting coefficients [13].

Several other systems have been proposed for multidisciplinary design optimization (MDO) of complex systems. In particular, collaborative optimization (CO) [14], based on concepts introduced by Sobieski [15], contains a similar form of minimizing deviations between targets and responses using the square of the l_2 norm. CO formulations so far have dealt only with bilevel problems, although multilevel extensions seem possible. Moreover, it has been observed by Alexandrov and Lewis [16] and reemphasized by Kim [17] that CO cannot, in general, produce KKT points because of constraint qualification failures, whereas ATC has proven convergence properties. ATC is different from MDO frameworks, such as multidisciplinary feasible (MDF) and individual discipline feasible (IDF) [18], or the bilevel integrated system synthesis (BLISS) approach [19], where analysis models at a single level are integrated under a master problem introduced as an authority to achieve the overall design goal. Furthermore, ATC should not be confused with strategies for nonhierarchical systems, such as concurrent subspace optimization (CSSO) [20], or formulation choices for design optimization statements at individual problem elements, such as simultaneous analysis and design (SAND) or nested analysis and design (NAND) [21]. In contrast, ATC represents a multilevel decision-making hierarchy for complex systems design consisting of an arbitrarily large hierarchy of levels of analysis and design models representing systems, subsystems, and components.

The global convergence theory of ATC [2] asserts that weighting coefficients can be found such that consistency deviation terms converge to zero. However, we will show that for problems with attainable targets, strictly consistent designs can be found with any positive finite weighting coefficients, but for problems with unattainable targets, strict design consistency cannot be achieved with finite weighting coefficients. Thus, the selection of proper weighting coefficients is necessary to achieve a solution within acceptable inconsistency tolerances. This result is particularly relevant when intentionally using "stretch targets" or "stretch goals," terms used in management communities to describe setting very high, usually unattainable, goals in order to motivate employees [22].

In this paper the issue of consistency for unattainable targets is discussed, and an iterative approach is proposed to find weighting coefficients that achieve solutions with user-specified inconsistency tolerances. The method is then generalized and demonstrated with several examples.

Consistency for Unattainable Targets

In the ATC global convergence proof [2], Michelena et al. proved that when elements of the ATC hierarchy [Eq. (4)] are solved separately and iteratively using certain coordination strategies, the system will converge to the solution of the relaxed AAO formulation, Eq. (3). They go further to assert "given that consis-

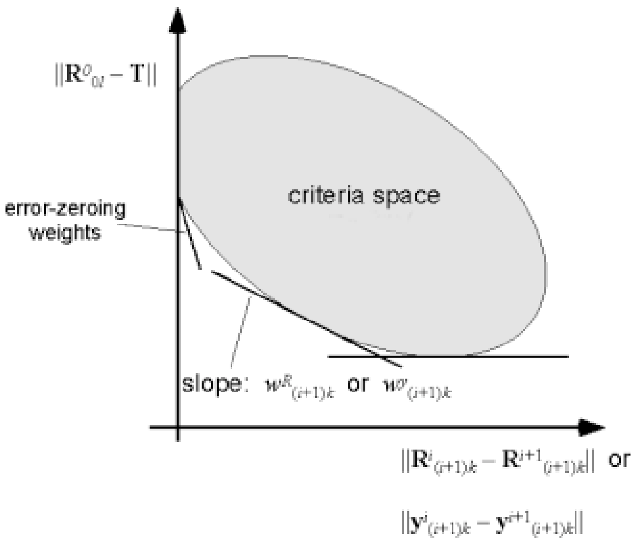

Fig. 2 Existence of error-zeroing weights proposed by Michelena et al. [2]

tency and feasibility are assumed for the original design target problem, it is possible to find weights $w^{\mathrm{R}}_{(i+1)k}$ and $w^{\mathrm{y}}_{(i+1)k}$ such that $\varepsilon^{\mathrm{R}}_{(i+1)k}$ and $\varepsilon^{\mathrm{y}}_{(i+1)k}$ . . . converge to zero . . . This implies that the ATC process, recursively applied to the problem hierarchy, produces an optimum solution of the original design target problem.''

The concepts of feasibility and consistency deserve further discussion here. Feasibility of the original design target problem means that a design exists that satisfies all constraints. Feasibility of the ATC elements means a local design exists at each ATC element P_{ij} that satisfies all of the constraints at that element. Consistency of the ATC formulation further supposes a solution exists such that $\mathbf{R}^i_{(i+1)k}=\mathbf{R}^{i+1}_{(i+1)k}$ and $\mathbf{y}^i_{(i+1)k}=\mathbf{y}^{i+1}_{(i+1)k}$ for all i, $j \in E_i$, $k \in C_{ij}$, which implies that $\varepsilon^{\mathrm{R}}=0$ and $\varepsilon^{\mathrm{y}}=0$ for all elements. Feasibility of the original design target formulation implies that a design exists in the decomposed ATC formulation that is feasible at all elements and consistent among elements.

In this section it is demonstrated that despite existence of a feasible, consistent design, the ATC formulation will not find this design with finite weighting coefficients unless the design meets the top-level targets exactly. Specifically, if a feasible solution to the original problem exists that meets the top-level targets exactly, then any choice of positive, finite weighting coefficients in the ATC formulation will yield a consistent solution. If such a solution does not exist, the ATC formulation will not yield a consistent solution for any finite weighting coefficients. However, an ATC solution can be found with arbitrarily small inconsistency deviations if weights are chosen appropriately.

Michelena et al. [2] proposed a Pareto optimization analogy to illustrate the existence of error-zeroing weights, as shown in Fig. 2. They observed that Eq. (4) contains a weighted sum of deviation metric terms, and they visualized the solution as a Pareto set between terms in the objective function, showing how larger weighting coefficients for parent-child deviation terms yield points with lower consistency deviation between parent and child at the expense of minimizing deviation from the top-level target. However, this figure could be misleading. Note that if a consistent, feasible design exists that meets the top-level targets, then the design would map to the origin in Fig. 2 and any other design would be either dominated by or equivalent to it in this space. Therefore, in this case the Pareto surface degenerates to a single point—the origin—which can be achieved with any positive weighting coefficients. If such a design does not exist, then it will be shown in Eqs. (7), (13), and (16) that, in general, the consistency deviation approaches zero only as the weighting coefficients for consistency approach infinity. So, in this case the vertical axis is tangent to the Pareto surface, and there are no finite error-zeroing weights. This is important for applications where unattainable targets are used purposefully or when the designer is uncertain if targets can be achieved.

A simple example will demonstrate this situation. Let us examine an unconstrained level-0 element with a single level-1 child. The level-0 element is called l, and the level-1 element is called k. There are no linking variables, and we consider only a single top-level target T. Following Eq. (4), the level-0 problem (P_{0l}) is written as

$$\underset{\{\mathbf{x}^0_{0l},\mathbf{R}^0_{1k}\}}{\text{minimize}}\|T-r_{0l}(\mathbf{x}^0_{0l},\mathbf{R}^0_{1k})\|_2^2+\|\mathbf{w}^{\mathrm{R}}_{1k}\circ(\mathbf{R}^0_{1k}-\mathbf{R}^1_{1k})\|_2^2 \tag{5}$$

Writing out the squared l_2 norm in terms of vector elements by using the angle bracket symbol $\langle\rangle$ to denote vector elements indexed with α, and dropping the functional dependency notation for r_{0l}, the objective function f_{0l} at level-0 is

$$f_{0l}=(T-r_{0l})^2+\sum_{\alpha}\ (\langle\mathbf{w}^{\mathrm{R}}_{1k}\rangle_\alpha\langle\mathbf{R}^0_{1k}-\mathbf{R}^1_{1k}\rangle_\alpha)^2 \tag{6}$$

The first-order necessary conditions for optimality of an unconstrained problem require that if a (local) solution to Eq. (6) exists, then the gradient of the objective function with respect to the response targets $\mathbf{R}^0_{1k}$ at that point must be zero

$$\begin{aligned}
\frac{\partial f_{0l}}{\partial\mathbf{R}^0_{1k}} &= 2(r_{0l}-T)\frac{\partial r_{0l}}{\partial\mathbf{R}^0_{1k}}+2\sum_{\alpha}\ \langle\mathbf{w}^{\mathrm{R}}_{1k}\rangle^2_\alpha\langle\mathbf{R}^0_{1k}-\mathbf{R}^1_{1k}\rangle_\alpha\frac{\partial\langle\mathbf{R}^0_{1k}\rangle_\alpha}{\partial\mathbf{R}^0_{1k}}\\
&= 2(r_{0l}-T)\frac{\partial r_{0l}}{\partial\mathbf{R}^0_{1k}}\\
&\quad +2\begin{pmatrix}\langle\mathbf{w}^{\mathrm{R}}_{1k}\rangle^2_1\langle\mathbf{R}^0_{1k}-\mathbf{R}^1_{1k}\rangle_1[1,0,\ldots,0]^T\\ +\langle\mathbf{w}^{\mathrm{R}}_{1k}\rangle^2_2\langle\mathbf{R}^0_{1k}-\mathbf{R}^1_{1k}\rangle_2[0,1,\ldots,0]^T\\ +\ \ldots\\ +\langle\mathbf{w}^{\mathrm{R}}_{1k}\rangle^2_n\langle\mathbf{R}^0_{1k}-\mathbf{R}^1_{1k}\rangle_n[0,0,\ldots,1]^T\end{pmatrix}\\
&= 2(r_{0l}-T)\frac{\partial r_{0l}}{\partial\mathbf{R}^0_{1k}}+2\mathbf{w}^{\mathrm{R}}_{1k}\circ\mathbf{w}^{\mathrm{R}}_{1k}\circ(\mathbf{R}^0_{1k}-\mathbf{R}^1_{1k})=0
\end{aligned}$$

$$\therefore\langle\mathbf{R}^0_{1k}-\mathbf{R}^1_{1k}\rangle_\alpha=\left(\frac{T-r_{0l}}{(\langle\mathbf{w}^{\mathrm{R}}_{1k}\rangle_\alpha)^2}\right)\frac{\partial r_{0l}}{\partial\langle\mathbf{R}^0_{1k}\rangle_\alpha} \tag{7}$$

This last equation shows that the optimal design will not be strictly consistent ($\mathbf{R}^0_{1k}\neq\mathbf{R}^1_{1k}$) for positive, finite weights unless the top-level target is met exactly or the derivative of the response function with respect to $\mathbf{R}^0_{1k}$ happens to be zero at the optimum. If top-level targets are unattainable $(T-r_{0l})\neq 0$, then the inconsistency deviation error ($\mathbf{R}^0_{1k}-\mathbf{R}^1_{1k}$) will be nonzero, except in the special case where the derivative of the response function is zero at the optimum, which can happen mostly by coincidence. Thus, in general, ($\mathbf{R}^0_{1k}-\mathbf{R}^1_{1k}$) approaches zero only as the terms of $\mathbf{w}^{\mathrm{R}}_{1k}$ approach infinity.

At this point one is tempted to simply set large weights. However, apart from the ATC convergence requirement, the size of the weights will also have a scaling effect on the nonlinear programing algorithm used to solve the element problem. Adverse scaling will increase computational time or altogether prevent solution of the element problem. Additionally, as will be shown later, in multilevel hierarchies the resulting deviations at any particular element depend on ratios of the weights at that element to weights at the parent element, and there are interactions between weights for linking variables $\mathbf{w}^{\mathrm{y}}$ and for response variables $\mathbf{w}^{\mathrm{R}}$. So, simply setting all weighting coefficients to large values will not necessarily result in small inconsistency deviation values. The task then is to find appropriate weights such that the resulting inconsistency deviation is acceptable. One way to approach this task is to use the results of Eq. (7) to calculate estimates of the weighting terms $\mathbf{w}_{1k}^{\mathrm{R}}$ required to achieve acceptable consistency errors $\boldsymbol{\theta}_{1k}^{\mathrm{R}}$ for each of the response targets $\mathbf{R}_{1k}^{0}$. To do this, we set the left-hand side of the equation to the desired inconsistency $\boldsymbol{\theta}_{1k}^{\mathrm{R}}$ and solve for the weights

$$\langle \mathbf{w}_{1k}^{\mathrm{R}} \rangle_{\alpha} = \left| \left(\frac{r_{0l} - T}{\langle \boldsymbol{\theta}_{1k}^{\mathrm{R}} \rangle_{\alpha}} \right) \frac{\partial r_{0l}}{\partial \langle \mathbf{R}_{1k}^{0} \rangle_{\alpha}} \right|^{1/2} \tag{8}$$

Thus, in this example the weighting update method for finding appropriate weights to achieve consistency error tolerances $\boldsymbol{\theta}_{1k}^{\mathrm{R}}$ would follow these steps:

1. Set initial-guess weights (say $\mathbf{w}_{1k}^{\mathrm{R}} = [1,1,\ldots,1]^{T}$).
2. Solve the ATC problem and calculate the top-level target deviation and the derivative of the response function at the solution.
3. If the deviation tolerance is not satisfied at the solution, then use Eq. (8) to find new weighting terms and return to step 2.

Generalization of the Weighting Update Method

The goal of the weighting update method is to automatically identify appropriate weighting coefficients that achieve designs with acceptable deviation tolerance values for the response variables at each element $\boldsymbol{\theta}_{ij}^{\mathrm{R}}$ and for the linking variables at each parent coordinating element $\boldsymbol{\theta}_{(i+1)j}^{\mathrm{y}}$. The problem is first solved using starting values for weighting coefficients. The solution to that problem is used to calculate a linear approximation of the weighting coefficients needed to achieve the desired tolerances. Weights are updated with this approximation, and the problem is solved again. This process is repeated until the inconsistency deviation tolerance is achieved, namely, the final solution satisfies the conditions

$$\begin{gathered} |\mathbf{R}_{ij}^{i-1} - \mathbf{R}_{ij}^{i}| \leq \boldsymbol{\theta}_{ij}^{\mathrm{R}} \\ |\mathbf{y}_{(i+1)k}^{(i+1)} - \mathbf{y}_{(i+1)k'}^{(i+1)}| \leq \boldsymbol{\theta}_{(i+1)j}^{\mathrm{y}} \\ \forall k,k' \in C_{ij}, \quad j \in E_i, \quad i = 0,1,\ldots,N \end{gathered} \tag{9}$$

To generalize the method presented in the previous section, we examine one of the KKT first-order necessary conditions for optimality of constrained nonlinear problems, which involves the Lagrangian. From Eq. (4), the Lagrangian L_{ij} of element j at level i is

$$\begin{aligned} L_{ij} = {} & \|\mathbf{w}_{ij}^{\mathrm{R}} \circ (\mathbf{r}_{ij} - \mathbf{R}_{ij}^{i-1})\|_2^2 + \|\mathbf{S}_j \mathbf{w}_{ip}^{\mathrm{y}} \circ (\mathbf{S}_j \mathbf{y}_{ip}^{i-1} - \mathbf{y}_{ij}^{i})\|_2^2 \\ & + \sum_{k \in C_{ij}} \|\mathbf{w}_{(i+1)k}^{\mathrm{R}} \circ (\mathbf{R}_{(i+1)k}^{i} - \mathbf{R}_{(i+1)k}^{i+1})\|_2^2 \\ & + \sum_{k \in C_{ij}} \|\mathbf{S}_k \mathbf{w}_{(i+1)j}^{\mathrm{y}} \circ (\mathbf{S}_k \mathbf{y}_{(i+1)j}^{i} - \mathbf{y}_{(i+1)k}^{(i+1)})\|_2^2 + \boldsymbol{\mu}_{ij}^{T} \mathbf{g}_{ij} + \boldsymbol{\lambda}_{ij}^{T} \mathbf{h}_{ij} \end{aligned} \tag{10}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ are the vectors of Lagrange multipliers for the inequality and equality constraints respectively. Expressing the norms with vector terms indexed with the symbol α, we have

$$\begin{aligned} L_{ij} = {} & \sum_{\alpha_1} (\langle \mathbf{w}_{ij}^{\mathrm{R}} \rangle_{\alpha_1} \langle \mathbf{r}_{ij} - \mathbf{R}_{ij}^{i-1} \rangle_{\alpha_1})^2 + \sum_{\alpha_2} (\langle \mathbf{S}_j \mathbf{w}_{ij}^{\mathrm{y}} \rangle_{\alpha_2} \langle \mathbf{S}_j \mathbf{y}_{ip}^{i-1} \\ & - \mathbf{y}_{ij}^{i} \rangle_{\alpha_2})^2 + \sum_{k \in C_{ij}} \sum_{\alpha_3} (\langle \mathbf{w}_{(i+1)k}^{\mathrm{R}} \rangle_{\alpha_3} \langle \mathbf{R}_{(i+1)k}^{i} - \mathbf{R}_{(i+1)k}^{i+1} \rangle_{\alpha_3})^2 \\ & + \sum_{k \in C_{ij}} \sum_{\alpha_4} (\langle \mathbf{S}_k \mathbf{w}_{(i+1)j}^{\mathrm{y}} \rangle_{\alpha_4} \langle \mathbf{S}_k \mathbf{y}_{(i+1)j}^{i} - \mathbf{y}_{(i+1)k}^{(i+1)} \rangle_{\alpha_4})^2 \\ & + \boldsymbol{\mu}_{ij}^{T} \mathbf{g}_{ij} + \boldsymbol{\lambda}_{ij}^{T} \mathbf{h}_{ij} \end{aligned} \tag{11}$$

If a feasible solution to Eq. (4) exists, then one property of the KKT first-order necessary conditions states that at the (local) solution the gradient of the Lagrangian with respect to each term β of the response target vector $\mathbf{R}_{(i+1)\gamma}^{i}$ for element γ is zero

$$\begin{aligned} \frac{\partial L_{ij}}{\partial \langle \mathbf{R}_{(i+1)\gamma}^{i} \rangle_{\beta}} = {} & 2 \sum_{\alpha_1} \left[(\langle \mathbf{w}_{ij}^{\mathrm{R}} \rangle_{\alpha_1})^2 \langle \mathbf{r}_{ij} - \mathbf{R}_{ij}^{i-1} \rangle_{\alpha_1} \frac{\partial \langle \mathbf{r}_{ij} \rangle_{\alpha_1}}{\partial \langle \mathbf{R}_{(i+1)\gamma}^{i} \rangle_{\beta}} \right] \\ & + 2 (\langle \mathbf{w}_{(i+1)\gamma}^{\mathrm{R}} \rangle_{\beta})^2 \langle \mathbf{R}_{(i+1)\gamma}^{i} - \mathbf{R}_{(i+1)\gamma}^{i+1} \rangle_{\beta} \\ & + \boldsymbol{\mu}_{ij}^{T} \frac{\partial \mathbf{g}_{ij}}{\partial \langle \mathbf{R}_{(i+1)\gamma}^{i} \rangle_{\beta}} + \boldsymbol{\lambda}_{ij}^{T} \frac{\partial \mathbf{h}_{ij}}{\partial \langle \mathbf{R}_{(i+1)\gamma}^{i} \rangle_{\beta}} = \mathbf{0} \end{aligned} \tag{12}$$

Therefore, at the solution the deviation between response variable copies at parent and child level is

$$\begin{aligned} & \langle \mathbf{R}_{(i+1)\gamma}^{i} - \mathbf{R}_{(i+1)\gamma}^{i+1} \rangle_{\beta} \\ & = \frac{1}{(\langle \mathbf{w}_{(i+1)\gamma}^{\mathrm{R}} \rangle_{\beta})^2} \sum_{\alpha_1} \left((\langle \mathbf{w}_{ij}^{\mathrm{R}} \rangle_{\alpha_1})^2 \langle \mathbf{R}_{ij}^{i-1} - \mathbf{r}_{ij} \rangle_{\alpha_1} \frac{\partial \langle \mathbf{r}_{ij} \rangle_{\alpha_1}}{\partial \langle \mathbf{R}_{(i+1)\gamma}^{i} \rangle_{\beta}} \right) \\ & \quad - \frac{1}{2 (\langle \mathbf{w}_{(i+1)\gamma}^{\mathrm{R}} \rangle_{\beta})^2} \left(\boldsymbol{\mu}_{ij}^{T} \frac{\partial \mathbf{g}_{ij}}{\partial \langle \mathbf{R}_{(i+1)\gamma}^{i} \rangle_{\beta}} + \boldsymbol{\lambda}_{ij}^{T} \frac{\partial \mathbf{h}_{ij}}{\partial \langle \mathbf{R}_{(i+1)\gamma}^{i} \rangle_{\beta}} \right) \end{aligned} \tag{13}$$

Note that this equation holds for all elements except the top-level element. To achieve desired response variable deviation tolerances within $\boldsymbol{\theta}_{(i+1)\gamma}^{\mathrm{R}}$ for each element in $\mathbf{R}_{(i+1)\gamma}^{i}$, each weighting term β in $\mathbf{w}_{(i+1)\gamma}^{\mathrm{R}}$ should be updated as

$$\begin{aligned} & \langle \mathbf{w}_{(i+1)\gamma}^{\mathrm{R}} \rangle_{\beta} = \left| \frac{\Psi_{\gamma\beta}}{\langle \boldsymbol{\theta}_{(i+1)\gamma}^{\mathrm{R}} \rangle_{\beta}} \right|^{1/2} \\ & \text{where } \Psi_{\gamma\beta} = \sum_{\alpha_1} \left((\langle \mathbf{w}_{ij}^{\mathrm{R}} \rangle_{\alpha_1})^2 \langle \mathbf{R}_{ij}^{i-1} - \mathbf{r}_{ij} \rangle_{\alpha_1} \frac{\partial \langle \mathbf{r}_{ij} \rangle_{\alpha_1}}{\partial \langle \mathbf{R}_{(i+1)\gamma}^{i} \rangle_{\beta}} \right) \\ & \qquad - \frac{1}{2} \left(\boldsymbol{\mu}_{ij}^{T} \frac{\partial \mathbf{g}_{ij}}{\partial \langle \mathbf{R}_{(i+1)\gamma}^{i} \rangle_{\beta}} + \boldsymbol{\lambda}_{ij}^{T} \frac{\partial \mathbf{h}_{ij}}{\partial \langle \mathbf{R}_{(i+1)\gamma}^{i} \rangle_{\beta}} \right) \end{aligned} \tag{14}$$

Note again that this equation holds for all levels except the top level, where the weighting coefficient vector is not updated. Top-level response deviations reflect failure of the design to meet the top-level targets, rather than inconsistencies in the design. The top-level weighting coefficient vector is set by the modeler to express the relative importance of matching each top-level target; it is not updated. While all weighting coefficient vectors reflect the relative importance of matching variable copies, the lower-level vectors are updated such that the final preference reflects that which is needed to achieve user-defined inconsistency tolerances.

Additionally, at the solution, the gradient of the Lagrangian with respect to the linking variables of element j is zero

$$\begin{aligned} \frac{\partial L_{ij}}{\partial \langle \mathbf{y}_{ij}^{i} \rangle_{\beta}} = {} & 2 \sum_{\alpha_1} (\langle \mathbf{w}_{ij}^{\mathrm{R}} \rangle_{\alpha_1})^2 \langle \mathbf{r}_{ij} - \mathbf{R}_{ij}^{i-1} \rangle_{\alpha_1} \frac{\partial \langle \mathbf{r}_{ij} \rangle_{\alpha_1}}{\partial \langle \mathbf{y}_{ij}^{i} \rangle_{\beta}} \\ & + 2 (\langle \mathbf{S}_j \mathbf{w}_{ip}^{\mathrm{y}} \rangle_{\beta})^2 \langle \mathbf{y}_{ij}^{i} - \mathbf{S}_j \mathbf{y}_{ip}^{i-1} \rangle_{\beta} + \boldsymbol{\mu}_{ij}^{T} \frac{\partial \mathbf{g}_{ij}}{\partial \langle \mathbf{y}_{ij}^{i} \rangle_{\beta}} \\ & + \boldsymbol{\lambda}_{ij}^{T} \frac{\partial \mathbf{h}_{ij}}{\partial \langle \mathbf{y}_{ij}^{i} \rangle_{\beta}} = \mathbf{0} \end{aligned} \tag{15}$$

Therefore, the deviation between linking variable term β in $\mathbf{y}_{ij}^{i}$ and the parent coordination copy in $\mathbf{y}_{ip}^{i-1}$ is

$$\langle \mathbf{y}_{ij}^{i} - \mathbf{S}_j \mathbf{y}_{ip}^{i-1} \rangle_\beta = \frac{1}{(\langle \mathbf{S}_j \mathbf{w}_{ip}^{\mathrm{y}} \rangle_\beta)^2} \sum_{\alpha_1} \left((\langle \mathbf{w}_{ij}^{\mathrm{R}} \rangle_{\alpha_1})^2 \langle \mathbf{R}_{ij}^{i-1} - \mathbf{r}_{ij} \rangle_{\alpha_1} \frac{\partial \langle \mathbf{r}_{ij} \rangle_{\alpha_1}}{\partial \langle \mathbf{y}_{ij}^{i} \rangle_\beta} \right) - \frac{1}{2(\langle \mathbf{S}_j \mathbf{w}_{ip}^{\mathrm{y}} \rangle_\beta)^2} \left(\boldsymbol{\mu}_{ij}^T \frac{\partial \mathbf{g}_{ij}}{\partial \langle \mathbf{y}_{ij}^{i} \rangle_\beta} + \boldsymbol{\lambda}_{ij}^T \frac{\partial \mathbf{h}_{ij}}{\partial \langle \mathbf{y}_{ij}^{i} \rangle_\beta} \right) \quad (16)$$

This term represents deviation between linking variable copies at element j and the parent coordination copy. Recall that linking variables are shared by elements at the same level and coordinated at the parent level. To achieve a desired deviation tolerance between elements at the same level, the weight for each term β must be set high enough so that the difference between copies at any two child elements is less than or equal to the tolerance. The updating calculation for the linking variable weighting coefficients is then

$$\langle \mathbf{w}_{ip}^{\mathrm{y}} \rangle_\beta = \underset{\forall j,j' \in C_{(i-1)p}^{\beta}}{\text{maximum}} \left| \frac{\Psi_{j\beta} - \Psi_{j'\beta}}{\langle \boldsymbol{\theta}_{ip}^{\mathrm{y}} \rangle_\beta} \right|^{1/2}$$

$$\text{where} \quad \Psi_{j\beta} = \sum_{\alpha_1} \left((\langle \mathbf{w}_{ij}^{\mathrm{R}} \rangle_{\alpha_1})^2 \langle \mathbf{R}_{ij}^{i-1} - \mathbf{r}_{ij} \rangle_{\alpha_1} \frac{\partial \langle \mathbf{r}_{ij} \rangle_{\alpha_1}}{\partial \langle \mathbf{S}_j^T \mathbf{y}_{ij}^{i} \rangle_\beta} \right) - \frac{1}{2} \left(\boldsymbol{\mu}_{ij}^T \frac{\partial \mathbf{g}_{ij}}{\partial \langle \mathbf{S}_j^T \mathbf{y}_{ij}^{i} \rangle_\beta} + \boldsymbol{\lambda}_{ij}^T \frac{\partial \mathbf{h}_{ij}}{\partial \langle \mathbf{S}_j^T \mathbf{y}_{ij}^{i} \rangle_\beta} \right) \quad (17)$$

and where $C_{(i-1)p}^{\beta}$ is the set of children of parent element p that contain linking variable β (i.e., Ψ drops out for children where $\langle \mathbf{S}_j^T \mathbf{y}_{ij}^{i} \rangle_\beta = 0$).

In summary, the generalized weighting update method involves iteratively solving the ATC formulation and updating the weighting coefficient vectors of each element (which express relative preferences for meeting each target) to achieve a solution with user-specified inconsistency deviation tolerances for each response variable $\boldsymbol{\theta}^{\mathrm{R}}$ and each linking variable $\boldsymbol{\theta}^{\mathrm{y}}$. The method is implemented with the following steps:

1. Set an acceptable inconsistency deviation tolerance for each response variable and each linking variable and set initial weights (for example, set all weights to 1).
2. Solve the ATC Problem
3. If the inconsistency deviation tolerance is not satisfied at the solution, then update each term in each weighting coefficient vector using Eqs. (14) and (17) and return to step 2.

Demonstration

To illustrate the topic of strict consistency for unattainable targets, a simple example is used where the target (zero in this case) is unattainable

$$\underset{z_1}{\text{minimize}} \|z_1\|_2^2 \qquad \text{subject to} \quad z_1 \geq 1 \quad (18)$$

The solution to this problem is $z_1 = 1$. In the relaxed formulation of this problem, copies of z_1 are made at level-0 element l and at level-1 element k, using the R notation to designate responses (there are no local variables or linking variables), and the weighted deviation between the copies is constrained less than or equal to ε. The positive, finite weight w is used as the weighting term. The relaxed AAO problem (before decomposition) is then

$$\underset{R_{1k}^0, R_{1k}^1, \varepsilon}{\text{minimize}} \|R_{0k}^0\|_2^2 + \varepsilon$$

$$\text{subject to} \quad g_1 = \|w(R_{1k}^0 - R_{1k}^1)\|_2^2 - \varepsilon \leq 0$$

$$g_2 = 1 - R_{1k}^1 \leq 0,$$

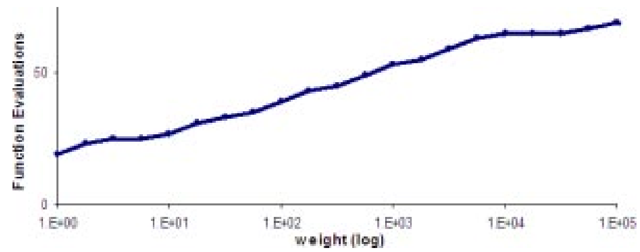

Fig. 3 Number of function evaluations required to find the solution as a function of the weighting term

$$\text{where} \quad R_{0l}^0 = r_{0l}(R_{1k}^0) = R_{1k}^0 \quad (19)$$

Note that the relaxed AAO problem is used in the remainder of this example, and the problem is not decomposed for ATC, since Michelena et al. [2] showed that these formulations yield equivalent solutions. At a KKT point, the gradient of the Lagrangian is zero

$$\nabla f + \mu_1 \nabla g_1 + \mu_2 \nabla g_2 = \mathbf{0}$$

$$\begin{bmatrix} 2R_{1k}^0 \\ 0 \\ 1 \end{bmatrix} + \mu_1 \begin{bmatrix} w^2(R_{1k}^0 - R_{1k}^1) \\ 2w^2(R_{1k}^1 - R_{1k}^0) \\ -1 \end{bmatrix} + \mu_2 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \mu_1 = 1, \quad (g_1 \text{is active}) \quad \therefore R_{1k}^0 = \frac{w^2}{1+w^2} R_{1k}^1$$

$$\therefore \mu_2 = \left(\frac{2w^2}{1+w^2} \right) R_{1k}^1 \quad (g_2 \text{ is active})$$

$$\therefore R_{1k}^1 = 1, \quad R_{1k}^0 = \frac{w^2}{1+w^2}, \quad R_{1k}^1 - R_{1k}^0 = \frac{1}{1+w^2} \neq 0$$

$$\therefore \varepsilon = \left(\frac{w}{1+w^2} \right)^2 \neq 0 \quad (20)$$

This shows that ε is nonzero at the KKT point for any finite weight; however, w can be found to achieve ε arbitrarily close to zero. It is important to note that ε approaches zero as w approaches infinity *or* zero, and the goal is to ensure that the inconsistencies between the responses at each level are within an acceptable tolerance, rather than focusing on the value of ε. The inconsistency $(R_{1k}^1 - R_{1k}^0)$ approaches zero only as w approaches infinity.

In addition, to demonstrate the need to avoid setting arbitrarily large weights, this problem was implemented in MATLAB® 6.5.0 using the *fmincon* function with the feasible starting point $[R_1^0\ R_1^1\ \varepsilon]^T = [2\ 5\ 5]^T$ and the following parameters: TolCon $=$ TolFun $=$ TolX $= 10^{-10}$. The algorithm and parameters are specified here because the algorithm behavior depends on the parameters and starting point; however, this example serves to show the basic trends. Figure 3 shows the number of function evaluations needed to converge to a solution for each value of w. The figure shows an upward trend, emphasizing the need to avoid large weighting terms when possible.

Figure 4 shows the resulting inconsistency deviation $(R_1^1 - R_1^0)$ at the optimum for each value of w. The graph shows a trend of reduced error as the weighting term is increased, although the error never reaches zero.

In general, it is difficult to set appropriate weights simply by guessing. The weighting update method is applied to this example to show how appropriate weights are found. In this example, the response function r_{0l} is a linear function of R_1^0, so the derivative of the response function is a constant $(=1)$, therefore, the use of the weighting update method to find appropriate weights yields

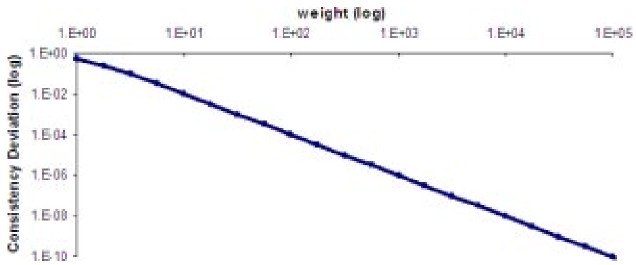

Fig. 4 Inconsistency between responses at the optimum as a function of the weighting term

$$w=\left|\left(\frac{r_0-T}{\theta}\right)\frac{\partial r_0}{\partial R_1^0}\right|^{1/2}=\left|\frac{R_1^0}{\theta}\right|^{1/2} \tag{21}$$

The update procedure was implemented for this example with an inconsistency tolerance goal of $\theta=10^{-2}$, and a starting weight of $w=1$. The proper weight needed to achieve this inconsistency tolerance, $w=9.95$, was found after three weighting update iterations and a total of 89 function evaluations.

Geometric Programing Example

The geometric programing example, proposed by Kim [1], is used here as a multilevel example with linking variables to demonstrate the weighting update method. The original design target problem is

$$\begin{aligned}
&\underset{z_1,z_2,\ldots,z_{14}}{\text{minimize}}\ f=z_1^2+z_2^2\\
&\text{subject to}\ \ g_1=z_3^{-2}+z_4^2-z_5^2\leqslant 0\\
&g_2=z_5^2+z_6^{-2}-z_7^2\leqslant 0\\
&g_3=z_8^2+z_9^2-z_{11}^2\leqslant 0\\
&g_4=z_8^{-2}+z_{10}^2-z_{11}^2\leqslant 0\\
&g_5=z_{11}^2+z_{12}^{-2}-z_{13}^2\leqslant 0\\
&g_6=z_{11}^2+z_{12}^2-z_{14}^2\leqslant 0\\
&h_1=z_1^2-z_3^2-z_4^{-2}-z_5^2=0\\
&h_2=z_2^2-z_5^2-z_6^2-z_7^2=0\\
&h_3=z_3^2-z_8^2-z_9^{-2}-z_{10}^{-2}+z_{11}^2=0\\
&h_4=z_6^2-z_{11}^2-z_{12}^2-z_{13}^2+z_{14}^2=0\\
&z_1,z_2,\ldots,z_{14}\geqslant 0
\end{aligned} \tag{22}$$

The original problem will be decomposed first as a two-level ATC hierarchy with three elements, as proposed by Kim [1], and second, as a three-level ATC hierarchy with five elements, as proposed by Tzevelekos et al. [13]. The feasible starting point $\mathbf{z}$ = [5,5,2.76,0.25,1.26,4.64,1.39,0.67,0.76,1.70,2.26,1.41,2.71, 2.66]^T is used for all trials, and the acceptable inconsistency tolerance value of 10^{-2} is used for all response variables and linking variables.

Two-Level Decomposition. In the two-level decomposition, following Kim [1], the problem is partitioned into three elements: one level-0 element, A, with two level-1 children, B and C. The equality constraints of the original problem h_1, h_2, h_3, and h_4 are solved for z_1, z_2, z_3, and z_6, respectively, and used as response functions of elements A, A, B, and C, respectively. The objective function of each element is then to minimize deviation between targets and responses at that element, as in Eq. (4), where the top-level targets are both zero. The variable z_{11} is treated as a linking variable between elements B and C, variables z_4, z_5, and z_7, are local variables of element A, variables z_8, z_9, and z_{10} are local variables of element B, and variables z_{12}, z_{13}, and z_{14} are local variables of element C. The constraints g_1, g_2, g_3, g_4, g_5, and g_6 are associated with elements A, A, B, B, C, and C, respectively. Kim [1] provides a picture of this decomposition for reference.

Table 1 Results of the two-level and three-level geometric programing examples

		Two-Level		Three-Level	
		Default weights	Weighting update method	Default weights	Weighting update method
Weighting Coefficients (at soln.)	$\mathbf{w}^y_{1A}$	1	27.72	1	109.70
	$\mathbf{w}^R_{1B}$	1	14.61	1	99.34
	$\mathbf{w}^R_{1C}$	1	16.64	1	103.59
	$\mathbf{w}^R_{2D}$	--	--	1	85.96
	$\mathbf{w}^R_{2E}$	--	--	1	98.05
Resulting Inconsistency	z_1	--	--	1.47	0.01
	z_2	--	--	1.26	0.01
	z_3	0.69	0.01	0.78	0.01
	z_5	--	--	0.80	0.01
	z_6	0.65	0.01	1.05	0.01
	z_{11}	0.96	0.01	--	--

The problem was first solved with default weights $\mathbf{w}^{\mathrm{R}}_{1B}=\mathbf{w}^{\mathrm{R}}_{1C}=\mathbf{w}^{\mathrm{y}}_{1A}=[1]$. At the solution, resulting inconsistency deviations are 0.688, 0.649, and 0.961 for z_3, z_6, and z_{11}, respectively, all of which are larger than the acceptable tolerance value of 10^{-2}. Using the weighting update method, the weights are updated with Eqs. (14) and (17), and the new problem is solved. This process of updating and solving is repeated four times before converging. The final weights, $\mathbf{w}^{\mathrm{R}}_{1B}=14.534$, $\mathbf{w}^{\mathrm{R}}_{1C}=16.561$, and $\mathbf{w}^{\mathrm{y}}_{1A}=27.572$, yield inconsistencies of 10^{-2} for z_3, z_6, and z_{11}. The weighting update method successfully found the weighting coefficients that yield a solution with the desired inconsistency tolerance. These results are summarized in Table 1.

Three-Level Decomposition. In the three-level decomposition, following Tzevelekos et al. [13], the problem is partitioned into five elements: one level-0 element A with two level-1 children, B and C, and two level-2 elements, D and E, which are children of B and C, respectively. In the formulation, z_5 is treated as a linking variable between elements B and C, z_{11} is set as a parameter with known (optimal) value 1.30, the equality constraints of the original problem h_1, h_2, h_3, and h_4 are used to calculate z_1, z_2, z_3, and z_4 as response functions of elements B, C, D, and E, respectively. The response function of element A is $f=(z_1^2+z_2^2)$, with the top level target set to zero. The variable z_4 is a local variable of element B, variable z_7 is a local variable of element C, variables z_8, z_9, and z_{10} are local variables of element D, and variables z_{12}, z_{13}, and z_{14} are local variables of element E. The constraints g_1, g_2, g_3, g_4, g_5, and g_6 are associated with elements B, C, D, D, E, and E, respectively.

The problem was first solved with default weights $\mathbf{w}^{\mathrm{y}}_{1A}=\mathbf{w}^{\mathrm{R}}_{1B}=\mathbf{w}^{\mathrm{R}}_{1C}=\mathbf{w}^{\mathrm{R}}_{2D}=\mathbf{w}^{\mathrm{R}}_{2E}=[1]$. At the solution, resulting inconsistencies are 1.47, 1.26, 0.78, 0.80, and 1.05 for z_1, z_2, z_3, z_5, and z_6, respectively, all of which are larger than the acceptable tolerance value of 10^{-2}. Using the weighting update method, the weights are updated with Eqs. (14) and (17), and the new problem is solved. This process of updating and solving is repeated five times before converging. The final weights, $\mathbf{w}^{\mathrm{y}}_{1A}=109.70$, $\mathbf{w}^{\mathrm{R}}_{1B}=99.34$,

Table 2 Optimal solution to original, two-level ATC, and three-level ATC formulations

	Original	2-Level ATC		3-Level ATC	
	AAO	Default Weights	WUM (10^{-2})	Default Weights	WUM (10^{-2})
z_1	2.84	2.25	2.83	0.75*	2.82*
z_2	3.09	2.04	3.07	0.64*	3.07*
z_3	2.36	1.53*	2.35*	1.58*	2.35*
z_4	0.76	0.76	0.76	0.90	0.76
z_5	0.87	1.00	0.87	0.70*	0.87*
z_6	2.81	1.21*	2.79*	1.76*	2.80*
z_7	0.94	1.30	0.94	0.64	0.93
z_8	0.97	0.93	0.97	0.97	0.97
z_9	0.87	1.07	0.87	0.86	0.86
z_{10}	0.79	0.93	0.80	0.80	0.80
z_{11}	1.30	0.94*	1.30*	1.30†	1.30†
z_{12}	0.84	0.84	0.84	0.84	0.84
z_{13}	1.77	1.27	1.76	1.76	1.76
z_{14}	1.55	0.96	1.54	1.55	1.55

$\mathbf{w}^R_{1C}=103.59$, $\mathbf{w}^R_{2D}=85.96$, and $\mathbf{w}^R_{2E}=98.05$, yield inconsistencies of 10^{-2} for z_1, z_2, z_3, z_5, and z_6. The weighting update method successfully found weights that yield a solution with the desired inconsistency tolerance. These results are summarized in Table 1.

Accuracy. It is important to stress that inconsistencies in response and linking variables affect the entire solution, not only the copied variables themselves. Table 2 summarizes the solutions to the original, two-level ATC, and three-level ATC formulations. For the two-level and three-level formulations, results are shown for default weights (all weights$=1$) and for the weighting update method within consistency tolerances of 10^{-2} for all variables. In the table, the* indicates that the variable has nonzero inconsistency at the solution, and the value of the variable copy at the parent level is reported. The † indicates that the variable was treated as a static parameter. Notice that solutions using the default weights are far from the solution to the original problem, whereas solutions using the weighting update method are close for all variables. Smaller inconsistency tolerances result in solutions closer to the solution of the original problem.

Local Convergence. One purpose of using the weighting update method is to avoid setting weights arbitrarily high to avoid costly iterations; however, the weighting update method requires additional update iterations to converge on the desired weights, so it is worthwhile to examine and compare the convergence efficiency. The two-level geometric programing problem was solved using the required weights directly as starting weights, thus achieving the desired tolerance without any weighting update iterations. This represents the best possible case that could be attained by guessing weights. Still, in this case the algorithm required almost twice as many function evaluations per element to converge as did the weighting update method. These results are summarized in Table 3. Note that the MATLAB function *fmincon*, based on SQP, was used in all cases.

It took longer to converge when starting with the desired weights because the starting point is not close to the solution. Large weighting coefficients act to slow progress of the algorithm by restricting the deviation between parent and child elements at each ATC iteration. Conceptually, this can be thought of as an effect similar to that of a trust-region algorithm, where high weighting coefficients have the effect of tight trust regions, preventing large moves at each iteration. In contrast, the weighting update method first solves the problem with small weighting coefficients, allowing the algorithm to move quickly in the design space and converge to a point close to the final solution. The weights are then updated (increased), and the new problem is solved starting at the solution to the problem with the previous weighting coefficients. In this way, the weighting update method first moves quickly to the proximity of the solution, then tightens tolerances and closes in precisely on the final solution. Results vary based on the problem, acceptable inconsistency tolerance, and the starting point; however, this example shows that using the weighting update method can sometimes be substantially more efficient than even best-case-scenario guessing.

Table 3 Speed of convergence statistics for the geometric programming problem

	Number of Function Evaluations	Num. Weight Updates
Original Problem AAO	25,173	- -
2 Level ATC Default Weights	***A***: 241 ***B***: 110 ***C***: 115	- -
2 Level ATC Weighting Update	***A***: 18,002 ***B***: 8,639 ***C***: 8,517	4
2 Level ATC Required Weights	***A***: 31,777 ***B***: 15,316 ***C***: 16,297	- -
3 Level ATC Default Weights	***A***: 195 ***B***: 158 ***C***: 152 ***D***: 24 ***E***: 19	- -
3 Level ATC Weighting Update	***A***: 45,092 ***B***: 34,087 ***C***: 35,449 ***D***: 984 ***E***: 905	5

Further study on local convergence properties of ATC and the weighting update method is needed before these results can be generalized. Note that in contrast with notions of asymptotic local convergence developed for AAO algorithms (e.g., standard nonlinear programing), local convergence concepts have not been rigorously defined for any system optimization method relying on decomposition, including ATC.

Table 3 also shows that the ATC decomposition can be solved with fewer function evaluations per element than the original AAO formulation. It is difficult to compare these cases directly because the objective function of each element is different from the objective function of the AAO formulation. However, generally, each decomposed element will take less computational time per function evaluation than the AAO formulation, and decomposition allows additional possibilities of parallel computing. These results are encouraging because they show that in some situations the decomposed formulation can be solved in less time than the AAO formulation.

Conclusions

This article showed that it is important to set weights appropriately to achieve inconsistency deviations within an acceptable tolerance when top-level targets are unattainable. Setting appropriate weights is nontrivial, particularly for multilevel hierarchies where weights at various levels influence each other in complex ways. Setting weights too small can result in solutions far from the solution of the original problem, and setting weights too large can result in excessive computational cost and numerical problems. The weighting update method can automatically find weighting coefficients required for generating a solution with user-specified inconsistency tolerances. This method can help ATC users to achieve acceptable solutions without the burden of trial-and-error searching for appropriate weighting coefficients, which can be intractable for multilevel problems. Despite the added computation involved in iteratively updating the weights, the total computational cost can sometimes be lower than solving the problem directly with the desired weights or solving the problem AAO. Future work is needed to define and understand local convergence properties of coordination strategies for hierarchical partitioned systems and bring more rigor to solution efficiency definitions for decomposed optimization strategies.

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Nomenclature

$\|\cdot\|$ = vector norm
$\langle \cdot \rangle_\alpha$ = vector element α (where α indexes the vector elements, ranging from 1 to the length of the vector)
$\circ$ = element-by-element vector multiplication, for example $[a_1,a_2]^T \circ [b_1,b_2]^T = [a_1b_1,a_2b_2]^T$ and $\mathbf{a}\circ\mathbf{b} = \mathrm{diag}(\mathbf{ab}^T)$
c_{ij} = number of elements that are children of element j at level i
C_{ij} = set of elements that are children of element j at level i
E_i = set of elements at level i of the hierarchy
f_{ij} = objective function for element j at level i
$\mathbf{g}_{ij}$ = vector function of inequality constraints for element j at level i in negative null form
$\mathbf{h}_{ij}$ = vector function of equality constraints for element j at level i in null form
i = ATC hierarchy level index (starts at level 0)
j = ATC element index
k = ATC element index, used to designate children of element j
l = ATC element index designating the top level element
L_{ij} = the Lagrangian for the formulation of element j at level i
p = ATC element index, used to designate the parent of element j
P_{ij} = problem formulation of element j at level i
$\mathbf{r}_{ij}$ = vector function that calculates responses for element j at level i
$\mathbf{R}_{ij}^i$ = vector of response variable copies at level i for element j
$\mathbf{R}_{ij}^{i-1}$ = the $(i-1)$th level parent-copy of the vector of responses that function as targets for element j at level i
$\mathbf{S}_j$ = binary selection matrix for element j specifying which terms in the parent coordination vector are relevant to element j
$\mathbf{T}$ = vector of top level targets $(=\mathbf{R}_{0l}^{-1})$
$\overline{\mathbf{x}}_{ij}^i$ = aggregation vector for all input variables to the response function of element j at level i
$\mathbf{x}_{ij}^i$ = vector of local decision variables for element j at level i
$\mathbf{y}_{ij}^i$ = vector of linking variables for element j at level i
$\mathbf{y}_{(i+1)j}^i$ = vector of coordinating variables for the linking variables in the children of element j at level i. This vector includes one copy of each linking variable from all of element j's children
$\mathbf{w}_{ij}^{\mathrm{R}}$ = weighting coefficient vector for the deviation of responses between element j at level i and its parent
$\mathbf{w}_{(i+1)j}^{\mathrm{y}}$ = weighting coefficient vector for the deviation of linking variables coordinated at element j level i
α = index for terms in a vector
β = index for a specific term in a vector
$\varepsilon_{ij}^{\mathrm{R}}$ = tolerance variable for consistency of targets set at element j level i and the responses of j's children
$\varepsilon_{ij}^{\mathrm{y}}$ = tolerance variable for consistency of linking variables coordinated at element j level i for child elements at the $(i+1)$th level
γ = ATC element index, used to designate a specific child of element j
$\boldsymbol{\mu}_{ij}$ = vector of Lagrange multipliers for inequality constraints at element j level i
$\boldsymbol{\lambda}_{ij}$ = vector of Lagrange multipliers for equality constraints at element j level i
$\boldsymbol{\theta}_{ij}^{\mathrm{R}}$ = vector of user specified tolerances for inconsistency deviation between response variables of element j at level i and targets set by j's parent
$\boldsymbol{\theta}_{(i+1)j}^{\mathrm{y}}$ = vector of user specified tolerances for inconsistency deviation between linking variables at level $i+1$ that are coordinated at parent element j at level i

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