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Weights, Norms, and Notation in Analytical Target Cascading

Jeremy J. Michalek
e-mail: michalek@umich.edu

Panos Y. Papalambros

Department of Mechanical Engineering, University of Michigan, Ann Arbor, MI 48109-2125

This technical note provides clarification, modification, and generalization of the notation used to describe analytical target cascading, a model-based hierarchical optimization methodology for systems design. [DOI: 10.1115/1.1862674]

Introduction

Analytical target cascading (ATC) is a model-based, hierarchical optimization methodology for systems design. ATC requires a set of analysis or simulation models that predict responses (the characteristics) of each system, subsystem, and component as a function of the design variables (the decisions). The analysis models are organized using design optimization models that are the elements or building blocks of the hierarchy, as shown in Fig. 1 with the standard index notation. The top level represents the overall system and each lower level represents a subsystem or component of its parent element. In the ATC process, top-level system design targets are propagated down to lower subsystem and component level targets that are then optimized to meet the targets as closely as possible. The resulting responses are rebalanced at higher levels by iteratively adjusting targets and designs to achieve consistency.

Following the general notation for an elemental problem in the ATC hierarchy used by Michelena et al. [1], the problem P_{ij} for element j at level i is stated as

$$\underset{\bar{\mathbf{x}}_{ij}}{\text{minimize}}\ w_{ij}^{\mathrm{R}}\|\mathbf{R}_{ij}^{i}-\mathbf{R}_{ij}^{i-1}\| + w_{ij}^{\mathrm{y}}\|\mathbf{y}_{ij}^{i}-\mathbf{y}_{ij}^{i-1}\| + \varepsilon_{ij}^{\mathrm{R}} + \varepsilon_{ij}^{\mathrm{y}}$$

$$\text{subject to} \sum_{k\in C_{ij}} w_{(i+1)k}^{\mathrm{R}}\|\mathbf{R}_{(i+1)k}^{i}-\mathbf{R}_{(i+1)k}^{i+1}\| \leqslant \varepsilon_{ij}^{\mathrm{R}}$$

$$\sum_{k\in C_{ij}} w_{(i+1)k}^{\mathrm{y}}\|\mathbf{y}_{(i+1)k}^{i}-\mathbf{y}_{(i+1)k}^{i+1}\| \leqslant \varepsilon_{ij}^{\mathrm{y}}$$

$$\mathbf{g}_{ij}(\mathbf{R}_{ij}^{i},\mathbf{x}_{ij}^{i},\mathbf{y}_{ij}^{i}) \leqslant \mathbf{0}$$

$$\mathbf{h}_{ij}(\mathbf{R}_{ij}^{i},\mathbf{x}_{ij}^{i},\mathbf{y}_{ij}^{i}) = \mathbf{0}$$

$$\mathbf{R}_{ij}^{i} - \mathbf{r}_{ij}(\mathbf{R}_{(i+1)k_1}^{i},\ldots,\mathbf{R}_{(i+1)k_{c_{ij}}}^{i},\mathbf{x}_{ij}^{i},\mathbf{y}_{ij}^{i}) = \mathbf{0} \qquad (1)$$

where $\bar{\mathbf{x}}_{ij} \in [\mathbf{x}_{ij}^{i},\mathbf{y}_{ij}^{i},\mathbf{y}_{(i+1)k_1}^{i},\ldots,\mathbf{y}_{(i+1)k_{c_{ij}}}^{i},\mathbf{R}_{(i+1)k_1}^{i},\ldots,\mathbf{R}_{(i+1)k_{c_{ij}}}^{i},\varepsilon_{ij}^{\mathrm{R}},\varepsilon_{ij}^{\mathrm{y}}]^T$, $\mathbf{x}_{ij}^{i}$ is the vector of local variables for element j at level i, $\mathbf{y}_{ij}^{i}$ is the vector of linking variables for element j at level i, $\mathbf{y}_{ij}^{i-1}$ is the copy of the vector of linking variables at element j level i coordinated at level $(i-1)$, $\mathbf{R}_{ij}^{i}$ is the vector of responses at element j level i, $\mathbf{R}_{ij}^{i-1}$ is the vector of response targets for element j at level i that are set at level $(i-1)$, $\mathbf{r}_{ij}$ is the vector-valued response function of element j at level i, $\mathbf{g}_{ij}$ is the vector of inequality constraints at element j level i, $\mathbf{h}_{ij}$ is the vector of equality constraints at element j level i, $\varepsilon_{ij}^{\mathrm{R}}$ is the response deviation tolerance variable for element j level i, $\varepsilon_{ij}^{\mathrm{y}}$ is the linking deviation tolerance variable for element j level i, w_{ij}^{R} is the response deviation weighting coefficient for element j level i, and w_{ij}^{y} is the linking deviation weighting coefficient for element j level i.

ATC Notational Modifications

Several modifications to the modeling notation of Eq. (1) are needed for clarity and rigor in order to properly develop a mathematically rigorous update method for weighting coefficients, which is covered in a separate work [2]. First, it is implied that pure norms are used as deviation metrics, while in previous applications (for example, see Refs. [3–6]) the squares of the norms are used. Second, the response function $\mathbf{r}$ is written as an equality constraint, while it is used as an embedded definition. Third, local constraints $\mathbf{g}$ and $\mathbf{h}$ should be written as functions of targets for the child elements rather than the local responses. Finally, the linking variable notation may be modified to clarify the coordination of linking variables. These modifications update the notation in Ref. [1] and are consistent with implementations of actual problem solutions reported in the literature.

Deviation Metrics. The use of the "norm" symbol to represent target-response deviations has generated some confusion. In all applications of ATC, including Kim's dissertation [7], the square of the l_2 norm is used, which is not strictly a norm. The ATC convergence proof [1] does not require response deviation to be measured with vector norms, and the square of the l_2 norm is used in place of a norm in ATC applications. Direct use of the l_1, l_2, or l_∞ norm results in derivative discontinuities and numerical difficulties. Therefore, the expression $\|\mathbf{x}\|_2^2$ will be used henceforth to designate the square of the l_2 norm of the vector $\mathbf{x}$. Use of true norms in ATC is not recommended if gradient-based algorithms such as SQP are to be used. Instead, a more appropriate general requirement for the deviation function is that the function and its derivative be continuous over the domain and the function be monotonically increasing in all directions away from the point where target deviation is zero. The square of the l_2 norm satisfies

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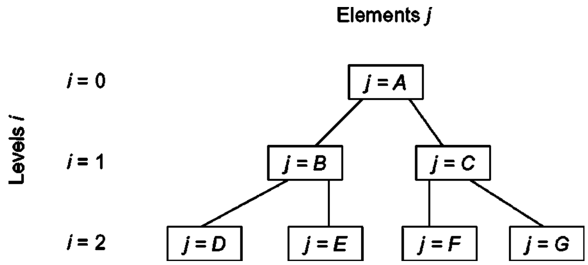

Fig. 1 Example of index notation for a hierarchically partitioned design problem

these criteria, and its Hessian is constant over the domain, so it is the recommended choice for measuring target deviations.

Response Function Variable. In the ATC convergence proof [1] the symbol $\mathbf{R}_{ij}^{i}$ is included in the constraint $\mathbf{R}_{ij}^{i}-\mathbf{r}_{ij}(\overline{\mathbf{x}}_{ij})=\mathbf{0}$, but it is not identified as a decision variable. In Kim's dissertation [7], this relationship is represented as an embedded substitution, using the statement "where $\mathbf{R}_{ij}^{i}=\mathbf{r}_{ij}(\overline{\mathbf{x}}_{ij})$" rather than "subject to $\mathbf{R}_{ij}^{i}-\mathbf{r}_{ij}(\overline{\mathbf{x}}_{ij})=\mathbf{0}$." To be rigorous, either the substitution statement "where" should be used, or $\mathbf{R}_{ij}^{i}$ should be included as a decision variable. We use the substitution statement for the remainder of this note.

Constraint Functions. The constraint functions for problem j at level i are written incorrectly as a function of the response $\mathbf{R}_{ij}^{i}$, whereas they should be written instead as a function of the targets set for the children $k_1 \ldots k_{c_{ij}}$, so that

$$
\begin{aligned}
&\mathbf{g}_{ij}(\mathbf{x}_{ij}^{i},\mathbf{y}_{ij}^{i},\mathbf{R}_{(i+1)k_1}^{i},\ldots,\mathbf{R}_{(i+1)k_{c_{ij}}}^{i})\leqslant\mathbf{0}\\
&\mathbf{h}_{ij}(\mathbf{x}_{ij}^{i},\mathbf{y}_{ij}^{i},\mathbf{R}_{(i+1)k_1}^{i},\ldots,\mathbf{R}_{(i+1)k_{c_{ij}}}^{i})=\mathbf{0}
\end{aligned}
\qquad (2)
$$

Constraints are placed on the *input* of $\mathbf{r}_{ij}$, not the output. This is consistent with the way ATC has been implemented in practice (for example, in Refs. [3–6]), but the notation has deviated.

Linking Variables. The current notation defines linking variable copies at upper $\mathbf{y}_{(i+1)k}^{i}$ and lower $\mathbf{y}_{(i+1)k}^{i+1}$ levels for each child, and constrains deviation between the upper and lower level copies using the ε terms. However, this notation dictates that a separate copy is created for each linking variable vector of each child, and deviations among elements of the parent copies are not explicitly constrained. For example, if child B at level 1 has linking variables $\mathbf{y}_{1B}^{1}$, and child C at level 1 has linking variables $\mathbf{y}_{1C}^{1}$, then notation dictates that the parent A at level 0 has linking variable copies $\mathbf{y}_{1B}^{0}$ and $\mathbf{y}_{1C}^{0}$ with no explicit specification stating that $\mathbf{y}_{1B}^{0}=\mathbf{y}_{1C}^{0}$. This requirement is implied in the definition of C_{ij}, and case studies in the literature have combined copies at the parent level into a single vector (for example, see Ref. [7], p. 80) rather than creating multiple copies at the parent level. However, for clarity, it is important that the notation reflect the linking variable intent unambiguously and be flexible enough to accommodate multiple children with interspersed linking variables, some of which may be shared by some children and not others.

One effective way to denote linking variables in the general case is to use a single coordination vector $\mathbf{y}_{(i+1)j}^{i}$ at parent element j (level i) that aggregates copies of the linking variables of all of j's children (at level $i+1$) such that it contains exactly one copy of each variable. Each child k in C_{ij} uses a binary selection matrix $\mathbf{S}_k$ to define which linking variables in the parent coordination vector $\mathbf{y}_{(i+1)j}^{i}$ are elements of $\mathbf{y}_{(i+1)k}^{i+1}$ in child k. The selection matrix is defined so that the number of columns in $\mathbf{S}_k$ equals the number of terms in $\mathbf{y}_{(i+1)j}^{i}$ and the number of rows in $\mathbf{S}_k$ equals the number of terms in $\mathbf{y}_{(i+1)k}^{i+1}$, which is less than or equal to the number of terms in $\mathbf{y}_{(i+1)j}^{i}$. Each element of $\mathbf{S}_k$ is either 1 or 0, where each row of $\mathbf{S}_k$ sums to 1, such that the product $\mathbf{S}_k\mathbf{y}_{i(i+1)j}^{i}$ extracts the elements of $\mathbf{y}_{(i+1)j}^{i}$ corresponding to $\mathbf{y}_{(i+1)k}^{i+1}$.

To illustrate this concept, suppose an ATC hierarchy consists of parent element A at level 0 and child elements B, C, and D at level 1. Suppose linking variable y_1 is shared between elements B and C, y_2 is shared between elements C and D, and y_3 is shared among elements B, C, and D. In this case, the parent coordination vector is $\mathbf{y}_{1A}^{0}=[y_1,y_2,y_3]^T$, element B has $\mathbf{y}_{1B}^{1}=[y_1,y_3]^T$ and $\mathbf{S}_B=[1\ 0\ 0;0\ 0\ 1]$, element C has $\mathbf{y}_{1C}^{1}=[y_1,y_2,y_3]^T$ and $\mathbf{S}_C=[1\ 0\ 0;0\ 1\ 0;0\ 0\ 1]$, and element D has $\mathbf{y}_{1D}^{1}=[y_2,y_3]^T$ and $\mathbf{S}_D=[0\ 1\ 0;0\ 0\ 1]$, where the semicolon is used to represent a new matrix row. Using this notation, the linking variable constraints can be written as

$$
\sum_{k\in C_{ij}} w_{(i+1)j}^{\mathrm{y}}\|\mathbf{S}_k\mathbf{y}_{(i+1)j}^{i}-\mathbf{y}_{(i+1)k}^{(i+1)}\|_2^2\leqslant\varepsilon_{ij}^{\mathrm{y}}
\qquad (3)
$$

This more general notation may appear to add complexity, but it formalizes the way linking variables have been used in ATC case studies in practice.

Weighting Coefficients. The weighting coefficient scheme used in Eq. (1) can be generalized by providing a weighting coefficient for each term of each vector rather than having only one weighting coefficient for each vector. Individual weighting coefficients for each term allow the designer to express relative preferences for meeting each target, and the generalization is important for using the weighting update method to achieve acceptable levels of inconsistency between elements when top level targets are unattainable [2]. This weighting coefficient scheme can be written as

$$
\|\mathbf{w}_{ij}^{\mathrm{R}}\circ(\mathbf{R}_{ij}^{i}-\mathbf{R}_{ij}^{i-1})\|_2^2
\qquad (4)
$$

where $\mathbf{w}_{ij}^{\mathrm{R}}$ is a vector of weighting coefficients and the $\circ$ symbol is used to denote term by term multiplication of vectors, so that each element of the weight vector is multiplied by its corresponding element in the deviation vector resulting in a weighted vector (i.e., $[a_1a_2\ldots a_n]^T\circ[b_1b_2\ldots b_n]^T=[ab_1ab_2\ldots ab_n]^T$). Note that the weighting coefficient scalars in Eq. (1) are special cases of the weighting coefficient vectors in Eq. (4), where all terms in each vector are equal (accounting for the square).

Revised ATC Problem Statements

Using the notational modifications earlier, the relaxed non-decomposed problem from Ref. [1] is written as

$$
\begin{aligned}
&\underset{\overline{\mathbf{x}}_{ij},\mathbf{y}_{(i+1)j}^{i},\varepsilon_{ij}^{\mathrm{R}},\varepsilon_{ij}^{\mathrm{y}}}{\text{minimize}}\ \|\mathbf{R}_{0l}^{0}-\mathbf{T}\|_2^2+\sum_{i=0}^{N-1}\sum_{j\in\mathcal{E}_i}\varepsilon_{ij}^{\mathrm{R}}+\sum_{i=0}^{N-1}\sum_{j\in\mathcal{E}_i}\varepsilon_{ij}^{\mathrm{y}}\\
&\text{subject to}\ \sum_{k\in C_{ij}}\|\mathbf{w}_{(i+1)k}^{\mathrm{R}}\circ(\mathbf{R}_{(i+1)k}^{i}-\mathbf{R}_{(i+1)k}^{i+1})\|_2^2\leqslant\varepsilon_{ij}^{\mathrm{R}}\\
&\sum_{k\in C_{ij}}\|\mathbf{S}_k\mathbf{w}_{(i+1)j}^{\mathrm{y}}\circ(\mathbf{S}_k\mathbf{y}_{(i+1)j}^{i}-\mathbf{y}_{(i+1)k}^{(i+1)})\|_2^2\leqslant\varepsilon_{ij}^{\mathrm{y}}\\
&\mathbf{g}_{ij}(\overline{\mathbf{x}}_{ij})\leqslant\mathbf{0}\\
&\mathbf{h}_{ij}(\overline{\mathbf{x}}_{ij})=\mathbf{0}\\
&\text{where}\ \mathbf{R}_{ij}^{i}=\mathbf{r}_{ij}(\overline{\mathbf{x}}_{ij})\\
&\overline{\mathbf{x}}_{ij}=[\mathbf{x}_{ij}^{i},\mathbf{y}_{ij}^{i},\mathbf{R}_{(i+1)k_1}^{i},\ldots,\mathbf{R}_{(i+1)k_{c_{ij}}}^{i}]^T\\
&\forall j\in\mathcal{E}_i, i=0,1,\ldots,N
\end{aligned}
\qquad (5)
$$

where $\mathcal{E}_i$ is the set of elements at level i. Note that the ε terms drop out for elements that do not have children. The revised statement for P_{ij}, the problem element j at level i, is then

$$\underset{\overline{\mathbf{x}}_{ij},\mathbf{y}^i_{(i+1)j},\varepsilon^{\mathrm{R}}_{ij},\varepsilon^{\mathrm{y}}_{ij}}{\text{minimize}} \|\mathbf{w}^{\mathrm{R}}_{ij} \circ (\mathbf{R}^i_{ij} - \mathbf{R}^{i-1}_{ij})\|^2_2 + \|\mathbf{S}_j \mathbf{w}^{\mathrm{y}}_{ip} \circ (\mathbf{S}_j \mathbf{y}^{i-1}_{ip} - \mathbf{y}^i_{ij})\|^2_2 + \varepsilon^{\mathrm{R}}_{ij} + \varepsilon^{\mathrm{y}}_{ij}$$

$$\text{subject to} \sum_{k \in C_{ij}} \|\mathbf{w}^{\mathrm{R}}_{(i+1)k} \circ (\mathbf{R}^i_{(i+1)k} - \mathbf{R}^{i+1}_{(i+1)k})\|^2_2 \leqslant \varepsilon^{\mathrm{R}}_{ij}$$

$$\sum_{k \in C_{ij}} \|\mathbf{S}_k \mathbf{w}^{\mathrm{y}}_{(i+1)j} \circ (\mathbf{S}_k \mathbf{y}^i_{(i+1)j} - \mathbf{y}^{(i+1)}_{(i+1)k})\|^2_2 \leqslant \varepsilon^{\mathrm{y}}_{ij}$$

$$\mathbf{g}_{ij}(\overline{\mathbf{x}}_{ij}) \leqslant \mathbf{0}$$

$$\mathbf{h}_{ij}(\overline{\mathbf{x}}_{ij}) = \mathbf{0}$$

$$\text{where } \mathbf{R}^i_{ij} = \mathbf{r}_{ij}(\overline{\mathbf{x}}_{ij})$$

$$\overline{\mathbf{x}}_{ij} = [\mathbf{x}^i_{ij}, \mathbf{y}^i_{ij}, \mathbf{R}^i_{(i+1)k_1}, \ldots, \mathbf{R}^i_{(i+1)k_{c_{ij}}}]^T \tag{6}$$

where p is the parent of element P_{ij}. Note again that the epsilon terms drop out for elements that do not have children, and the linking variable terms in the objective function drop out for elements that do not have linking variables. Using monotonicity analysis [8], the constraints containing epsilon terms can be shown to be active (epsilon terms are monotonic in the objective function and each is constrained only by its epsilon constraint), so problem element j at level i can be alternatively written as

$$\underset{\overline{\mathbf{x}}_{ij},\mathbf{y}^i_{(i+1)j}}{\text{minimize}} \|\mathbf{w}^{\mathrm{R}}_{ij} \circ (\mathbf{R}^i_{ij} - \mathbf{R}^{i-1}_{ij})\|^2_2 + \|\mathbf{S}_j \mathbf{w}^{\mathrm{y}}_{ip} \circ (\mathbf{S}_j \mathbf{y}^{i-1}_{ip} - \mathbf{y}^i_{ij})\|^2_2$$

$$+ \sum_{k \in C_{ij}} \|\mathbf{w}^{\mathrm{R}}_{(i+1)k} \circ (\mathbf{R}^i_{(i+1)k} - \mathbf{R}^{i+1}_{(i+1)k})\|^2_2$$

$$+ \sum_{k \in C_{ij}} \|\mathbf{S}_k \mathbf{w}^{\mathrm{y}}_{(i+1)j} \circ (\mathbf{S}_k \mathbf{y}^i_{(i+1)j} - \mathbf{y}^{(i+1)}_{(i+1)k})\|^2_2$$

$$\text{subject to } \mathbf{g}_{ij}(\overline{\mathbf{x}}_{ij}) \leqslant \mathbf{0}$$

$$\mathbf{h}_{ij}(\overline{\mathbf{x}}_{ij}) = \mathbf{0}$$

$$\text{where } \mathbf{R}^i_{ij} = \mathbf{r}_{ij}(\overline{\mathbf{x}}_{ij})$$

$$\overline{\mathbf{x}}_{ij} = [\mathbf{x}^i_{ij}, \mathbf{y}^i_{ij}, \mathbf{R}^i_{(i+1)k_1}, \ldots, \mathbf{R}^i_{(i+1)k_{c_{ij}}}]^T \tag{7}$$

Both of these forms are equivalent; however, Eq. (7) has fewer independent variables and may exhibit different numerical properties depending on the algorithm used to solve P_{ij}. Either formulation may be used to solve individual elements in the ATC hierarchy.

Conclusions

The modeling and notational enhancements presented here with respect to norms, response functions, constraint functions, and linking variables, should supersede previous notation in the literature. The generalization of weighting coefficients from scalars to vectors, so that a separate weighting coefficient is assigned to each response variable and each linking variable, offers additional flexibility and is important for the weighting update method, presented in a separate article [2]. The ATC problem is complex by nature: Rigor and clarification in notation and modeling reduce confusion and enhance usability.

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