SUAMI 2016 - Turán numbers of vertex-disjoint cliques in r-partite graphs

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Abstract

The Turán number of a pair of graphs G and H is denoted $ex(G, H)$, and is the maximum number of edges a subgraph of G may have and still contain no copy of H. In this paper, we determine $ex(K_{a_1,a_2,...,a_r}, mK_r)$, where $K_{a_1,a_2,...,a_r}$ denotes a complete r-partite graph with part sizes $a_1, ..., a_r$ and mK_r denotes m vertex-disjoint copies of K_r, the complete graph on r vertices. We prove that for any integers $1 \leq m \leq a_1 \leq a_2 \leq ... \leq a_r$ we have

$$ex(K_{a_1,a_2,...,a_r}, mK_r) = \sum_{1 \leq i < j \leq r} a_i a_j - a_1 a_2 + a_2(m-1).$$

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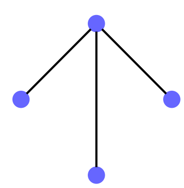
1. Definitions

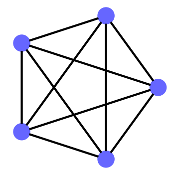
Definition 1. *A **graph** G is a pair of sets $G = (V, E)$, where V is a fixed set of vertices, and the edge set E is a set of pairs of distinct elements from V. We often write V as $V(G)$ and E as $E(G)$.*



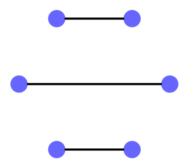
An example of a graph. The nodes represent vertices and the lines represent edges.

Definition 2. *Let G be a graph. A **subgraph** H of G is a pair of sets $H = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$, which is itself a graph. If H is a subgraph of G, we write $H \subseteq G$.*

Definition 3. *A graph $G = (V, E)$ is called **complete** if for every pair $x \neq y$ in V we have $xy \in E$. If $|E(G)| = n$, this graph is denoted K_n.*

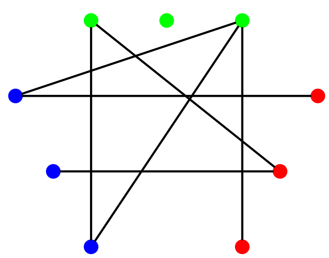


Definition 4. *A graph $G = (V, E)$ is called a **matching** if every vertex is incident to exactly one edge. That is, for any $v \in V$, there is exactly one vertex $w \neq v$ such that $vw \in E$. If every vertex in G is incident to an edge, then the graph is a **perfect matching**.*



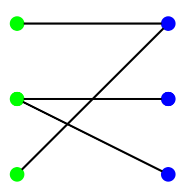
A perfect matching

Definition 5. *A graph G is called **r-partite** if one can partition the vertex set into r parts $V(G) = V_1 \cup V_2 \cup ... \cup V_r$ such that if x and y are both in the same V_i, then $xy \notin E(G)$.*



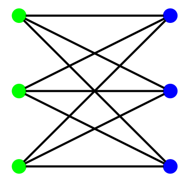
A 3-partite (commonly called tripartite) graph

Definition 6. *A graph G is called **bipartite** if it is 2-partite.*



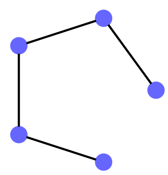
A bipartite graph

Definition 7. *A graph G which is bipartite or r-partite is called* ***complete bipartite*** *or* ***complete r-partite*** *if every pair of vertices among different vertex sets in the partition (commonly called parts) are adjacent (connected by an edge). If $|V_i| = m_i$, then this graph is denoted $K_{m_1,m_2,...,m_r}$.*

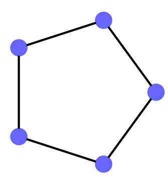


A complete bipartite graph

Definition 8. *A graph G is called a* ***path*** *if the graph is an alternating sequence of vertices and edges $G = v_1e_1v_2e_2...v_{n-1}e_{n-1}v_n$, where $e_i = v_iv_{i+1}$, and if $i \neq j$ then $v_i \neq v_j$*



Definition 9. *A graph G is called a* ***cycle*** *if the graph is an alternating sequence of vertices and edges $G = v_1e_1v_2e_2...v_{n-1}e_{n-1}v_ne_nv_{n+1}$, where $e_i = v_iv_{i+1}$, and if $i < j \leq n$ then $v_i \neq v_j$, and $v_{n+1} = v_1$.*



Definition 10. *Let G be a graph. The* ***neighborhood*** *of a particular vertex $v \in G$ is the set of all vertices adjacent to v. The set of neighbors of a vertex v is denoted $N(v)$. If $A \subseteq V(G)$, then $N(A) := \bigcup_{v \in A} N(v)$.*

Definition 11. *The* ***extremal (Turán) number*** *of a pair of graphs S and G with $S \subseteq G$ is the maximum number of edges which a subgraph of G may have and still contain no copy of S. This number is denoted $ex(G, H)$.*

Definition 12. *The* ***rainbow number*** *of a pair of graphs S and G with $S \subseteq G$ is defined as the minimum number of colors so that any coloring of the edges of G will contain a rainbow copy of S.*

2. Basic Results

Theorem 1. ***Hall's Theorem*** *Let G be a bipartite graph, with vertex set partitioned as $V(G) = A \cup B$. Then there exists a matching with every vertex in A incident to an edge if and only if for any $V \subseteq A$ we have $|N(V)| \geq |V|$.*

Theorem 2. *Let $H \subseteq G$ be graphs. Let*

$$\mathcal{H} := \{h \subseteq G : h = H - e \text{ for some edge } e \in E(H)\}$$

and define

$$\mathrm{ex}(G, \mathcal{H}) := \max\{n \in \mathbb{Z} : \exists L \subseteq G \textit{ such that } |E(L)| = n \textit{ and } h \not\subseteq L \textit{ for any } h \in \mathcal{H}\}.$$

Then we have the following:

$$\mathrm{ex}(G, \mathcal{H}) + 2 \leq \mathrm{rb}(G, H) \leq \mathrm{ex}(G, H) + 1 \tag{1}$$

Proof. We first prove the inequality $rb(G, H) \leq ex(G, H) + 1$. Let $m = ex(G, H)$, and color G with $m + 1$ colors. Construct $F \subseteq G$ by picking one edge from each color class. Since each color is used, then $|E(F)| = m + 1$, and by the definition of F, F is rainbow. Since $m + 1 > ex(G, H)$, that means $H \subseteq F$, and since F is rainbow in this coloring, so is this copy of H. Since the coloring was arbitrary, that means every coloring of G with $m + 1$ colors has a rainbow H, so $rb(G, H) \leq m + 1$.

Now we prove $ex(G, \mathcal{H}) + 1 < rb(G, H)$. Let $ex(G, \mathcal{H}) = n$, and let $L \subseteq G$ such that $|E(L)| = n$ and $h \not\subseteq L$ for any $h \in \mathcal{H}$. We begin to color G by coloring every edge in $E(G) \setminus E(L)$ with color 1, which is never used again. Note this set of edges is clearly nonempty, as if not, then $L = G$ and $H \not\subseteq G$, so color 1 is used. We then color every edge in L uniquely. Clearly this is a coloring of G with $n + 1$ colors, and we claim that this coloring yields no rainbow H. Any rainbow subgraph of G must have at most one edge in $E(G) \setminus E(L)$. Also, any copy of H in G must contain at least one edge in $E(G) \setminus E(L)$, as $H \not\subseteq L$. Thus, to find a rainbow copy of $H \subseteq G$ we must select exactly one edge in $E(G) \setminus E(L)$, and the rest of the edges from $E(L)$. That is to say, we must select a subgraph of the form $J = F \cup e$ where $F \subseteq L$ and $e \in E(H)$. But since for any edge $e \in E(H)$ we have $H \setminus e \not\subseteq L$, we have $F \cup e \not\subseteq H$. Thus, no rainbow H exists, and the inequality is proven.

□

Theorem 3. *Chen, Li, Tu (2009)*

$$ex(K_{a_1,a_2}, mK_2) = a_2(m - 1)$$

3. Results

Theorem 4. *For all integers $m \geq 1, z \geq 0, r \geq 2$ we have*

$$ex(K_{\underbrace{m+z, m+z, ..., m+z}_{r\ times}}, mK_r) = \binom{r}{2}(m+z)^2 - (m+z)(z+1)$$

For notational simplicity we let $K_{(m+z)_r} := K_{\underbrace{m+z, m+z, ..., m+z}_{r \text{ times}}}$, and we let $ex(m, z, r) := ex(K_{(m+z)_r}, mK_r)$. We proceed by proving a series of lemmas.

Lemma 1. *For all $m \geq 1, z \geq 0, r \geq 2$ we have $ex(m, z, r) \geq \binom{r}{2}(m+z)^2 - (m+z)(z+1)$.*

Proof. We construct a graph $G \subseteq K_{(m+z)_r}$ satisfying

1. $|E(G)| = \binom{r}{2}(m+z)^2 - (z+1)(m+z)$
2. $mK_r \nsubseteq G$

We let $K_{(m+z)_r}$ be r-paritioned as $A_1 \cup A_2 \cup ... \cup A_r$. We consider a single part A_1, and a fixed set of $z+1$ vertices in that part. We connect these vertices to no vertex in part A_2, and include every other edge from the host graph. We claim this graph G satisfies conditions 1 and 2. In order to form a single K_r, we must select at most one vertex from each part, as there are no edges within a part, and since we must select r vertices, we must select exactly one from each part, and all the edges between these r vertices must be present. Thus, in order to form mK_r, we would need to select exactly m vertices from each part, and they would all need to be part of some $K_r \subseteq G$. However, there are only $m + z - (z+1) = m-1$ vertices in A_1 connected to any vertex in A_2, meaning that if we select m vertices from A_1, then at least one of them has no edge to A_2, and so can be involved in no K_r.

□

We now prove base cases for an inductive argument in the following lemmas:

Lemma 2. *Our theorem holds for $z = 0$. That is, for $m \geq 1, z = 0$, we have the following:*

$$ex(m, 0, r) = \binom{r}{2}m^2 - m$$

Proof. By Lemma 1 we know that $ex(m,0,r) \geq \binom{r}{2}m^2 - m$, so we show $ex(m,0,r) < \binom{r}{2}m^2 - m + 1$. We note that this is clearly true for $m = 1$, as any graph $G \subseteq K_{(1)_r} = K_r$ with $|E(G)| = \binom{r}{2}(1)^2 - 1 + 1 = \binom{r}{2}$ is exactly K_r, as desired.

Now we proceed inductively on m. We first let $G \subseteq K_{(m)_r}$ satisfying $|E(G)| = ex(m,0,r) + 1 = \binom{r}{2}m^2 - m + 1$. Note that

$$\begin{aligned}|E(G)| - (ex(m-1,0,r)+1) &= \binom{r}{2}m^2 - m + 1 - \left(\binom{r}{2}(m-1)^2 - (m-1) + 1\right)\\ &= \binom{r}{2}(m^2 - (m-1)^2) - 1\\ &= \binom{r}{2}(2m-1) - 1\end{aligned}$$

Thus, if we find a $K_r \subset G$ such that removing K_r removes no more than $\binom{r}{2}(2m-1) - 1$ edges, then this will guarantee that $(m-1)K_r \subseteq G \setminus K_r$, so $mK_r \subseteq G$. Now we show that $K_r \subseteq G$. We define an r-tuple as $a_1a_2...a_r$, with $a_i \in A_i$. We define the weight of an r-tuple $W(a_1a_2...a_r)$ to be the number of edges in G incident to two of the vertices in the r-tuple. We use this definition to show $K_r \subset G$. We suppose to the contrary that $K_r \nsubseteq G$, meaning that

$$\sum_{a_1a_2...a_r} W(a_1a_2...a_r) \leq m^r \left(\binom{r}{2} - 1\right)$$

where the sum is taken over all possible r-tuples $a_1...a_r$. Now we relate $|E(G)|$ to $\sum\limits_{a_1...a_r} W(a_1...a_r)$. Summing over all r-tuples means that each edge is counted m^{r-2} times, so that

$$|E(G)|m^{r-2} = \sum_{a_1...a_r} W(a_1...a_r) \leq m^r \left(\binom{r}{2} - 1\right)$$

That is,

$$|E(G)|m^{r-2} \leq m^r \left(\binom{r}{2} - 1\right) \iff |E(G)| \leq m^2 \left(\binom{r}{2} - 1\right)$$

And since $m \geq 2$, then $m^2\left(\binom{r}{2} - 1\right) < \binom{r}{2}m^2 - m + 1 = |E(G)|$, a contradiction. Thus $K_r \subseteq G$. We recall that if the number of edges incident to the r vertices in this copy of K_r is at most $\binom{r}{2}(2m-1)-1$, then we have $(m-1)K_r \subseteq G \setminus K_r$ which implies $mK_r \subseteq G$. This motivates a definition - let the degree of an r-tuple, $d(a_1...a_r)$ be the number of edges incident to any vertex in the r-tuple. We suppose to the contrary that if $a_1...a_r$ forms a copy of $K_r \subseteq G$, then $d(a_1...a_r) \geq \binom{r}{2}(2m-1)$.

We investigate the maximum value that $d(a_1...a_r)$ can attain. For any vertex a_i in the r-tuple, it may be connected to at most m vertices in each of the other $r-1$ parts. Thus, we have $d(a_1...a_r) \leq r(r-1)m - \binom{r}{2}$, over-counting then subtracting the edges counted twice. We note that $r(r-1)m - \binom{r}{2} = \binom{r}{2}(2m-1)$. Thus, our assumption to the contrary means that if $a_1...a_r$ forms a copy of K_r in G, then $d(a_1,...,a_r) = \binom{r}{2}(2m-1)$, that is, $N(a_k) = \bigcup_{j=1...r:\, j\neq k} A_j$ for $k=1,...,r$. We now show that this implies $G = K_{(m)_r}$. We let $a_{i,j}$ be the i^{th} vertex in the j^{th} part, so $1 \leq i \leq m$, and $1 \leq j \leq r$. We let $a_{1,1}, a_{1,2}, ...a_{1,r}$ be a copy of K_r in G (which we have already shown exists). We consider the r-tuples formed by varying a single vertex in a single part. That is, for $i=1,...,m$ and $k=1,...,r$, we consider the r-tuples $a_{1,1}, a_{1,2}, ..., a_{i,k}, ..., a_{1,r}$. By our assumption that $N(a_{1,k}) = \bigcup_{j=1...r:\, j\neq k} A_j$ for $k=1,...,r$, we have that all these r-tuples form copies of K_r in G. Since for any vertex $a_{i,k}$ involved in a copy of K_r in G satisfies $N(a_{i,k}) = \bigcup_{j=1...r:\, j\neq k} A_j$, and every vertex is involved in some K_r in G, then $G = K_{(m)_r}$, a contradiction.

□

Lemma 3. *Our theorem holds for $m=1, z\geq 0, r\geq 2$. That is, we have:*

$$ex(1,z,r) = \binom{r}{2}(1+z)^2 - (1+z)^2 = \left(\binom{r}{2}-1\right)(z+1)^2$$

Proof. We already have by Lemma 1 that $ex(1,z,r) \geq \left(\binom{r}{2}-1\right)(z+1)^2$, so now we show $ex(1,z) < \left(\binom{r}{2}-1\right)(z+1)^2+1$. Let $G \subseteq K_{(1+z)_r}$ with $|E(G)| = \left(\binom{r}{2}-1\right)(z+1)^2+1$. If $K_r \not\subseteq G$, then we have (using the same definition for weight as before), that $W(a_1...a_r) \leq \binom{r}{2}-1$ for all r-tuples.

Then we have (by the same argument as in Lemma 2) that

$$\begin{aligned}|E(G)|(1+z)^{r-2} &= \sum_{a_1...a_r} W(a_1...a_r)\\ &\leq \left(\binom{r}{2}-1\right)(1+z)^r\\ &\implies |E(G)| \leq \left(\binom{r}{2}-1\right)(1+z)^2\end{aligned}$$

contradicting $|E(G)| = (\binom{r}{2}-1)(1+z)^2+1$.

□

We now use this information to prove our main theorem.

Proof. We proceed by inducting on m and z, with $m \geq 2, z \geq 1$ and $r \geq 2$.

1. $ex(m-1,z,r) = \binom{r}{2}(m-1+z)^2 - (m-1+z)(z+1)$
2. $ex(m,z-1,r) = \binom{r}{2}(m+z-1)^2 - (m+z-1)(z)$

as the base case $z=0$ was proven in Lemma 2, and the base case $m=1$ was proven in Lemma 3. Let $G \subseteq K_{(m+z)_r}$ with $|E(G)| = \binom{r}{2}(m+z)^2 - (m+z)(z+1)+1$ and suppose to the contrary that $mK_r \not\subseteq G$. If the r-tuple forms a K_r in G, then we know $|E(G \setminus (a_1...a_r))| \leq ex(m-1,z,r)$ because if $|E(G \setminus (a_1...a_r))| > ex(m-1,z,r)$, then $(m-1)K_r \subseteq (G \setminus (a_1...a_r))$, and adjoining $a_1...a_r$ with this $(m-1)K_r$ yields $mK_r \subseteq G$, a contradiction. If $a_1...a_r$ does not form a copy of K_r in G, then we know $|E(G \setminus (a_1...a_r))| \leq ex(m,z-1,r)$, because if $|E(G \setminus (a_1...a_r))| > ex(m,z-1)$, then $mK_r \subseteq G \setminus (a_1...a_r)) \subseteq G$, a contradiction.

We would like to use this information to construct an upper bound on $|E(G)|$, which contradictions the known number of edges in G. To aid in this, we will first show a lower bound on the number of (not necessarily disjoint) copies of K_r within G.

Suppose there are exactly q copies of K_r in G. Then we should have, summing over all r-tuples, (using the same definition for weight as before)

$$\begin{aligned}\sum_{a_1...a_r} W(a_1...a_r) &\leq \binom{r}{2}q + ((m+z)^r - q)\left(\binom{r}{2}-1\right)\\ &= q + \left(\binom{r}{2}-1\right)(m+z)^r\end{aligned}$$

Thus, by the same argument as in the lemmas, we have

$$|E(G)|(m+z)^{r-2} = \sum_{a_1...a_r} W(a_1...a_r) \leq q + \left(\binom{r}{2} - 1\right)(m+z)^r$$

So $q \geq |E(G)|(m+z)^{r-2} - \left(\binom{r}{2} - 1\right)(m+z)^r$, and we let $p := |E(G)|(m+z)^{r-2} - \left(\binom{r}{2} - 1\right)(m+z)^r$, so that $q \geq p$.

Now we bound $|E(G)|$ from above. We consider the following sum over all r-tuples:

$$\begin{aligned}\sum_{a_1...a_r} |E(G \setminus (a_1...a_r))| &\leq q(ex(m-1,z,r)) + ((m+z)^r - q)(ex(m,z-1,r)) \\ &\leq p(ex(m-1,z,r)) + ((m+z)^r - p)(ex(m,z-1,r))\end{aligned}$$

The first inequality is justified by the paragraph immediately following the proof of Lemma 3. The second inequality is justified by the fact that $ex(m,z-1,r) - ex(m-1,z,r) = m+z-1 \geq 0$, so if G had more copies of K_r than our lower bound, then the upper bound for $\sum_{a_1...a_r} |E(G \setminus (a_1...a_r))|$ could only get smaller.

We now relate this sum to $|E(G)|$. For any edge $a_{i,j}a_{k,l} \in E(G)$, we note that we only count $a_{i,j}a_{k,l}$ in $\sum_{a_1..a_r} |E(G \setminus (a_1...a_r))|$ if we select neither $a_{i,j}$ nor $a_{k,l}$ to remove, and select any vertex in each of the remaining $r-1$ part to remove. That is, we count $a_{i,j}a_{k,l}$ exactly $(m+z-1)^2(m+z)^{r-2}$ times. Thus, we have

$$\begin{aligned}|E(G)|(m+z-1)^2(m+z)^{r-2} &= \sum_{a_1...a_r} |E(G \setminus (a_1...a_r))| \\ &\leq p(ex(m-1,z,r)) + ((m+z)^r - p)(ex(m,z-1,r))\end{aligned}$$

Thus, we should have that

$$|E(G)|(m+z-1)^2(m+z)^{r-2} - p(ex(m-1,z,r)) + ((m+z)^r - p)(ex(m,z-1,r)) \leq 0$$

Let $s = m+z$. Substituting $m = s-z$ into this inequality and simplifying yields $s^{r-1}(s-1) \leq 0$. Since $m \geq 2, z \geq 1$, and $r \geq 2$, we have $s \geq 3$, and clearly $s^{r-1}(s-1) > 0$, a contradiction. Thus, $mK_r \subseteq G$, and since G was arbitrary, $ex(m,z,r) < \binom{r}{2}(m+z)^2 - (m+z)(z+1) + 1$. Since Lemma 1 shows $ex(m,z,r) \geq \binom{r}{2}(m+z)^2 - (m+z)(z+1)$, we have

$$ex(K_{\underbrace{m+z,m+z,...,m+z}_{r\text{ times}}}, mK_r) = \binom{r}{2}(m+z)^2 - (m+z)(z+1)$$

□

Theorem 5. *For any integers $1 \leq m \leq a_1 \leq a_2 \leq ... \leq a_r$ we have*

$$ex(K_{a_1,a_2,...,a_r}, mK_r) = \sum_{1\leq i<j\leq r} a_ia_j - a_1a_2 + a_2(m-1)$$

We let $K_{a_1,a_2,...,a_r}$ be r-partitioned as $A_1 \cup A_2 \cup ... \cup A_r$. For notational simplicity we define $ex(a_1,a_2,...,a_r,m) = ex(K_{a_1,a_2,...,a_r}, mK_r)$.

Lemma 4. *For any integers $1 \leq m \leq a_1 \leq a_2 \leq ... \leq a_r$ we have*

$$ex(K_{a_1,a_2,...,a_r}, mK_r) \geq \sum_{1\leq i<j\leq r} a_ia_j - a_1a_2 + a_2(m-1)$$

Proof. We construct a graph $F \subseteq K_{a_1,...,a_r}$ with $|E(F)| = \sum\limits_{1\leq i<j\leq r} a_ia_j - a_1a_2 + a_2(m-1)$ and $mK_r \nsubseteq F$. We consider the part A_1 and a fixed set of $a-(m-1)$ in that part. We connect these vertices to no vertex in part A_2 and include every other edge from the host graph. In order to form a single K_r, we must select at most one vertex from each part, and all edges between these r vertices must be present. Thus, to form mK_r, we would need exactly m vertices from each part. However, we there are only $a-(a-(m+1)) = m-1$ vertices in A_1 connected to any vertex in A_2, and so can be involved in no K_r. To count the number of edges, we note that between A_1 and A_2 there are $a_2(m-1)$ edges, and all other pairs have all edges present, so $|E(F)| = \sum\limits_{1\leq i<j\leq r} a_ia_j - a_1a_2 + a_2(m-1)$.

□

Proposition 1. *For $1 \leq m \leq a_1 \leq a_2 = a_3 = ... = a_r$ we have*

$$ex(K_{a_1,a_2...a_2}, mK_r) = a_1a_2(r-2) + a_2^2\binom{r-1}{2} + a_2(m-1)$$

The proof is by induction on a_1+m, and we prove base cases for this induction in the following two lemmas.

Lemma 5. *For $1 = m \leq a_1 \leq a_2 = a_3 = ... = a_r$ we have*

$$ex(K_{a_1,a_2...a_2}, K_r) = a_1a_2(r-2) + \binom{r-1}{2}a^2$$

Proof. We have a lower bound by the previous lemma. Now to show an upper bound we let $G \subseteq K_{a_1,a_2,...,a_2}$ with $|E(G)| = a_1a_2(r-2) + \binom{r-1}{2}a^2 + 1$ and suppose to the contrary that $K_r \nsubseteq G$.

$$\sum_{v_1...v_r} W(v_1...v_r) \leq \left(\binom{r}{2} - 1\right) a_1a_2^{r-1}$$

We now relate $\sum\limits_{v_1...v_r} W(v_1...v_r)$ to $|E(G)|$, by finding how many times an edge is counted in the sum. We define $e(A_i, A_j)$ as the number of edges in G between parts A_i and A_j. Then we have

$$\sum_{v_1...v_r} W(v_1...v_r) = \sum_{j=2}^{r} e(A_1, A_j)a_2^{r-2} + \sum_{i,j\neq 1} e(A_i, A_j)(a_1a_2^{r-3})$$

We then have

$$\begin{aligned}
0 &\geq \sum_{v_1...v_r} W(v_1...v_r) - \left(\binom{r}{2} - 1\right) a_1a_2^{r-1} \\
&= \sum_{j=2}^{r} e(A_1, A_j)a_2^{r-2} + \sum_{i,j\neq 1} e(A_i, A_j)(a_1a_2^{r-3}) - \left(\binom{r}{2} - 1\right) a_1a_2^{r-1} \\
&= a_1a_2^{r-3}|E(G)| + \sum_{j=2}^{r} e(A_1, A_j)(a_2^{r-2} - a_1a_2^{r-3}) - \left(\binom{r}{2} - 1\right) a_1a_2^{r-1} \\
&= (r-2)(a_1^2a_2^{r-2} - a_1a_2^{r-1}) + a_1a_2^{r-3} + \sum_{j=2}^{r} e(A_1, A_j)(a_2^{r-2} - a_1a_2^{r-3}) \\
&:= b
\end{aligned}$$

If $a_2 = a_1$, then $0 \geq b = a_1^{r-2} > 0$, a contradiction. If $a_2 > a_1$, then

$$\begin{aligned}
b &\geq (r-2)(a_1^2a_2^{r-2} - a_1a_2^{r-1}) + a_1a_2^{r-3} + |E(G)| - |E(K\underbrace{a_2, a_2, ..., a_2}_{r-1 \text{ times}})| \\
&= a_2^{r-2} > 0
\end{aligned}$$

a contradiction.

□

Lemma 6. *For $1 < m = a_1 \leq a_2 = ... = a_r$ we have*

$$ex(K_{a_1,a_2,...,a_2}, a_1K_r) = a_1a_2(r-2) + a_2^2\binom{r-1}{2} + a_2(a_1-1)$$

Proof. We already know the extremal number is bounded below by $a_1a_2(r-2) + a_2^2\binom{r-1}{2} + a_2(a_1-1)$ by Lemma 4. Now we show the extremal number is strictly less than $a_1a_2(r-2) + a_2^2\binom{r-1}{2} + a_2(a_1-1) + 1$.

Let $G \subseteq K_{a_1,a_2,...,a_2}$ with $|E(G)| = a_1a_2(r-2) + a_2^2\binom{r-1}{2} + a_2(a_1-1) + 1$. We begin by noting that

$$|E(G)| - (ex(K_{a_1-1,a_2-1,...,a_2-1}, (a_1-1)K_r) + 1) = a_2(r-1) + (r-1)(a_1 + a_2(r-2)) - \binom{r}{2} - 1,$$

which is one fewer than full degree of a K_r. Thus, if we find a single K_r with strictly less than full degree, we may remove it and conclude

$$|E(G \setminus K_r)| \geq ex(K_{a_1-1,a_2-1,...,a_2-1}, (a_1-1)K_r) + 1.$$

Thus, $(a_1-1)K_r \subseteq G \setminus K_r \implies a_1K_r \subseteq G$. We know $a_1 > 1$, so $|E(G)| > a_1a_2(r-2) + a_2^2\binom{r-1}{2} + 1 = ex(K_{a_1,a_2...a_2}, K_r) + 1$, where the last equality holds by Lemma 5. Therefore, $K_r \subseteq G$ by the definition of an extremal number. By previous reasoning, if any K_r has less than full degree, $mK_r \subseteq G$, so we suppose to the contrary that every copy of $K_r \subseteq G$ has full degree.

Thus, our assumption to the contrary means that if $v_1...v_r$ forms a copy of K_r in G, then $N(v_k) = \bigcup_{j=1...r:\, j \neq k} A_j$ for $k = 1, ..., r$. We now show that this implies $G = K_{a_1,a_2,...,a_2}$. We label the vertices, where $v_{i,j}$ is the i^{th} vertex in the j^{th} part. We let $v_{1,1}, v_{1,2}, ...v_{1,r}$ be a copy of K_r in G (which we have already shown exists). We consider the r-tuples formed by varying a single vertex in a single part. That is, we consider the r-tuples $v_{1,1}, v_{1,2}, ..., v_{i,k}, ..., v_{1,r}$. By our assumption that $N(v_{1,k}) = \bigcup_{j=1...r:\, j \neq k} A_j$ for $k = 1, ..., r$, we have that all these r-tuples form copies of K_r in G. Since for any vertex $v_{i,k}$ involved in a copy of K_r in G satisfies $N(v_{i,k}) = \bigcup_{j=1...r:\, j \neq k} A_j$, and every vertex is involved in some K_r in G, then $G = K_{a_1,a_2,...,a_2}$, a contradiction.

□

We now use these two lemmas to prove our proposition:

Proof. The proof is by induction on $a_1 + m$.

Let $G \subseteq K_{a_1,a_2,\dots,a_2}$ with $|E(G)| = a_1a_2(r-2) + a_2^2\binom{r-1}{2} + (m-1)a_2 + 1$, and suppose to the contrary that $mK_r \nsubseteq G$. By the previous two lemmas, we know that this assumption implies $m > 1$ and $a_1 > m$. We first obtain a lower bound on the number of (not necessarily disjoint) copies of K_r in G. We suppose there are exactly q such copies, and see that

$$
\begin{aligned}
\sum_{v_1\dots v_r} W(v_1\dots v_r) \leq q\binom{r}{2} &+ (a_1a_2^{r-1} - q)\left(\binom{r}{2} - 1\right) \\
\implies q &\geq \sum_{v_1\dots v_r} W(v_1\dots v_r) - a_1a_2^{r-1}\left(\binom{r}{2} - 1\right) \\
&= \sum_{j=2}^{r} e(A_1, A_j)a_2^{r-2} + \sum_{i,j\neq 1} e(A_i, A_j)a_1a_2^{r-3} - a_1a_2^{r-1}\left(\binom{r}{2} - 1\right) \\
&:= p
\end{aligned}
$$

We use this lower bound to get an upper bound on the $|E(G)|$. We note that:

$$
\begin{aligned}
\sum_{v_1\dots v_r} |E(G \setminus v_1\dots v_r) \leq\ & q\big(ex(K_{a_1-1,a_2-1,\dots,a_2-1}, (m-1)K_r)\big) \\
&+ (a_1a_2^{r-1} - q)\big(ex(K_{a_1-1,a_2-1,\dots,a_2-1}, mK_r)\big) \\
=\ & q(1-a_2) + (a_1a_2^{r-1})\Big[(a_1-1)(a_2-1)(r-2) \\
&+ (a_2-1)^2\binom{r-1}{2} + (m-1)(a_2-1)\Big] \\
\leq\ & p(1-a_2) + (a_1a_2^{r-1})\Big[(a_1-1)(a_2-1)(r-2) \\
&+ (a_2-1)^2\binom{r-1}{2} + (m-1)(a_2-1)\Big]
\end{aligned}
$$

We define $t := (p + a_1 a_2^{r-1})(1 - a_2)$, so that the above inequality reduces to

$$\sum_{v_1 \ldots v_r} |E(G \setminus v_1 ... v_r) \le t + a_1 a_2^{r-1}(a_2 - 1)\left[(a_1 - 1)(r-2) + (a_2 - 1)\binom{r-1}{2} + (m-1) + \left(\binom{r}{2} - 1\right)\right].$$

A simple counting argument shows that

$$\sum_{v_1 \ldots v_r} |E(G \backslash v_1 ... v_r)| = \sum_{i=2}^{r} e(A_1, A_j)(a_1 - 1)(a_2 - 1)(a_2)^{r-2} + \sum_{i,j \neq 1} e(A_i, A_j)(a_2 - 1)^2 a_1 a_2^{r-3}$$

So that

$$\begin{aligned}\sum_{v_1 \ldots v_r} |E(G \setminus v_1 ... v_r)| - t &= \sum_{i=2}^{r} e(A_1, A_i)(a_2 - 1) a_2^{r-2} a_1 + \sum_{i,j \neq 1} e(A_i, A_j)(a_2 - 1) a_2^{r-2} a_1 \\ &= |E(G)|(a_2 - 1) a_2^{r-2} a_1\end{aligned}$$

Plugging this result into the above inequality, we obtain

$$\begin{aligned}|E(G)| &\le a_2 \left[(a_1 - 1)(r-2) + (a_2 - 1)\binom{r-1}{2} + (m-1) + \binom{r}{2} - 1\right] \\ &= a_2 a_1 (r-2) + a_2^2 (r-1) + a_2(m-1) \\ &\quad - a_2^2 (r-2) - a_2 \binom{r-1}{2} + a_2(r-1) + a_2\left(\binom{r}{2} - 1\right) \\ &= |E(G)| - 1 - a_2\left[a_2(r-2) + \binom{r-1}{2} - (m-1) + \left(\binom{r}{2} - 1\right)\right] \\ &\le |E(G)| - 1 - a_2\left[\binom{r-1}{2} + \left(\binom{r}{2} - 1\right) + 1\right] \\ &< |E(G)|\end{aligned}$$

a contradiction.

$\square$

We use this proposition to prove our theorem.

Proof. Since Lemma 4 proves the desired lower bound for $ex(K_{a_1,a_2,\ldots,a_r}, mK_r)$, all that remains to be shown is

$$ex(K_{a_1,a_2,\ldots,a_r}, mK_r) < \sum_{1\le i<j\le r} a_i a_j - a_1 a_2 + a_2(m-1) + 1$$

We fix a particular m and r, and induct on the total number of vertices in the host graph. Let $G \subseteq K_{a_1,\ldots,a_r}$ with

$$|E(G)| = \sum_{1\le i<j\le r} a_i a_j - a_1 a_2 + a_2(m-1) + 1.$$

We suppose to the contrary there is a graph that is a counterexample to our theorem, consider the minimum number of vertices in the host graph for which there is a counterexample. Let the graph G be a minimum counterexample. By the proposition, the fact that G is a counterexample implies $a_r > a_2$. We consider a fixed vertex $v \in A_r$ and see that

$$\begin{aligned} |E(G \setminus \{v\})| &\le ex(a_1, \ldots, a_{r-1}, a_r - 1, m) \\ &= \sum_{1\le i<j\le r} a_i a_j - \sum_{i\ne r} a_i - a_1 a_2 + a_2(m-1) \end{aligned}$$

Where the extremal number is as in our theorem (by induction), and has the above form as $a_1 \le a_2 \le a_r - 1$ and $a_2 \le a_i$ for $3 \le i \le r-1$.

We then have

$$\begin{aligned} |E(G)| &= |E(G \setminus \{v\})| + d(v) \\ &\le \sum_{1\le i<j\le r} a_i a_j - \sum_{i\ne r} a_i - a_1 a_2 + a_2(m-1) + d(v) \\ &\le \sum_{1\le i<j\le r} a_i a_j - \sum_{i\ne r} a_i - a_1 a_2 + a_2(m-1) + \sum_{i\ne r} a_i \\ &= \sum_{1\le i<j\le r} a_i a_j - a_1 a_2 + a_2(m-1) \\ &= |E(G)| - 1 \end{aligned}$$

which is a contradiction.

Thus, we must have $mK_r \subseteq G$, so

$$ex(K_{a_1,a_2,\ldots,a_r}, mK_r) = \sum_{1 \le i < j \le r} a_i a_j - a_1 a_2 + a_2(m-1)$$

$\square$

[1] Chen, He, Xueliang Li, and Jianhua Tu. "Complete Solution for the Rainbow Numbers of Matchings." Discrete Mathematics 309.10 (2009): 3370-380. Web.