EXPLORING ORIGAMI GENERATED STRUCTURES IN $\mathbb{C}$

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ABSTRACT. The origami generator problem can be described as follows: given a set U of angles, and a set S of points containing 0 and 1, construct lines at angles in U from each point in S and from all possible intersection points of constructed lines, the set $R(U)$ is the closure of all possible points (including initial points) generated by such action. Previous research has shown many properties regarding the algebraic and geometric structures of $R(U)$, given U and S satisfying certain conditions. In particular, results have been proven for cases where $1 \in U$ and $S = \{0, 1\}$. In this paper, we venture beyond these restrictions to explore results for more general cases of U and S. Our main results hold for cases where U does not contain 1, and when $|S| \geq 2$. We will state and prove the conditions in those cases for $R(U)$ to be a lattice or a ring.

1. INTRODUCTION

The origami generator problem was originally suggested by Erik Demaine, drawing inspiration from reference points in paper folding. In origami, we sometimes attempt to obtain reference points by taking the intersection of creases. As such, we are curious about the following problem: given a set U of angles, and a set S of points containing 0 and 1, if we are allowed to fold a plane at angles in U and from points in S, what does the closure set of all possible points generated in this way look like? This, translated into mathematical language, leads us to investigate properties of a generated point set $R(U)$ in the complex plane $\mathbb{C}$. We are particularly interested in the geometric and algebraic structures of $R(U)$: what kinds of mathematical properties does $R(U)$ does have? When is $R(U)$ a lattice, or a ring?

Previous research on the topic has produced many important results. Buhler et al. showed that if $|U| \geq 3$ and U contains angles that are equally spaced (i.e. if $|U| = n$ then $k\pi/n \in U$, $0 \leq k < n, k \in \mathbb{N}$), then $R(U) = \mathbb{Z}[\zeta_n]$ if $|U| = n$ is prime; $R(U) = \mathbb{Z}[1/n, \zeta_n]$ if $|U| = n$ is not prime, where $\zeta_n = exp(2\pi i/n)$. Bahr et al. arrived at results for cases where $1 \in U$, showing that when $|U| = 3$, $R(U)$ is a ring if and only if points constructed in the first step are quadratic integers. Nedrenco explored the question of whether $R(U)$ could still be a ring when U is not a semi group, and proposed several examples. While these results, some of which we will discuss in detail in the BACKGROUND section of this paper, provide solid foundation and great insight into further research, previous work have mainly addressed special cases of the problem, leaving out more general cases of U and other possibilities of S. This paper attempts to generalize certain cases of the origami generator problem, while exploring structures generated by alternative initial conditions.

In particular, I will begin by listing the necessary definitions, notations, and basic properties of the problem in the BACKGROUND section. In the following section, I will state and prove the conditions under which $R(U)$ is a lattice in $\mathbb{C}$ when $|U| = 3$ and $1 \notin U$, and provide examples. I will also consider the conditions where $R(U)$ is a ring. Next, I will explore the possibilities of $R(U)$ when S contains more than two initial points. This section will include some examples, as well as a conjecture. Finally, I will list some possible directions for future research related to this problem. All examples provided in this paper are produced with the help of a data visualization algorithm I wrote with collaboration.

2. BACKGROUND

We begin by introducing some definitions and notations that will be used throughout this paper.

Let u, v be distinct angles, and p, q be distinct points in $\mathbb{C}$. Denote $I_{u,v}(p, q)$ as the intersection point of the lines $l_1 : p+ru$ and $l_2 : q+sv$. Regarding this notation, we have the following properties, proven by Buhler et al.:

Proposition 1. *Let u, v be distinct angles, and p, q be distinct points in $\mathbb{C}$.*

(I) (Symmetry) $I_{u,v}(p, q)$=$I_{v,u}(q, p)$

(II) (Reduction) $I_{u,v}(p, q) = I_{u,v}(p, 0) + I_{v,u}(q, 0)$

(III) (Linearity) $I_{u,v}(p+q, 0) = I_{u,v}(p, 0)+I_{u,v}(q, 0)$ *and for all* $r \in \mathbb{R}$, $I_{u,v}(rp, 0) = rI_{u,v}(p, 0)$.

(IV) (Rotation) For $w \in \boldsymbol{T}$, $wI_{u,v}(p, q) = I_{wu,wv}(wp, wq)$.

We define an iteration to be the construction of new lines and their intersections of the existing set of generated points.

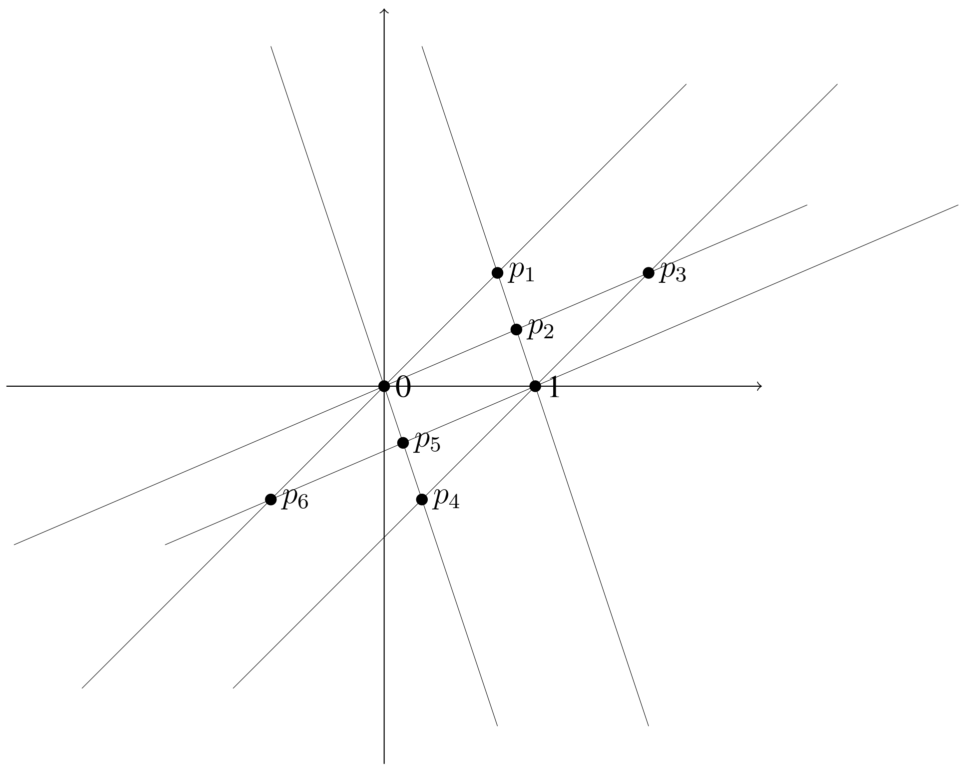


FIGURE 1

3. WHEN $1 \notin U$

Theorem 1. *($\mathbb{Z}$ Generalization for $|U| = 3$.) Let $U = \{u, v, w\}$ with $\arg(u) < \arg(v) < \arg(w)$ and $\arg(u) \neq 0$, then $R(U)$ is a lattice in $\mathbb{C}$ of the form $z_1\mathbb{Z} + z_2\mathbb{Z}$ if and only if $\frac{sinv}{sinw} \cdot \frac{sin(w-u)}{sin(v-u)} \in \mathbb{Q}$.*

Proof. ($\Leftarrow$):

We begin with some definitions of points and lengths as follows. As shown in FIGURE 1, let $p_1 = I_{v,w}(0,1), p_2 = I_{u,w}(0,1), p_3 = I_{u,v}(0,1); p_4 = I_{w,v}(0,1), p_5 = I_{w,u}(0,1), p_6 = I_{v,u}(0,1)$. Let the length of the line segment AB where $A, B \in R(U)$ be denoted $\overline{AB}$.

By the Law of Sines, we have $\overline{p_1 0} = \frac{sinw}{sin(w-u)}$, and $\overline{p_3 1} = \frac{sinv}{sin(v-u)}$. Since $\frac{\overline{p_1 0}}{\overline{p_3 1}} = \frac{sinw \cdot sin(v-u)}{sinv \cdot sin(w-u)} \in \mathbb{Q}$, $\frac{\overline{p_1 0}}{\overline{p_3 1}} = \frac{b}{a}$ for some $a, b \in \mathbb{N}$ where $(a, b) = 1$.

Case 1 We first consider the case where $a = b = 1$. Then $\overline{p_1 0} = \overline{p_3 1}$, as shown in FIGURE 2.We will show that $R(U)$ is a lattice in $\mathbb{C}$ of the form $R(U) = p_1\mathbb{Z} + p_2\mathbb{Z}$.

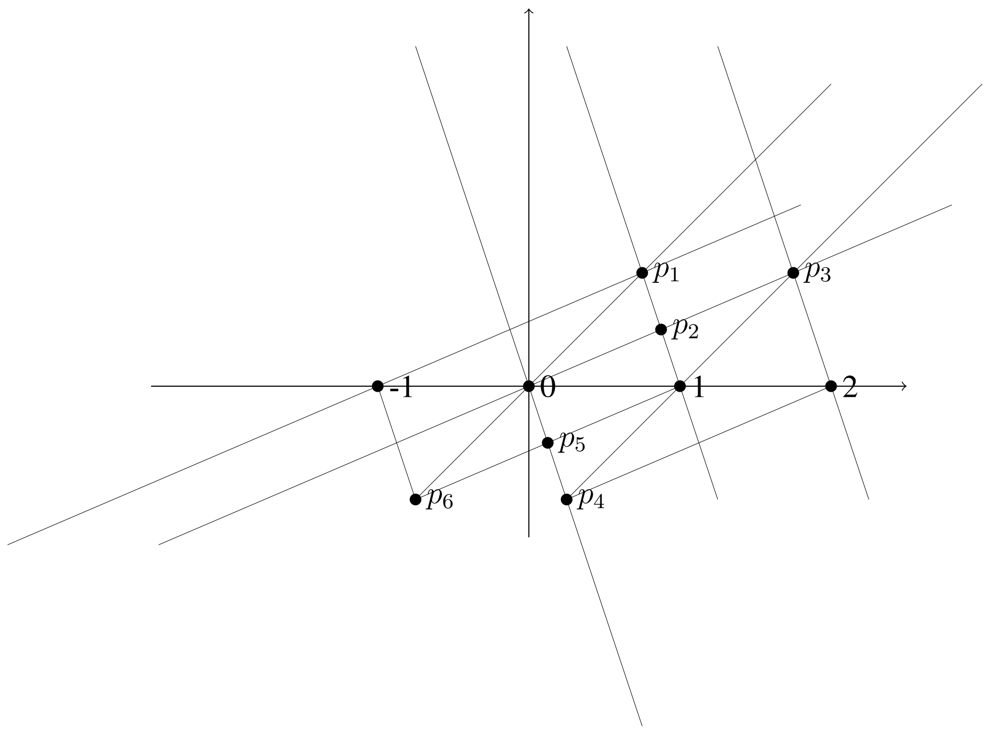


FIGURE 2

We begin by showing that $R(U)$ is closed under addition and taking additive inverses. First, we claim that we can obtain the points 2 and -1. We claim that $I_{w,u}(p_3, p_4) = 2$ and $I_{u,w}(p_1, p_6) = -1$. As shown in FIGURE 2, since $\overline{p_1 0} = \overline{p_3 1}$, $p_1, 0, 1, p_3$ form a parallelogram, hence $\overline{p_3 0} = 2\overline{p_2 0}$. Similarly, $\overline{p_4 0} = 2\overline{p_5 0}$. Therefore $\overline{I_{w,u}(p_3, p_4) 0} = 2 \cdot \mathbf{1} = 2$. By a symmetric argument, $I_{u,w}(p_1, p_6) = -1$.

Since we can obtain that $k+2, k-1 \in R(U)$ given k and $k+1 \in R(U)$ using the method above, we can show that all $n \in \mathbb{N}$ are in $R(U)$. Moreover, given that p and $q \in R(U)$, we can obtain $p+q$ by constructing q starting from p and $p+1$; given that $p \in R(U)$, we can construct $-p$ by constructing p from 0 and -1, going in the negative direction. Therefore $R(U)$ is closed under addition and taking additive inverses.

Since $R(U)$ is a subgroup of $\mathbb{C}$ with addition, and $p_1, p_2 \in R(U)$, it is obvious that $p_1\mathbb{Z} + p_2\mathbb{Z} \subseteq R(U)$.

Now we will show that $R(U) \subseteq p_1\mathbb{Z} + p_2\mathbb{Z}$. Since $0, 1, p_1, p_2, \ldots, p_6 \in p_1\mathbb{Z} + p_2\mathbb{Z}$, it suffices to show that $p_1\mathbb{Z} + p_2\mathbb{Z}$ is closed under intersections.

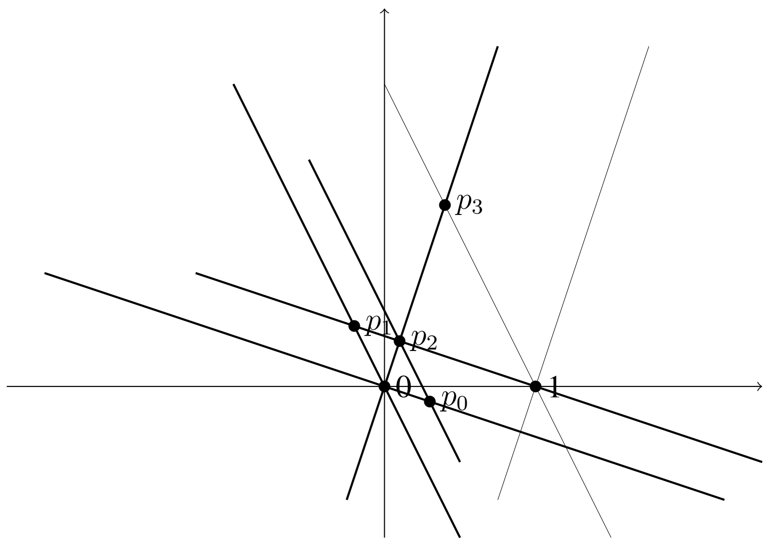


FIGURE 3

Let $z = I_{\alpha,\beta}(p,q)$, where $\alpha, \beta \in \{u, v, w\}, \alpha \neq \beta$ and $p, q \in \{ap_1 + bp_2 | a, b \in \mathbb{Z}\}$. We want to show that $z \in p_1\mathbb{Z} + p_2\mathbb{Z}$. Since $I_{\alpha,\beta}(p,q) = I_{\alpha,\beta}(p,0) + I_{\beta,\alpha}(q,0)$ by reduction, it suffices to show that $I_{\alpha,\beta}(ap_1 + bp_2, 0) \in p_1\mathbb{Z} + p_2\mathbb{Z}$. By linearity, $I_{\alpha,\beta}(ap_1 + bp_2, 0) = aI_{\alpha,\beta}(p_1, 0) + bI_{\alpha,\beta}(p_2, 0)$. It now suffices to show that $I_{\alpha,\beta}(p_1, 0)$ and $I_{\alpha,\beta}(p_2, 0) \in p_1\mathbb{Z} + p_2\mathbb{Z}$.

There are 4 possibilities of $I_{\alpha,\beta}(p_1, 0)$: $I_{u,w}(p_1, 0) = p_2$, $I_{u,v}(p_1, 0) = 0$, $I_{v,w}(p_1, 0) = I_{v,u}(p_1, 0) = p_1$, $I_{w,v}(p_1, 0) = 0$, $I_{w,u}(p_1, 0) = p_1 - p_2$, all of which are in $p_1\mathbb{Z} + p_2\mathbb{Z}$. Similarly, $I_{\alpha,\beta}(p, q) \in p_1\mathbb{Z} + p_2\mathbb{Z}$ Therefore $z = I_{\alpha,\beta}(p, q) \in p_1\mathbb{Z} + p_2\mathbb{Z}$.

The initial points in $U = 0, 1$ are obviously in $p_1\mathbb{Z} + p_2\mathbb{Z}$. Therefore we have proven *Case 1*.

Case 2 We then consider the case where $a > b = 1$, and show that it can be reduced, by a linear transformation, to the case where an element of U is 1.

Since $a > b = 1$, $\overline{p_3 1} = a\overline{p_1 0}$ for some $a > 0, a \in \mathbb{Z}$. Then since $\overline{p_3 1}$ is parallel to $\overline{p_1 0}$, $\overline{p_3 0} = (a+1)\overline{p_2 0}$ and $\overline{p_1 1} = (a+1)\overline{p_1 p_2}$. As shown in FIGURE 3, construct $p_0 = I_{v,w}(p_2, 0)$. Now we examine the highlighted lines in FIGURE 3, as well as points $0, p_1, p_2$. Define $\overrightarrow{0p_2}$ as a new $x'-$axis, and $\mathbf{p_2}$ as a new unit $\mathbf{1}'$, then we have the initial structure of a new $U' = \{1, v - u, w - u\}$ and $S' = \{0, 1'\}$. Bahr et al. have shown that $R(U')$ would form a lattice in the form of $1'\mathbb{Z} + x\mathbb{Z}$, where $1' = p_2$ and $x = p_1$. Hence $R(U') = p_1\mathbb{Z} + p_2\mathbb{Z} \subseteq R(U)$.

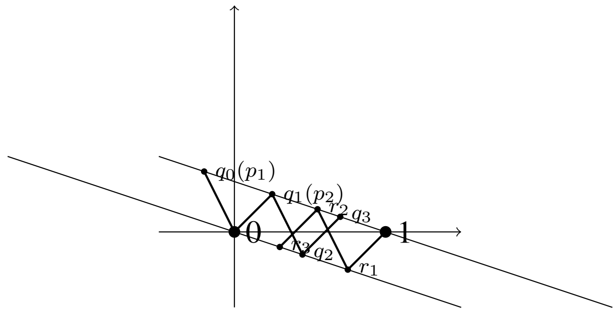


FIGURE 4

We now show that $R(U) \subseteq p_1\mathbb{Z}+p_2\mathbb{Z}$. We resort to the points in FIGURE 3 (initial points of $R(U)$) that are not in $R(U')$, namely, 1 and p_3. Since $\overline{p_3 0} = (a+1)\overline{p_2 0}$ and $\overline{p_1 1} = (a+1)\overline{p_1 p_2}$ for some $a \in \mathbb{Z}$, $p_3 = (a+1)p_2 \in p_1\mathbb{Z}+p_2\mathbb{Z}$, $1 = ap_2 - ap_1 \in p_1\mathbb{Z}+p_2\mathbb{Z}$. Similarly, we can show that $p_4, p_5, p_6 \in p_1\mathbb{Z}+p_2\mathbb{Z}$. Since the closure set obtained by taking intersections of parallels constructed from a set of initial points in the lattice $p_1\mathbb{Z}+p_2\mathbb{Z}$ cannot contain points outside the lattice, points in $R(U)$ must be in $p_1\mathbb{Z}+p_2\mathbb{Z}$, and hence $R(U) \subseteq p_1\mathbb{Z}+p_2\mathbb{Z}$.

Therefore we have proven *Case 2*.

Case 3

Finally, we prove the theorem for the general case, i.e. $a, b > 1$, by reducing it to the previous case.

Since $(a, b) = 1$, there exists $x, y \in \mathbb{Z}$ such that $ax + by = 1$. Using the method with which we constructed all integers n in *Case 1*, as shown in detail in FIGURE 4, we can construct $n(p_1 - p_2)$ beginning from both 0 and 1, in opposite directions. Since $\overline{q_0 q_1} = \overline{p_1 p_2} = \frac{b}{a+b}\overline{q_0 1}$, and $(a, b) = 1$, we can obtain q_i, r_j such that $\overline{q_i r_j} = \frac{1}{a+b}\overline{q_0 1}$.

Once we have a segment of length $\frac{1}{b}\overline{q_0 q_1} = \frac{1}{a+b}\overline{p_1 0}$, which we will set to be a new unit length $\mathbf{1}'$, all possible segments at angle v can only be integer multiples of $\mathbf{1}'$. Similarly, we can obtain a segment of length $\frac{1}{b}\overline{q_1 0} = \frac{1}{b}\overline{p_2 0}$ at angle u. We can therefore reduce *Case 3* to *Case 2*, where we have shown that such structure will produce a lattice of the form $\frac{1}{b}p_1\mathbb{Z} + \frac{1}{b}p_2\mathbb{Z}$.

($\Rightarrow$):

We claim that if $\frac{sinv}{sinw} \cdot \frac{sin(w-u)}{sin(v-u)}$ is not in $\mathbb{Q}$ then R(U) is dense in $\mathbb{C}$. Let $\frac{sinv}{sinw} \cdot \frac{sin(w-u)}{sin(v-u)} = \frac{\overline{p_3 1}}{\overline{p_1 0}} = t$ where $t \in \mathbb{Q}^c$, then as can be illustrated in FIGURE 4, $\frac{\overline{q_1 1}}{\overline{q_0 q_1}} = \frac{\overline{p_3 1}}{\overline{p_1 0}} = t$, hence $\overline{p_1 p_2} = \overline{q_1 q_0} = \frac{1}{1+t}\overline{q_0 1}$. Using the method described in the proof of *Case 3*, we can obtain $m\overline{q_0 1} - n\overline{p_1 p_0} = m(t+1)\overline{q_0 q_1} - n\overline{q_0 q_1} = (mt + m - n)\overline{q_0 q_1}$ for all $m, n \in \mathbb{Z}$. By Kronecker's Theorem, given $s \in \mathbb{Q}^c$, for any $\varepsilon > 0$, there exists $x, y \in \mathbb{Z}$ such that $|xs - y| < \varepsilon$. Let $s = t + 1$, $m = \overline{q_0 q_1}x, n = \overline{q_0 q_1}y$, then there exists $m, n \in \mathbb{Z}$ such that $|(mt + m - n)\overline{q_0 q_1}| < \varepsilon$ for all $\varepsilon > 0$. Similarly, we can obtain line segments at infinitesimal length at any angle. Therefore $R(U)$ is dense in $\mathbb{C}$.

□

Corollary 1. *(Definition of z_1 and z_2 with angles) Let $U = \{u, v, w\}$ with $\arg(u) < \arg(v) < \arg(w)$ and $\arg(u) \neq 0$, and $\frac{sinv}{sinw} \cdot \frac{sin(w-u)}{sin(v-u)} = \frac{b}{a}$ for $a, b \in \mathbb{Z}$ and $(a, b) = 1$, then $R(U) = \frac{1}{b}\frac{(i \cdot sinu + cosu)sinw}{sin(w-u)}\mathbb{Z} + \frac{1}{b}\frac{(i \cdot sinv + cosv)sinw}{sin(w-v)}\mathbb{Z}$.*

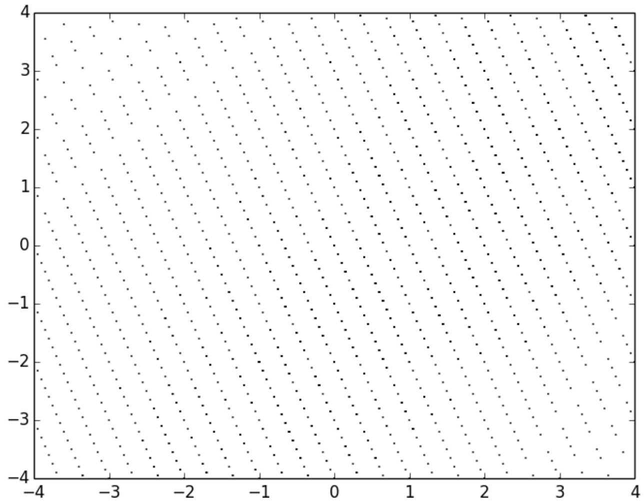
Example. (Lattice) Here is an example of $R(U)$ in the form of a lattice. FIGURE 5 shows $R(U)$ with $u = tan^{-1}(1), v = tan^{-1}(2), w = tan^{-1}(-3)$, after 5 iterations in (A), and after 6 iterations in (B). In this example, $\frac{sinv}{sinw} \cdot \frac{sin(w-u)}{sin(v-u)} = \frac{8}{3} \in \mathbb{Q}$. Notice that both plots reveal a lattice structure, and the iteration does not increase the "density" of the points.

Example. (Dense) Here is another example of $R(U)$ when it is dense in $\mathbb{C}$. FIGURE 6 shows $R(U)$ with $u = \pi/4, v = \pi/3, w = 2\pi/3$, after 5 iterations in (A), after 6 iterations in (B), and after 7 iterations in (C). $\frac{sinv}{sinw} \cdot \frac{sin(w-u)}{sin(v-u)} = \frac{\sqrt{3}(1+\sqrt{3})csc(120)}{2(\sqrt{3}-1)} \notin \mathbb{Q}$. With each iteration, the plotted points becomes noticeably "denser."

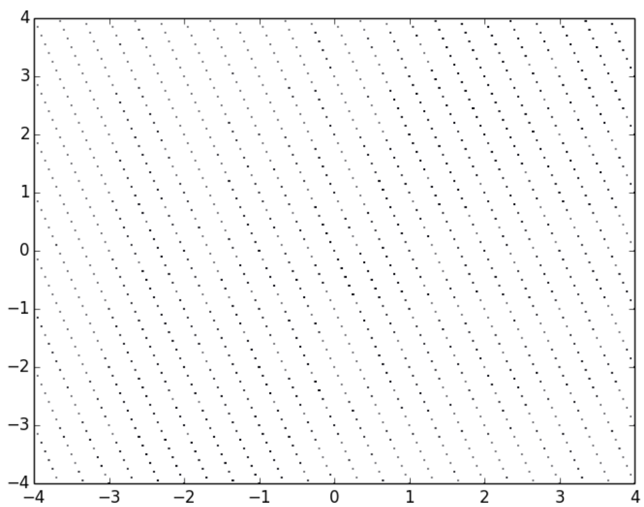
Theorem 2. *(Ring Generalization for $|U| = 3$.) Let $U = \{u, v, w\}$ with $arg(u) < arg(v) < arg(w)$ and $arg(u) \neq 0$, and $\frac{sinv}{sinw} \cdot \frac{sin(w-u)}{sin(v-u)} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $(a, b) = 1$. Then $R(U)$ is a ring if and only if $z = \frac{sin(w-v)}{sin(w-u)}(cos(v-u) + isin(v-u))$ is a non-real quadratic integer.*

Proof. In order for $R(U)$ to be a ring, it must first be closed under addition and taking additive inverses, and hence must be a lattice in $\mathbb{C}$. Corollary 1 gave the general form of $R(U)$ as a lattice: $R(U) = \frac{1}{b}\frac{(i \cdot sinu + cosu)sinw}{sin(w-u)}\mathbb{Z} + \frac{1}{b}\frac{(i \cdot sinv + cosv)sinw}{sin(w-v)}\mathbb{Z}$. We can think of such a lattice as generated by two base vectors: $z_1 = \frac{1}{b}\frac{(i \cdot sinu + cosu)sinw}{sin(w-u)}$ and $z_2 = \frac{1}{b}\frac{(i \cdot sinv + cosv)sinw}{sin(w-v)}$.

Bahr et al. solved the ring conditions for the lattice $R(U) = \mathbb{Z} + x\mathbb{Z}$, where x is a non-real number in $\mathbb{C}$. In order for $\mathbb{Z} + x\mathbb{Z}$ to be a ring, x must be a quadratic integer. We notice that our case for $z_1\mathbb{Z} + z_2\mathbb{Z}$ is essentially a change of basis from the $\mathbb{Z} + x\mathbb{Z}$ case. Therefore, if we let z_2 be a new



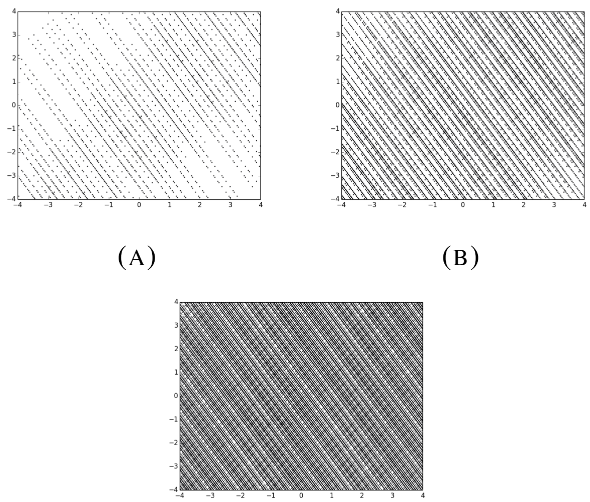
(A) 5th Iteration



(B) 6th Iteration

FIGURE 5

basis vector $\mathbf{1}'$, and set $\mathbf{z}$, which denotes z_1 with respect to z_2, as a new basis vector x', then $z_1\mathbb{Z} + z_2\mathbb{Z}$ is a ring if the conditions for x holds for z.



(A) (B)

(C)

FIGURE 6

Since $z_1 = \frac{1}{b}\frac{(i\cdot sinu+cosu)sinw}{sin(w-u)}$, $z_2 = \frac{1}{b}\frac{(i\cdot sinv+cosv)sinw}{sin(w-v)}$, $|z_1| = \frac{sinw}{b\cdot sin(w-u)}$, $|z_2| = \frac{sinw}{b\cdot sin(w-v)}$. Then $|z| = \frac{z_1}{z_2} = \frac{sin(w-v)}{sin(w-u)}$. Since $arg(z_1) = v$, $arg(z_2) = u$, $arg(z) = v - u$. Thus $z = \frac{sin(w-v)}{sin(w-u)}(cos(v-u) + i\cdot sin(v-u))$. Therefore $R(U) = z_1\mathbb{Z} + z_2\mathbb{Z} = 1'\mathbb{Z} + z\mathbb{Z}$ is a ring if and only if $z = \frac{sin(w-v)}{sin(w-u)}(cos(v-u) + i\cdot sin(v-u))$ is a non-real quadratic integer.

$\square$

4. WHEN $|S| > 2$

We now consider the case where the number of initial points is more than 2, beginning with the case of 3 initial points, and 2 angles. Without loss of generality, we let two of the points be $(0,0)$ *and* $(1,0)$*, and the third be* (x,y) *where* $x, y \in \mathbb{R}$ *and* $x, y > 0$*. Let* $\theta = arctan\frac{y}{x}$ *where* $\theta \in (0,\pi]$ *and* $\gamma = arctan\frac{y}{x-1}$ *where* $\gamma \in (0,\pi]$.

4.1. $|U|$=2.

Example. $|U|$=2

Here is an example of $R(U)$ with $u = \pi/4, v = 2\pi/3$, and $S = \{(0,0),(1,0),(3,1)\}$. $\theta, \gamma \notin U$. We can see that $R(U)$ is finite.

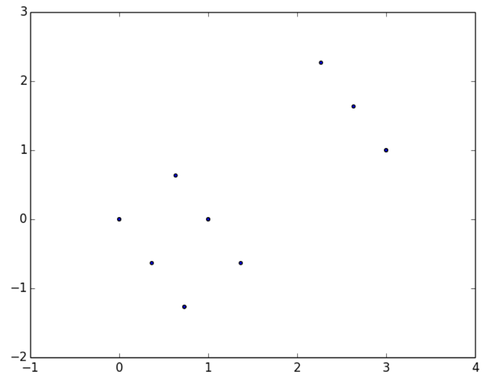


FIGURE 7

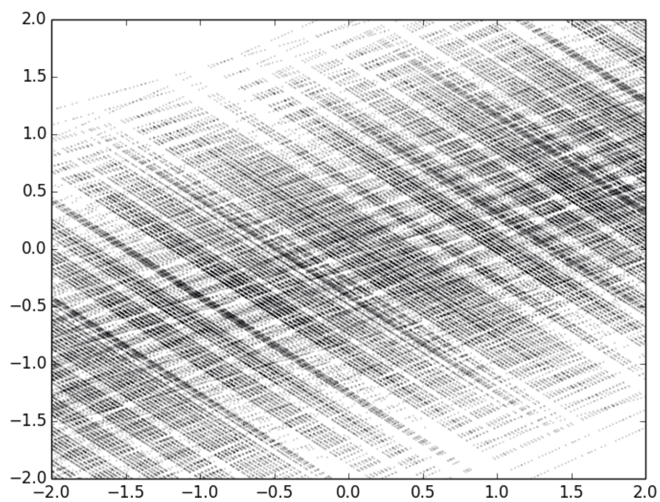


FIGURE 8

4.2. $|U|$**=3.**

Example. Here is an example of $R(U)$ with $u = \pi/9, v = \pi/6, w = 7\pi/9$, and $S = \{(0,0),(1,0),(2,1)\}$, after 4 iterations. $\theta, \gamma \notin U$. $R(U)$ seems to be much denser compared to when $S = 2$, even with few iterations. Observe that adding a point that cannot already be generated in with initial points $(0,0)$ and $(1,0)$ can be viewed as adding a new "dimension" to $R(U)$.

Hence, give the following conjecture:

Conjecture If both θ and γ are not in $U = \{u, v, w\}$, then $R(U)$ is dense in C.

5. Future Work

There are many questions left unanswered in this problem. Here, I will list a few, as directions for future work:

(I) For $S > 2$:
  (a) Prove the stated conjecture on when $R(U)$ is dense.
  (b) What if we allow $1 \in U$, or $n \in S$ for some $n \in \mathbb{R}$?
  (c) When is $R(U)$ a lattice, or a ring?

(II) For $U > 3$: what are the conditions of $R(U)$ being a lattice or a ring when $1 \notin U$?

(III) What subrings of $\mathbb{C}$ can be origami generated, and for what S and U?

6. Acknowledgements

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