RAINBOW NUMBERS WITH RESPECT TO 2-MATCHINGS AND 3-MATCHINGS

KATE BORST AND JÜRGEN KRITSCHGAU

Contents

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1. Abstract

Our results focus on the rainbow numbers of the various graphs with respect to M_2 and M_3. We find the rainbow numbers for all graphs with respect to M_2. From then on out, the number of troublesome cases increases for rainbow numbers with respect to M_3. We prove that the rainbow numbers of trees with a diameter of 6 or greater have $rb(T, M_3) = \Delta + 2$. We extend this result to all graphs with diameter 6 or greater. Our results suggest that $rb(G, M_3) = \Delta + 2$ for unconnected graphs G; this is an area for further study.

2. Introduction

In general, Anti-Ramsey Theory considers two given graphs, G and H, and determines the maximum number of colors that can be used to color the edges of G such that every subgraph of type H has at least 2 edges of the same color. This kind of problem is contrasted with Ramsey Theory problems in which the number of colors used is fixed, and the graphs are variable. Anti-Ramsey Theory problems have been explored with many different graph types [1]. Typically the super graph is a complete graph, however, this is not always the case.

Some work has been done to find the rainbow numbers of regular bipartite graphs, cycles, and paths with respect to matchings. In particular, Li and Xu found that $rb(B_k, M_m) = k(m-2)+2$ where B_k is a k-regular bipartite graph, and M_m is an m-matching [2]. The goal in this paper is to extend these results to graphs in general with respect to small matching sizes.

We will begin our paper by giving the definitions and corresponding notation. For the remainder of this paper we will assume that all graphs G are simple and undirected. We will let $V(G)$ denote the vertex set of G and $E(G)$ denote the edge set of G. As per convention, Δ is the maximum degree of G. We let $[r] = \{1, \ldots, r\}$. Furthermore, P_k denotes a path on k vertices with length $k - 1$, unless otherwise noted.

Definition 2.1. We say that diameter of G, denoted by $diam(G)$, is the length of the longest shortest path in between any two vertices in G. If G is disconnected, we have $diam(G) = \infty$.

Note that connectivity is implicitly included in the definition of diameter. That is, if a graph has a finite number as a diameter, then it must be connected.

Definition 2.2. Let G be a graph. Let $[r] = \{1, \ldots, r\}$. Then $c : E(G) \to [r]$ is an exact r-coloring of the edges of G if and only if for every $r' \in [r]$, there exists $e \in E(G)$ such that $c(e) = r'$.

Definition 2.3. Let G be a graph. Let $c : E(G) \to [r]$ be an exact r-coloring of the edges of G. We say G is rainbow under c if and only if for all $e, f \in E(G)$ where $e \neq f$, we have $c(e) \neq c(f)$.

Definition 2.4. Let G and H be graphs. Then the rainbow number of G with respect to H is the smallest positive integer r such that if G is exactly colored with r colors, then there exists a rainbow subgraph H. This is denoted by $rb(G, H)$.

The definition of rainbow is also defined negatively as r where $r - 1$ is the largest positive integer such that there exists an exact $r - 1$-coloring of G such that there does not exist a rainbow subgraph H. This part of the definition of $rb(G, H)$ is particularly useful when there is no subgraph $H \subseteq G$. In these cases $rb(G, H) = |E(G)| + 1$.

Definition 2.5. We say that a graph, M_k, is a k-matching if and only if $|E(M_k)| = k$, and for any $e, e' \in E(M_k)$, we have $e \cap e' = \{\}$.

Definition 2.6. Let G be a graph. We say that $D \subseteq V(G)$ is a vertex cover of G, if and only if for all $e \in E(G)$ there exists $v \in D$ such that $v \in e$.

Vertex covers are useful in proofs of rainbow numbers because if D is a vertex cover of G, then any k-matching in G must include at least k vertices in D.

To prove rainbow numbers of graphs, we need to give both an upper bound and a lower bound for the rainbow number. Since most of our results show that $rb(G, M) = \Delta + 2$, we offer Proposition 2.1 as a general lower bound for rainbow numbers. In most cases, we will assume that this is the lower bound for the result we are trying to prove. However, if it is unclear whether our proof is adequately exhaustive of possible rainbow numbers, we will explicitly reference this proposition.

Proposition 2.1. Let H be a graph with maximum degree Δ such that there exists a M_3 subgraph. Then $rb(H, M_3) \geq \Delta + 2$.

Proof. Let H be a graph with maximum degree Δ such that there exists a M_3 subgraph. We claim that the following construction yields an exact $\Delta+1$-coloring such that there does not exist a rainbow M_3 subgraph of H. Let $v \in V(H)$ be a vertex such that $\deg(v) = \Delta$. Let $c : E(H) \to [\Delta + 1]$ be an exact coloring such that if $v \notin e$ for $e \in E(H)$, then $c(e) = 1$ for all $e \in E(H)$. This implies that the remaining Δ edges incident upon v have unique colors. Any M_3 must contain at least 2 edges that are not incident upon v under c. Therefore, any M_3 subgraph of H is not rainbow under c. Thus, $rb(H, M_3) \geq \Delta + 2$. ■

As with Proposition 2.1, it is sometimes useful to have a general upper bound for rainbow numbers. Fortunately, the definition of $rb(G, M)$ gives us an intuitive upper bound.

Proposition 2.2. Let G be a graph such that there exists a M_k subgraph of G. Then $rb(G, M_3) \leq m$, where m is the number of edges in G.

Proof. Let G be a graph such that there exists a M_k subgraph of G. Let c color the edges of G such that every edge has a unique color. Then there must be a rainbow M_k. ■

3. Rainbow Numbers with Respect to 2-Matchings

By definition, if a graph G does not contain a subgraph H, then $rb(G, H) = |E(G)| + 1$. Since we know what the rainbow number $rb(G, M_2)$ is for the class of graphs not containing M_2 as a subgraph (stars, for example), we may assume from here on that G contains a 2-matching.

Theorem 3.1. If P_n is a path with $n > 4$, then $rb(P_n, M_2) = 2$.

Proof. By induction on the number of vertices in the path.

Base Case: Consider a path of length 4, $P_5 = (v_1, v_2, v_3, v_4, v_5)$ with edges $e_i = \{v_i, v_{i+1}\}$ for $i \in [4]$ and an exact 2-coloring $c : E(P_5) \to [2]$. Assume for the sake of contradiction that P_5 does not contain a rainbow 2-matching under any exact 2-coloring. Then it doesn't contain a rainbow 2-matching under c. Notice that the following pairs of edges are 2-matchings: $\{e_1, e_3\}$, $\{e_2, e_4\}$, $\{e_1, e_4\}$. Therefore each of these pairs must be monochromatic: $c(e_1) = c(e_3)$ and $c(e_2) = c(e_4)$ and $c(e_1) = c(e_4)$. Therefore $c(e_1) = c(e_2) = c(e_3) = c(e_4)$, which contradicts the assumption that c was an exact 2-coloring. So P_5 must have a rainbow M_2, and therefore $rb(P_5, M_2) = 2$.

Induction Hypothesis: Let P_n be a path such that $5 \leq n \leq N$ where $N \in \mathbb{N}$. Then $rb(P_n, M_2) = 2$.

Induction Step: Consider some path $P_{n+1} = (v_1, v_2, \ldots v_n, v_{n+1})$ with $5 \leq n \leq N$ and an exact 2-coloring $c : E(P_{n+1}) \to [2]$. The path $P' = P_{n+1} \setminus \{v_n, v_{n+1}\}$ is a path on n vertices, specifically the first n vertices of P_{n+1}, $v_1, v_2, \ldots, v_n$. The subpath P' is either monochromatic, or it is not. If it is not, then there is an exact 2-coloring $c' : E(P') \to [2]$, and by the induction hypothesis there exists a rainbow M_2 within P' which would imply there is also a rainbow M_2 in our host path P_{n+1}. Otherwise, the subpath P' is monochromatic (WLOG with color 1), in which case the edge $\{v_n, v_{n+1}\}$ must have color 2, since we assumed c to be an exact 2-coloring. Since this edge is adjacent only to the edge $\{v_{n-1}, v_n\}$, it forms a rainbow M_2 with any of the other edges. Therefore $rb(P_{n+1}, M_2) = 2$. ■

Theorem 3.2. If G is a disconnected graph with a 2-matching then $rb(G, M_2) = 2$.

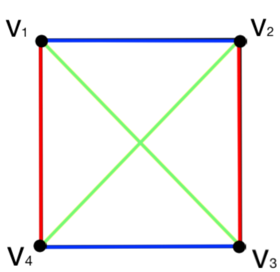
Proof. Let G be a disconnected graph with a 2-matching and $c : E(G) \to [2]$ be an exact 2-coloring on G. Assume for the sake of contradiction that there is no rainbow M_2 in G under any 2-coloring. Consider two edges $e, f \in E(G)$ such that e and f are not in the same component of G. That is, that there is no path between e and f. Certainly e and f are a 2-matching. Therefore $c(e)$ must equal $c(f)$, since we assumed none of G's 2-matchings are rainbow. Without loss of generality, say the $c(e) = c(f) = 1$. Since c is an exact 2-coloring, there must be some other edge $g \in E(G)$ such that $c(g) = 2$. We assumed there is no rainbow 2-matching in G, so g must be incident upon both e and f. This contradicts the assumption that e and f are from separate components. Therefore there must be a rainbow M_2 in G, and $rb(G, M_2) = 2$. ■

We have shown that $rb(G, M_2) = 2$ when G is a disconnected graph. Therefore, from here on, we may assume the G is connected.

Theorem 3.3. If G is the complete graph on four vertices, $rb(G, M_2) = 4$.

Proof.

Lower Bound: $rb(K_4, M_2) > 3$



There are three 2-matchings in this graph: $\{\{v_1, v_3\}, \{v_2, v_4\}\}, \{\{v_1, v_2\}, \{v_3, v_4\}\}, \{\{v_1, v_4\}, \{v_2, v_3\}\}$. Each gets a unique color. Notice that this 3-coloring of K_4 does not have a rainbow 2-matching. Therefore $rb(K_4, M_2)$ must be greater than three.

Upper Bound: $rb(K_4, M_2) \leq 4$

Let G be the complete graph on four vertices and $c : E(G) \to [4]$ be an exact 4-coloring on G. Every edge in $E(G)$ is in exactly one of three 2-matchings. Since there are four colors and three 2-matchings, by the Pigeon Hole Principle one of the 2-matchings must use two colors, and therefore be rainbow. ■

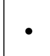
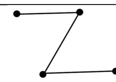
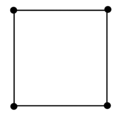
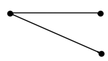
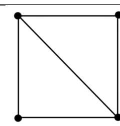
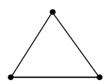
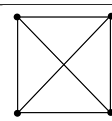
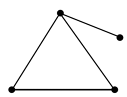
Graph G	$rb(G, M_2)$	Graph G	$rb(G, M_2)$
	1		3
	2		3
	3		4
	4		4
	4		

FIGURE 1. Rainbow numbers for connected graphs on four vertices or fewer

We have shown what the rainbow number is for all graphs on less than or equal to four vertices with respect to a 2-matching. Therefore from here on, we may assume that the graph G has more than four vertices.

Lemma 3.1. Let G be a connected graph on at least 5 vertices that contains a 2-matching. Then $rb(G, M_2) = 2$ if and only if no set of vertices $\{u, v\}$ is both an edge and a vertex cover.

Proof. Assume that G is a connected graph on at least 5 vertices that contains a 2-matching, such that no set of vertices $\{u, v\}$ is both an edge and a vertex cover. G must contain a path of length at least 4:

If the maximum path length is 1, G only has one edge which contradicts the assumption that no set of vertices $\{u, v\}$ is both an edge and a vertex cover. If the maximum path length is 2, G is a star, which again contradicts the assumption that no set of vertices $\{u, v\}$ is both an edge and a vertex cover. If the maximum path length is 3, consider a maximum path $P_{max} = (v_1, v_2, v_3, v_4)$. Any other edges in the graph must be adjacent to v_2 or v_3, since we assumed the graph to be connected and P_{max} to be of maximum length (the existence of an edge not in P_{max} adjacent to v_1 or v_4 would mean there is a longer path in G). This contradicts the assumption that no set of vertices $\{u, v\}$ is both an edge and a vertex cover because $\{v_2, v_3\}$ is both an edge in G and a vertex cover of G. Therefore G must contain a path of length at least 4.

Consider this path, P. Let $c : E(G) \to [2]$ be an exact 2-coloring of the edges of G. Under c, the path P is either 2-colored, in which case there is a rainbow M_2 by Lemma 3.1, or it is monochromatic. If the path is monochromatic, there must be an edge of the other color somewhere in the graph (else we would not have an exact two-coloring). We case on where this edge, $\{u, v\}$, could be:

We know that the other-colored edge, $\{u, v\}$, cannot be incident with either of the path's endpoints, since we assumed P to be maximum.

Furthermore, if there is any edge in the P that is not adjacent to $\{u, v\}$, then we have a rainbow M_2, and we are done. So we may assume that $\{u, v\}$ is adjacent to all edges in the path P. In a simple graph, an edge can be adjacent to at most 4 edges on a path. So if P has at least 5 edges, we must have a rainbow M_2. Therefore we may assume that P has exactly four edges, and may describe P as the path $(v_1, v_2, v_3, v_4, v_5)$, where the $\{u, v\}$ is the edge $\{v_2, v_4\}$. $\Rightarrow\Leftarrow$

This would mean that $\{v_2, v_3\}$ is a vertex cover, which contradicts the assumption that there is no set of vertices that is both an edge and a vertex cover. Therefore $rb(G, M_2) = 2$. ■

Lemma 3.2. Let G be a connected graph with $n \geq 5$. Let ℓ be the number of sets $D = \{u, v\}$ such that D is a vertex cover of G and $\{u, v\} \in E(G)$. Then $\ell \leq 2$, or G is a star.

Proof. Let G be a connected graph with $n \geq 5$ vertices. Assume that $\ell \geq 3$. This implies that there exists D_1, D_2, and D_3 such that they are all vertex covers of G. This implies that $D_1 \cap D_2 \cap D_3 \neq \{\}$. There are two cases; either D_1, D_2, and D_3 are given by a 3-cycle in G, or D_1, D_2, and D_3 are given by a star in G.

Case 1: Assume that D_1, D_2, and D_3 are given by a 3-cycle in G. Let this 3-cycle be given by $C = (v_1, v_2, v_3)$. Since $n \geq 5$, there must exist a vertex $v \notin V(C)$ such that v is adjacent to some vertex in $V(C)$. Without loss of generality, assume that $\{v, v_1\} \in V(G)$. In this case, the vertex cover given by $\{v_2, v_3\}$ does not cover $\{v, v_1\}$ which is a contradiction. Therefore, it cannot be the case that D_1, D_2, and D_3 are given by a 3-cycle in G.

Case 2: Assume that D_1, D_2, and D_3 is given by a star $S \subset G$; let S is a star the centered on a vertex v such that $D_i = \{v, v_i\}$ for vertices v_1, v_2, v_3. Since $n \geq 5$ and G is a connected graph, there must exist at least 2 other edges in G. There are two subcases; either all $e \in E(G)$ are incident up on v or there exist at least one $e \in E(G)$ such that it is not incident upon v.

Case 2.1: Assume that all $e \in E(G)$ are incident upon v. Then G is a star.

Case 2.2 Assume that there exists an edge $e \in E(G)$ such that e is not incident upon v. Now, e is incident on at most 2 leaves of S. Without loss of generality assume that e is incident upon at most two of the vertices v_1, v_2, but not v_3. This implies that D_3 is not a vertex cover of G. Since this is a contradiction, it cannot be the case that that there exists an edge $e \in E(G)$ such that e is not incident upon v.

Thus, $\ell \leq 2$, where ℓ is the number of sets $D = \{u, v\}$ such that D is a vertex cover of G and $\{u, v\} \in E(G)$, of G is a star. ■

Theorem 3.4. Let G be a graph on at least five vertices that contains a 2-matching. If G has $\ell \in \{0, 1, 2\}$ sets of vertices such that each is both a vertex cover and an edge in $E(G)$, then $rb(G, M_2) = 2 + \ell$.

Proof. Let G be a connected graph on at least five vertices that contains a 2-matching.

First, notice that by Lemma 3.1, we have the statement for $\ell = 0$. That is, if G contains no set of vertices that is both an edge and a vertex cover, then $rb(G, M_2) = 2 = 2 + 0$. ✓

We may assume hence forth that either $\ell = 1$ or $\ell = 2$.

Consider an exact $(2 + \ell)$-coloring on the edges of G, $c : E(G) \to [2 + \ell]$. Let $\{u_1, v_1\}$ and $\{u_2, v_2\}$ denote sets of vertices that are both vertex covers and edges in $E(G)$ (that is, if we only have one such set, it is $\{u_1, v_1\}$ and if we have two such sets they are $\{u_1, v_1\}$ and $\{u_2, v_2\}$). In either case, since every set of vertices that is both a vertex set and an edge is an edge adjacent to every edge in $E(G)$, none can be in a 2-matching. Therefore, we may color these with unique colors. Consider the graph G', generated by removing any set $\{u, v\}$ from the edge set of $E(G)$ if it is a vertex cover.

First, we will use this to establish a *lower bound* that $rb(G, M_2) \geq 2+\ell$. Any two matching in G must come from $E(G')$, so if we monochromatically color these edges, the result is an $(\ell + 1)$-coloring with no rainbow M_2. Therefore $rb(G, M_2) \geq 2 + \ell$. This same method also yields an *upper bound*, because G' satisfies the conditions of Lemma 3.1: it is a connected graph with $n \geq 5$ that contains a two-matching, and there is no set of vertices that is both an edge in $E(G')$ and a vertex cover of G'. Therefore, $rb(G', M_2)$ is 2, by Lemma 3.1. So every exact 2-coloring on G' has a rainbow M_2. Therefore $rb(G, M_2) = 2 + \ell$. ■

Corollary 3.2.1. Let G be a graph with $n \geq 5$ that contains a 2-matching. If G has exactly one set of vertices $\{u, v\}$ that is both a vertex cover and an edge in $E(G)$, then $rb(G, M_2) = 3$.

4. Rainbow Numbers of Disconnected Paths with Respect to 3-Matchings

When we start proving results about trees we will selectively delete edges whose colors are repeated. This will allow us to reduce a graph to a smaller case, making induction a promising approach. Unfortunately, there are complications that arise for trees that confound the selective deletion method. Therefore, we will consider the rainbow numbers of pairs of paths with respect to M_3 to clearly illustrate the selective deletion method.

To prove an a rainbow number we need both an upper bound and a lower bound. Proposition 4.1 follows directly from Proposition 2.1.

Proposition 4.1. Let $P_{n_1} \cap P_{n_2}$ denote the disconnected graph consisting of two paths, one of length n and one of length m. Then $rb(P_{n_1} \cap P_{n_2}, M_3) \geq 4$ for all $n \geq 5, m \geq 4$.

Theorem 4.1. Let $P_{n_1} \cap P_{n_2}$ denote the disconnected graph consisting of two paths, one on n_1 vertices and one on n_2 vertices. Then $rb(P_{n_1} \cap P_{n_2}, M_3) = 4$ for all $n \geq 3$ and $m \geq 1$ except $(n_1, n_2) = (4, 3)$ or $(n_1, n_2) = (4, 4)$.

Proof. By Proposition 4.1 we already have a the lower bound $rb(P_{n_1} \cap P_{n_2}, M_3) \geq 4$ for all $n_1 \geq 3$ and $n_2 \geq 1$.We will show Theorem 4.1 for $n_1 \geq 5, n_2 \geq 4$ by mathematical induction first on n_1 and then on n_2. Starting with $(n_1, n_2) = (5, 4)$ as a base case. All lower combinations of (n_1, n_2) can be shown by inspection.

Base Case (Part 1): Let $n_1 = 5$ and $n_2 = 4$. We know by inspection that $rb(P_5 \cap P_4, M_3) = 4$.

Induction Hypothesis (Part 1): $rb(P_{n_1} \cap P_4, M_3) = 4$ where $5 \leq n_1 \leq N$.

Induction Step (Part 1): Consider $G = P_{N+1} \cap P_4$. Let $c : E(G) \to [4]$ be a 4-coloring of the edges of $P_{N+1} \cap P_4$. Let the path $P_{N+1} = (v_1, v_2, \ldots v_N, v_{N+1})$ be the entire P_{N+1} component of G. There are two cases; either there exists an edge that shares a color with either $\{v_1, v_2\}$ or $\{v_N, v_{N+1}\}$, or $\{v_1, v_2\}$ and $\{v_N, v_{N+1}\}$ are uniquely colored.

Case 1: If there exists an edge $e \in E(G)$ such that $e \neq \{v_1, v_2\}$ and $c(e) = c(\{v_1, v_2\})$, or $e \neq \{v_N, v_{N+1}\}$ and $c(e) = c(\{v_N, v_{N+1}\})$ then let $G' \subset G$ created by deleting v_1 if $c(e) = c(\{v_1, v_2\})$ or deleting v_{N+1} if $c(e) = c(\{v_N, v_{N+1}\})$. Apply the induction hypothesis to G' to show that there exists a rainbow M_3 in G.

Case 2: If there does not exist an edge $e \in E(G)$ such that $e \neq \{v_1, v_2\}$ and $c(e) = c(\{v_1, v_2\})$, or $e \neq \{v_N, v_{N+1}\}$ and $c(e) = c(\{v_N, v_{N+1}\})$ then both $\{v_1, v_2\}$ and $\{v_N, v_{N+1}\}$ are uniquely colored under c. In this case the M_3 given by $\{v_1, v_2\}$,$\{v_N, v_{N+1}\}$, and some edge $e' \in E(G) \setminus E(P_{N+1})$ is rainbow. In either case, $rb(P_{n_1} \cap P_4, M_3) = 4$.

Therefore, by mathematical induction $rb(P_{n_1} \cap P_4) = 4$ for all $n_1 \geq 5$.

Now, for the second part of the induction, let $n_1 \geq 5$ be arbitrary but fixed.

Base Case (Part 2): $rb(P_n \cap P_4) = 4$ by Part 1.

Induction Hypothesis (Part 2): $rb(P_{n_1} \cap P_{n_2}) = 4$ for any $n_1 \geq 5$ and $4 \leq n_2 \leq M$.

Induction Step (Part 2): Consider the graph $G = P_{n_1} \cap P_{N+1}$. Let $c : E(G) \to [4]$ be a four coloring of the edges of $P_n \cap P_{M+1}$. Let the path $P_{N+1} = (v_1, v_2, \ldots, v_N, v_{N+1})$ be the entire P_{N+1} component of G. There are two cases; either there exists an edge that shares a color with either $\{v_1, v_2\}$ and $\{v_N, V_{N+1}\}$, or $\{v_1, v_2\}$ and $\{v_N, V_{N+1}\}$ are uniquely colored.

Case 1: If there exists an edge $e \in E(G)$ such that $e \neq \{v_1, v_2\}$ and $c(e) = c(\{v_1, v_2\})$, or $e \neq \{v_N, v_{N+1}\}$ and $c(e) = c(\{v_N, v_{N+1}\})$ then let $G' \subset G$ created by deleting v_1 if $c(e) = c(\{v_1, v_2\})$ or deleting v_{N+1} if $c(e) = c(\{v_N, v_{N+1}\})$. Apply the induction hypothesis to G' to show that there exists a rainbow M_3 in G.

Case 2: If there does not exist an edge $e \in E(G)$ such that $e \neq \{v_1, v_2\}$ and $c(e) = c(\{v_1, v_2\})$, or $e \neq \{v_N, v_{N+1}\}$ and $c(e) = c(\{v_N, v_{N+1}\})$ then both $\{v_1, v_2\}$ and $\{v_N, v_{N+1}\}$ are uniquely colored under c. In this case the M_3 given by $\{v_1, v_2\}$,$\{v_N, v_{N+1}\}$, and some edge $e' \in E(G) \setminus E(P)$ is rainbow.

In either case, $rb(P_{n_1} \cap P_{n_2}, M_3) = 4$. Therefore, by mathematical induction $rb(P_{n_1} \cup P_{n_2}, M_3) = \Delta + 2$ for all $n_1 \geq 5, n_2 \geq 4$. Thus, Theorem 4.1 has been shown. ■

Notice that some $P_{n_1} \cup P_{n_2}$ are excluded by Theorem 4.1. The lower bound, $rb(P_n \cup P_m, M_3) \geq 4$ can be constructed by coloring two edges in every M_3 subgraph the same color. Further inspection shows that $rb(P_4 \cup P_3, M_3) = 5$ and $rb(P_4 \cup P_4, M_3) = 5$.

5. Rainbow Numbers of Trees with Respect to 3-Matchings

Theorem 5.1. Let T be a tree with $diam(T) \leq 3$. Then $rb(T, M_3) = n$.

Proof. Let T be a tree with $diam(T) \leq 3$. Since T is a tree, T is connected. A maximum diameter of length 3 implies that all edges are incident upon at least one of two vertices. Therefore, there does not exist a $M_3 \subseteq T$. Thus, $rb(T, M_3) = n$. ■

Furthermore, the previous argument shows that a path on 5 vertices does not have a M_3 subgraph. Therefore, $rb(P_5, M_3) = n$.

Theorem 5.2. Let T be a tree with $n > 5$ and $diam(T) = 4$. Let $P = (v_1, v_2, v_3, v_4, v_5)$ be a path of length 4 in T. If $\deg(v_3) = 2$, then $rb(T, M_3) = n$.

Proof. Let T be a tree with $n > 5$ and $diam(T) = 4$. Let $P = (v_1, v_2, v_3, v_4, v_5)$ be a path of length 4 in T. Assume that there does not exist an edge incident with v_3 that is not in P. This implies that all other edges are incident with either v_2 or v_4. If this were not the case then T would either not be connected or T would not have $diam(T) = 4$. In this case, all edges are incident with v_2 or v_4, which means that there does not exist a M_3 subgraph of T. Therefore, $rb(T, M_3) = n$ by definition. Thus, the theorem is proven. ■

Lemma 5.1. Let $G = X \cup Y \cup Z$ be a graph where X, Y, Z are disconnected stars such that $|E(X)|, |E(Y)|, |E(Z)| \geq 1$. Then $rb(G, M_3) = \max\{|E(X)|, |E(Y)|, |E(Z)|\} + 2 = \Delta + 2$.

Proof. Let $G = X \cup Y \cup Z$ be a graph where X, Y, Z are disconnected stars such that $|E(X)|, |E(Y)|, |E(Z)| \geq 1$. By Proposition 2.1, we have $rb(G, M_3) \geq \Delta + 2$. Without loss of generality, assume $|E(X)| \geq |E(Y)| \geq |E(Z)|$. Let $c : E(G) \to [|E(X)| + 2]$ be an exact $|E(X)| + 2$-coloring of the edges of G. Without loss of generality, there must exist $e_1, e_2 \notin E(X)$ such that $c(e_1) = 1, c(e_2) = 2$ and the colors $1, 2$ do not appear in X when c is restricted to $E(X)$. There are two cases; either e_1 and e_2 are in different components, e_1 and e_2 both appear in the same component.

Case 1: Without loss of generality, assume $e_1 \in E(Y)$ and $e_2 \in E(Z)$. If $e_1 \in E(Y)$ and $e_2 \in E(Z)$ then $e \in E(X), e_1, e_2$ gives a rainbow M_3.

Case 2: Without loss of generality, assume $e_1, e_2 \in E(Y)$. Without loss of generality, there must exist $e_3 \in E(X)$ such that $c(e_3) = 3$. Either $c(e) \in \{1, 2, 3\}$ for all $e \in E(Z)$ or there exists $e \in E(Z)$ such that $c(e) \notin \{1, 2, 3\}$. If $c(e) \in \{1, 2, 3\}$ for all $e \in E(Z)$, then there exists $e_4 \in E(X)$ such that e_4 does not share a color with any edge in $E(Y)$ or $E(Z)$. In this case choose a rainbow M_3 given by e_4 and a M_2 in $Y \cup Z$, which must exist by Theorem 3.2. If there exists $e \in E(Z)$ such that $c(e) \notin \{1, 2, 3\}$ then choose e, e_1, e_3 to form a rainbow M_3.

In either case we have $rb(G, M_3) \leq \max\{|E(X)|, |E(Y)|, |E(Z)|\} + 2 = \Delta + 2$. With Proposition 2.1, we have $rb(G, M_3) = \max\{|E(X)|, |E(Y)|, |E(Z)|\} + 2 = \Delta + 2$. ■

We will use Lemma 5.1 to prove Theorem 5.3 by reducing graphs to the form $A = X \cup Y \cup Z$ and relating Δ_A to the maximum degree of the original graph. In particular, there will be a different answer depending on which disconnected component of A has the most edges. However, to preform the reduction we will need the following proposition.

Proposition 5.1. Let G and G' be graphs such that $M_3 \subseteq G' \subset G$ where $G' = G - e$ for some $e \in E(G)$. Then $rb(G, M_3) \leq rb(G', M_3) + 1$.

Proof. Let G be a graph. Let $c : E(G) \to [r]$ be an exact k-coloring of the edges of G. Let G' be the graph created by deleting some edge in G. Now, c when restricted to G' is at least an exact $r-1$-coloring of the edges of G'. Assume that there exist a rainbow M_3 in G' under c. This implies that $rb(G', M_3) \leq r - 1$. Then there exists a rainbow M_3 in G. Thus, $rb(G, M_3) \leq rb(G', M_3) + 1$. ■

Proposition 5.1 relates the rainbow number of a graph G to a particular kind of subgraph G'. Most importantly, it says that adding an edge This is particularly helpful when we can reduce the maximum degree of a graph or if we can delete edges that cannot possibly be in a M_3 subgraph.

Theorem 5.3. Let T be a tree with $n > 6$ and $diam(T) = 4$. Let $P = (v_1, v_2, v_3, v_4, v_5)$ be a path of length 4 in T where $deg(v_3) \geq 3$. If $D = \{v_2, v_3, v_4\}$ is a vertex cover of T, then $rb(T, M_3) = rb(A, M_3) + 3$ or $rb(T, M_3) = rb(A, M_3) + 2$, where A is the subgraph of T created by only using edges incident with exactly one vertex in D.

Proof. Let T be a tree with $n > 5$ and $diam(T) = 4$. Let $P = (v_1, v_2, v_3, v_4, v_5)$ be a path of length 4 in T. Let $D = \{v_2, v_3, v_4\}$ be a vertex cover of T. By Proposition 2.1, $rb(T, M_3) \geq \Delta + 2$. Let A be the subgraph of T only including all of the edges that are incident upon exactly one vertex $v \in D$. This means that A is of the form $V_2 \cup V_3 \cup V_4$ where V_i is the star centered on v_i. By Lemma 5.1, $rb(A, M_3) = \max\{|E(V_2)|, |E(V_3)|, |E(V_4)|\} + 2$. There are three cases; Either $|E(V_3)| = \max\{|E(V_2)|, |E(V_3)|, |E(V_4)|\}$, $|E(V_2)| = \max\{|E(V_2)|, |E(V_3)|, |E(V_4)|\}$

and $|E(V_2)| = |E(V_3)| + 1$, or $|E(V_2)| = \max\{|E(V_2)|, |E(V_3)|, |E(V_4)|\}$ and $|E(V_2)| > |E(V_3)| + 1$.

Case 1: Without loss of generality, assume that $|E(V_3)| = \max\{|E(V_2)|, |E(V_3)|, |E(V_4)|\}$. By construction $\Delta_T - 2 = |E(V_3)|$. Therefore, $rb(T, M_3) = \Delta_T + 2$.

Case 2: Without loss of generality, assume $|E(V_2)| = \max\{|E(V_2)|, |E(V_3)|, |E(V_4)|\}$ and $|E(V_2)| = |E(V_3)| + 1$. This implies that $deg(v_2) = deg(v_3)$ in T. Therefore, $rb(T, M_3) = \Delta_T + 3$.

Case 3: Without loss of generality, assume $|E(V_2)| = \max\{|E(V_2)|, |E(V_3)|, |E(V_4)|\}$ and $|E(V_2)| > |E(V_3)| + 1$. This implies that $deg(v_2) = \Delta_T$. Therefore, $rb(T, M_3) = \Delta_T + 3$.

Thus, Theorem 5.3 has been proven. ■

Corollary 5.1.1. Let T be a tree with $n > 5$ and $diam(T) = 4$. Let $P = (v_1, v_2, v_3, v_4, v_5)$ be a path of length 4 in T. Let $D = \{v_2, v_3, v_4\}$ be a vertex cover of T. If $deg(v_2) = \Delta \geq deg(v)$ for all other $v \in V(G)$ such that $v \neq v_2$, the $rb(T, M_3) = \Delta + 3$. Else, $rb(T, M_3) = \Delta + 2$.

Notice that all cases of T with $diam(T) = 4$ and $n = 6$ are covered in Theorem 5.3. However, there are trees T with $diam(T) = 4$ and $n > 6$ where the set D is not a vertex cover of T. Interestingly, these trees do not have an ambiguity in rainbow number. This happens because all edges in T can in fact be in a M_3 subgraph when the assumption about D is relaxed. To cover the remaining cases we need the following theorem.

Theorem 5.4. Let T be a tree with $n > 6$ and $diam(T) = 4$. Let $P = (v_1, v_2, v_3, v_4, v_5)$ be a path of length 4 in T. If $D = \{v_2, v_3, v_4\}$ is not a vertex cover of T, then $rb(T, M_3) = \Delta + 2$.

Proof. Let T be a tree with $n > 6$ and $diam(T) = 4$. Let $P = (v_1, v_2, v_3, v_4, v_5)$ be a path of length 4 in T. Assume that $D = \{v_2, v_3, v_4\}$ is not a vertex cover of T. By Proposition 2.1, $rb(T, M_3) \geq \Delta + 2$. We will prove Theorem 5.4 by mathematical induction on the number of vertices n.

Base Case: Assume T is a tree with $n = 7$ and $diam(T) = 4$. Let $P_1 = (v_1, v_2, v_3, v_4, v_5)$ be a path of length 4 in T. Let $P_2 = (v_1, v_2, v_3, v_6, v_7)$ be a path of length 4 in T. Let

$c : E(T) \to [5]$ be an exact 5-coloring of the edges of T. If there exists $e \in E(T)$ such that either $e \neq \{v_1, v_2\}$ but $c(e) = c(\{v_1, v_2\})$, or $e \neq \{v_4, v_5\}$ but $c(e) = c(\{v_4, v_5\})$, or $e \neq \{v_6, v_7\}$ but $c(e) = c(\{v_6, v_7\})$, then delete $\{v_1, v_2\}$, $\{v_4, v_5\}$, or $\{v_6, v_7\}$ respectively to form T'. Apply Theorem 5.3 to get $rb(T', M_3) = 5$ and, therefore, $rb(T, M_3) = 5$. If there does not exist $e \in E(T)$ such that either $e \neq \{v_1, v_2\}$ but $c(e) = c(\{v_1, v_2\})$, or $e \neq \{v_4, v_5\}$ but $c(e) = c(\{v_4, v_5\})$, or $e \neq \{v_6, v_7\}$ but $c(e) = c(\{v_6, v_7\})$, then $\{v_1, v_2\}$, $\{v_4, v_5\}$, and $\{v_6, v_7\}$ are uniquely colored under c and create a rainbow M_3. In either case we have $rb(T, M_3) = 5$.

Induction Hypothesis: For all trees T with $diam(T) = 4$, $7 \leq n \leq N$, and D is not a vertex cover of T, we have $rb(T, M_3) = \Delta + 2$.

Induction Step: Let T be a tree with $N + 1$ vertices and $diam(T) = 4$. Let $P_1 = (v_1, v_2, v_3, v_4, v_5)$ be a path of length 4 in T. Let $P_2 = (v_1, v_2, v_3, v_6, v_7)$ be a path of length 4 in T. Let $c : E(T) \to [\Delta + 2]$ be an exact coloring of the edges of T. By assumption, there exists $e \in E(T) \setminus E(P_1 \cup P_2)$ such that e is incident upon a leaf $v \in V(T)$. There are two cases; either there exists an edge that is in the same color class as e, or e is uniquely colored.

Case 1: Assume that there exists $e' \in E(T)$ such that $e' \neq e$ but $c(e) = c(e')$. If there exists $e' \in E(T)$ such that $e' \neq e$ but $c(e) = c(e')$ then delete e from T to form T'. Apply in the induction hypothesis to T' to get $rb(T', M_3) = \Delta + 2$. Therefore, $rb(T, M_3) = \Delta + 2$.

Case 2: Assume that there does not exist $e' \in E(T)$ such that $e' \neq e$ but $c(e) = c(e')$. This implies that e is uniquely colored under c in T. By assumption, $e = \{v, u\}$ where v is a leaf in T. Let T' be the graph given by deleting u and all the edges incident upon it. Since $deg(u) \leq \Delta$, we are guaranteed that c is at least an exact 2-coloring of T' when c is restricted to $E(T')$. Notice that $rb(T', M_2) = 2$ by Theorems 3.2, and 3.1. This implies that there exists a rainbow M_2 in T' such that $c(e') \neq c(e)$ for all $e' \in E(M_2)$. Therefore, the 3-matching given by $E(M_2) \cup \{\{e\}\}$ is a rainbow M_3 in T under c. Thus, $rb(T, M_3) = \Delta + 2$.

In either case we have $rb(T, M_3) = \Delta + 2$. Therefore, by mathematical induction we have proven Theorem 5.4. ■

Theorems 5.3 and 5.4 highlight a dichotomy between trees that are in some sense "nice" with respect to M_3 and tress that are not. The seemingly nice trees with respect to M_3 are the trees in which every edge is a part of some M_3 subgraph. Earlier in the section, we saw that smaller trees are generally not nice. Later in this section we will see that larger trees are nice. Unfortunately, we must spend more time in the gray zone by examining trees T with $diam(T) = 5$. Interestingly, the statements of the following two theorems are similar to those of the previous two.

Theorem 5.5. Let T be a tree with $diam(T) = 5$ with $n \geq 7$ vertices. Let $P = (v_1, v_2, v_3, v_4, v_5, v_6)$. If $D = \{v_2, v_3, v_5\}$ or $D = \{v_2, v_4, v_5\}$ is a vertex cover of T, then $rb(T, M_3) = \Delta + 3$ or $rb(T, M_3) = \Delta + 2$.

Proof. Let T be a tree with $diam(T) = 5$. Let $n \geq 6$ be arbitrary but fixed. Let $P = (v_1, v_2, v_3, v_4, v_5, v_6)$. Without loss of generality, assume that $D = \{v_2, v_3, v_5\}$ is a vertex cover of T. By Proposition 2.1, $rb(T, M_3) \geq \Delta + 2$. There are two cases; either $deg(v_5) = \Delta$, or $deg(v_5 \neq \Delta)$.

Case 1: Assume that $deg(v_5) = \Delta$. We will show that $rb(T, M_3) > \Delta + 2$. Let $c : E(T) \to [\Delta + 2]$ such that $c(e)$ is uniquely colored from 1 to Δ for e incident upon v_5, $c(\{v_2, v_3\}) = \Delta + 1$, and $c(e') = 2$ where e' is incident upon v_2 or v_3 but not both. Any M_3 contain an edge incident upon v_2 and a different edge incident upon v_3. Therefore, there does not exist a rainbow M_3 subgraph. Thus, $rb(T, M_3) > \Delta + 2$.

Let T' be created by deleting $\{v_2, v_3\}$ and $\{v_4, v_5\}$ from T. By Proposition 5.1, it suffices to show that $rb(T', M_3) = \Delta + 1$. Let V_i be the set of edges incident upon v_i in T'. Notice that $\Delta_{T'} = \Delta - 1$ and $T' = V_2 \cup V_3 \cup V_5$ where each component is a star. By Lemma 5.1 $rb(T', M_3) = \Delta_{T'} + 1 = \Delta$. Therefore, $rb(T, M_3) \leq \Delta + 3$. Thus, $rb(T, M_3) = \Delta + 3$.

Case 2: We will prove the remainder of the theorem by mathematical induction on $n \geq 7$ vertices.

Base Case: Let T be a tree with $diam(T) = 5$, and $n = 7$. Let $P = (v_1, v_2, v_3, v_4, v_5, v_6)$. Without loss of generality, assume that $D = \{v_2, v_3, v_5\}$ is a vertex cover of T. Assume that $deg(v_2) = \Delta > deg(v_5)$ or $deg(v_3) = \Delta > deg(v_5)$. This implies that $deg(v_3) = 3$. By inspection $rb(T, M_3) = 5$.

Induction Hypothesis: Let T be a tree with $diam(T) = 5$, and $7 \leq n \leq N$. Let $P = (v_1, v_2, v_3, v_4, v_5, v_6)$. Without loss of generality, $D = \{v_2, v_3, v_5\}$ is a vertex cover of T. Furthermore, $deg(v_2) = \Delta > deg(v_5)$ or $deg(v_3) = \Delta > deg(v_5)$. Then $rb(T, M_3) = \Delta + 2$.

Induction Step: Let T be a tree with $diam(T) = 5$, and $n = N + 1$. Let $P = (v_1, v_2, v_3, v_4, v_5, v_6)$. Without loss of generality, assume that $D = \{v_2, v_3, v_5\}$ is a vertex cover of T. Assume that $deg(v_2) = \Delta > deg(v_5)$ or $deg(v_3) = \Delta > deg(v_5)$. Let $c : E(T) \to [\Delta+2]$ be an exact coloring of the edges of T. Let $e \in E(T) \setminus E(P)$ such that e is incident upon a leaf. There are two cases; either there exists an edge in the same color class as e, or e is uniquely colored.

Case 2.1 Assume there exists $e' \in E(T)$ such that $e' \neq e$ but $c(e') = c(e)$. Delete e from T to form T'. If $\Delta_{T'} = deg(v_5)$, then $rb(T', M_3) = \Delta_{T'} + 3$ by the argument in case 1. However, because deleting e lowered the maximum degree of the graph, and preserved the exact coloring of c, this implies that $rb(T, M_3) = \Delta + 2$. If $deg(v_2) = \Delta_{T'} > deg(v_5)$ or $deg(v_3) = \Delta_{T'} > deg(v_5)$ holds, then we apply the induction hypothesis to get a rainbow M_3 in T' and, therefore, in T. Thus, $rb(T, M_3) = \Delta + 2$.

Case 2.2: Assume there does not exist $e' \in E(T)$ such that $e' \neq e$ but $c(e') = c(e)$. This means that e is uniquely colored under c. Let T' be the tree formed by deleting e and all edges adjacent to it. Since e is incident upon a leaf, it is adjacent to at most $\Delta - 1$ edges. Therefore, c when restricted to T' is at least an exact 2-coloring of the edges of T'. Notice that by Theorems 3.2 and 3.1, there exists a rainbow M_2 in T'. Let M_3 be the rainbow 3-matching in T under c given by the rainbow M_2 in T' and e. Therefore, $rb(T, M_3) = \Delta+2$.

In either case in induction, we have $rb(T, M_3) = \Delta+2$. Thus, by mathematical induction, if T is a tree with $diam(T) = 5$, such that T is dominated by $D = \{v_2, v_3, v_5\}$ or $D = \{v_2, v_4, v_5\}$ where $P = (v_1, v_2, v_3, v_4, v_5, v_6)$, and $deg(v_2) = \Delta > deg(v_5)$ or $deg(v_3) = \Delta > deg(v_5)$, then $rb(T, M_3) = \Delta + 2$.

Combining Case 1 and Case 2 shows that Theorem 5.5 holds true. ■

Corollary 5.1.2. Let T be a tree with $diam(T) = 5$ with $n \geq 7$ vertices. Let $P = (v_1, v_2, v_3, v_4, v_5, v_6)$. If $D = \{v_2, v_3, v_5\}$ or $D = \{v_2, v_4, v_5\}$ is a vertex cover of T. If $deg(v_5) = \Delta$ then $rb(T, M_3) = \Delta + 3$. Else $rb(T, M_3) = \Delta + 2$.

Theorem 5.6. Let T be a tree with $diam(T) = 5$. Let $P = (v_1, v_2, v_3, v_4, v_5, v_6)$. If $D = \{v_2, v_3, v_5\}$ or $D = \{v_2, v_4, v_5\}$ are not a vertex cover of T, then $rb(T, M_3) = \Delta + 2$.

Proof. By Proposition 2.1, $rb(T, M_3) \geq \Delta + 2$. We will proceed by induction on the number of vertices n. Notice that $n = 6$ and $n = 7$ are covered by Theorem 5.5. Therefore, our base case will start with $n = 8$. There are two base cases.

Base Case 1: Let T be a tree with $diam(T) = 5$ and $n = 8$. Let $P_1 = (v_1, v_2, v_3, v_4, v_5, v_6)$. Let $deg(v_3) = deg(v_4) = 3$. Let $c : E(T) \to [5]$ be an exact edge coloring of T. By Proposition 2.1, $rb(T, M_3) \geq \Delta + 2$. There are two cases; either there exists an edge $e \in E(T)$ in the same color class as $\{v_3, v_7\}$ or $\{v_4, v_8\}$, or $\{v_3, v_7\}$ and $\{v_4, v_8\}$ are both uniquely colored.

Case 1: Assume there exists $e \in E(T)$ such that either $e \neq \{v_3, v_7\}$ and $c(e) = c(\{v_3, v_7\})$ or $e \neq \{v_4, v_8\}$ and $c(e) = c(\{v_4, v_8\})$. Delete the either $\{v_3, v_7\}$ or $\{v_4, v_8\}$ accordingly, to form T'. By Theorem 5.5, there exists a rainbow M_3 in T'.

Case 2: Assume that $\{v_3, v_7\}$ and $\{v_4, v_8\}$ are both uniquely colored. Then $\{v_2, v_2\}$, $\{v_3, v_7\}$, and $\{v_4, v_8\}$ is a rainbow M_3 in T.

In either case, there exists a rainbow M_3 in T. Thus, $rb(T, M_3) = \Delta + 2$.

Base Case 2: Let T be a tree with $diam(T) = 5$ and $n = 8$. Let $P_1 = (v_1, v_2, v_3, v_4, v_5, v_6)$. Without loss of generality, let $P_2 = (v_1, v_2, v_3, v_7, v_8)$. Let $c : E(T) \to [5]$ be an exact edge coloring of T. By Proposition 2.1, $rb(T, M_3) \geq \Delta + 2$. There are two cases. Either there exists an edge in the same color class as $\{v_7, v_8\}$, or $\{v_7, v_8\}$ is uniquely colored.

Case 1: Without loss of generality, assume that there exists $e \in E(T)$ such that $e \neq \{v_7, v_8\}$ and $c(e) = c(\{v_7, v_8\}$. Delete $\{v_7, v_8\}$ from to T to form T'. By Theorem 5.5, $rb(T', M_3) = \Delta + 2$. Thus, $rb(T, M_3) = \Delta + 2$.

Case 2: Assume that there does not exist $e \in E(T)$ such that either $e \neq \{v_7, v_8\}$ and $c(e) = c(\{v_7, v_8\})$, or $e \neq \{v_1, v_2\}$ and $c(e) = c(\{v_1, v_2\})$. In this case, both $\{v_1, v_2\}$ and

$\{v_7, v_8\}$ are uniquely colored under c. Therefore, M_3 given by $\{v_5, v_6\}$, $\{v_1, v_2\}$, and $\{v_7, v_8\}$ is rainbow. Thus, $rb(T, M_3) = \Delta + 2$.

In both cases we have $rb(T, M_3) = \Delta + 2$. Thus, the base case has been demonstrated.

Induction Hypothesis: Let T be a tree with $diam(T) = 5$ and $8 \leq n \leq N$. Let $P_1 = (v_1, v_2, v_3, v_4, v_5, v_6)$. Without loss of generality, let $P_2 = (v_1, v_2, v_3, v_7, v_8)$. Then $rb(T, M_3) = \Delta + 2$.

Induction Step: Let T be a tree with $diam(T) = 5$ and $n = N + 1$. Let $P_1 = (v_1, v_2, v_3, v_4, v_5, v_6)$. Without loss of generality, let $P_2 = (v_1, v_2, v_3, v_7, v_8)$. By Proposition 2.1, $rb(T, M_3) \geq \Delta + 2$. Let $c : E(T) \to [\Delta + 2]$ be an exact edge coloring of T. Let $e \in E(T) \setminus E(P_1 \cup P_2)$ such that $e = \{v, u\}$ where v is a leaf in T. There are two cases; either there exists an edge in the same color class as e, or e is uniquely colored.

Case 1: There exists $e' \in E(T)$ such that $e' \neq e$ but $c(e') = c(e)$. In this case, delete e from T to form T'. By the induction hypothesis there exists a rainbow M_3 in T'. Therefore, there exists a rainbow M_3 in T. Thus, $rb(T, M_3) = \Delta + 2$.

Case 2: There does not exist $e' \in E(T)$ such that $e' \neq e$ and $c(e') = c(e)$. This means that e is uniquely colored under c. Let T' be the tree created by deleting e and all edges adjacent to it. Notice that c is at least an exact 2-coloring of the edges of T' when it is restricted to $E(T')$. Therefore, by Theorems 3.2 and 3.1, there exists a rainbow M_2 in T'. Thus, M_3 given by $E(M_2) \cup \{e\}$ is rainbow in T under c. That is to say, $rb(T, M_3) = \Delta + 2$.

In either case we have $rb(T, M_3) = \Delta + 2$. Thus, Theorem 5.6 has been proven by mathematical induction. ■

Theorem 5.7. Let T be a tree with $diam(T) \geq 6$. Then $rb(T, M_3) = \Delta + 2$.

Proof. Let T be a tree with $diam(T) \geq 6$. By Proposition 2.1, $rb(T, M_3) \geq \Delta + 2$. We will show that $rb(T, M_3) = \Delta + 2$ by induction on the number of vertices, n.

Base Case: Let P_7 be the base case. $rb(P_7, M_3) = \Delta + 2$ by inspection.

Induction Hypothesis: If T is a tree with $diam(T) \geq 6$ and with $7 \leq n \leq N$, then $rb(T, M_3) = \Delta + 2$.

Induction Step: Let T be a tree with $diam(T) \geq 6$ and $N+1$ vertices. Since T has more than 7 vertices, there exists $e \in T$ such that $diam(T-e) \geq 6$. Let v be a leaf of T and e be the edge incident upon it. Let $c : E(T) \to [\Delta+2]$ be an exact coloring of the edges of T. There are two cases; either there exists an edge in the same color class as e, or e is uniquely colored.

Case 1: There exists $e' \in E(T)$ such that $e' \neq e$ and $c(e) = c(e')$. In this case, delete e and v from T to form $T' \subset T$ with N vertices. By construction, the maximum degree of T' is less than or equal to the maximum degree of T. Now, c is an exact $\Delta+2$-coloring of T' where Δ is the maximum degree of T. Therefore, by the induction hypothesis, there exists a rainbow M_3 in T'. Thus, $rb(T, M_3) = \Delta + 2$.

Case 2: There does not exist $e' \in E(T)$ such that $e \neq e'$ and $c(e) = c`(e')$. That is to say that the color of e is unique under c. Delete all the edges adjacent to e in T to from T' where T' the subgraph of T that does not contain e or its endpoints. By construction T' is at least 2 colored under c. Therefore, by Theorems 3.2 and 3.1, there exists a rainbow M_2 in T'. Thus, we can choose a rainbow M_3 subgraph of T, namely the rainbow $M_2 \subset T'$ and e. Thus, $rb(T', M_3) = \Delta + 2$.

Thus, by mathematical induction, if T is a tree with $diam(T) \geq 6$, then $rb(T, M_3) = \Delta+2$.
■

6. Rainbow Numbers of Large Graphs with Respect to 3-Matchings

Interestingly, the argument for Theorem 5.7 is roughly generalizable to graphs with diameter length 6 or more. However, we must select the edge for deletion in a different manner than we did in the proof for Theorem 5.7.

Theorem 6.1. Let G be a graph. If $diam(G) \geq 6$, then $rb(G, M_3) = \Delta + 2$.

Proof. Let G be a graph with $diam \geq 6$. By Proposition 2.1, $rb(G, M_3) \geq \Delta + 2$. We will prove the upper bound for Theorem 6.1 by induction on the number of edges m.

Base Case: Let $G = P_7$. By Theorem 5.7, $rb(G, M_3) = \Delta + 2$.

Induction Hypothesis: Let G be a graph with $6 \leq m \leq M$ edges and $diam(G) \geq 6$. Then $rb(G, M_3) = \Delta + 2$.

Induction Step: Let G be a graph with $M + 1$ edges and $diam(G) \geq 6$. If G is a tree, then by Theorem 5.7, $rb(G, M_3) = \Delta + 2$. Therefore, we will assume that G is not a tree. That is, there exists $C \subseteq G$, where C is a cycle. Let $c : E(G) \to [\Delta + 2]$ be an exact edge coloring of G. There are two cases; either there exists an edge in the same color class as e, or e is uniquely colored.

Case 1: Assume there exists $e' \in E(G)$ and $e \in E(C)$ such that $e' \neq e$ but $c(e') = c(e)$. Because e is in a cycle, $G' = G - e$ is still connected; in fact, $diam(G') \geq 6$. Furthermore, c is an exact $[\Delta + 2]$-coloring of the edges of G' when c is restricted to $E(G')$. Therefore, by the induction hypothesis, there exists a rainbow M_3 in G'. Thus, there exists a rainbow M_3 in G.

Case 2: Assume that there does not exist $e' \in E(G)$ and $e \in E(C)$ such that $e' \neq e$ but $c(e') = c(e)$. Every edge $e \in E(C)$ is uniquely colored. Choose an $e \in E(C)$. Let G' be the subgraph of G created by deleting e and all the edges adjacent to it. There are two subcases.

Case 2.1: Assume that c when restricted to $E(G')$ is at least an exact 2-coloring; without loss of generality, say G' is colored with [2]. Either there exists a rainbow M_2 in G', in which case $M_3 = M_2 \cup e$ is a rainbow in G under c; or there does not exist a rainbow M_2 subgraph. Assume we are in the latter case. Since G' is 2-colored and does not have a rainbow M_2, $diam(G') \leq 3$. Since $diam(G) \geq 6$, $diam(G') \geq 3$ by construction. Thus, G' is a graph with $diam(G') = 3$ and $D = \{v_1, v_2\}$ is a vertex cover of G' for some $\{v_1, v_2\} \in E(G')$. This, in combination with the fact that every edge in C is uniquely colored, implies that there exists a rainbow M_2 in $E(G) \setminus E(G')$ such that $c(g) \notin [2]$ for some $g \in E(M_2)$. Therefore, $M_3 = M_2 \cup e$ for some $e \in E(G')$ is rainbow in G.

Case 2.2: Assume that c when restricted to $E(G')$ is monochromatic color 1. This implies that there are $\Delta + 1$ colors used on the edges adjacent to e. Therefore, there is a rainbow $M_2 \subset E(G) \setminus E(G')$, such that $c(e') \neq 1$ where $e' \in E(M_2)$. Since $diam(G) \geq 6$ there exists an edge $g \in E(G')$ such that $v \notin g$ for all $v \in V(M_2)$. Therefore, $M_3 = g \cup M_2$ is rainbow subgraph of G.

In either case we have a rainbow $M_3 \subset G$. Therefore, by mathematical induction $rb(G, M_3) = \Delta + 2$, proving Theorem 6.1. ■

7. Further Research

Areas of further research include classifying the rest of the rainbow numbers of graphs with respect to 3-matchings. In particular, this would involve proving the rainbow numbers of disconnected graphs. Furthermore, high matching numbers are also an interesting extension to this paper. In particular, we have the following conjecture.

Conjecture 7.1. *Let G be a graph with $diam(k * 2)$ where $k \geq 4$. Let D be a set of $k-2$ vertices of G, such that the order of the set, D_E, containing all edges incident to a vertex in D is maximized. Then $rb(G, M_k) = |D_E| + 2$.*

This conjecture comes from the briefly discussed notion of "nice" graphs where all edges are contained in the desired subgraph. Furthermore, this conjecture is motivated by a generalized version of Proposition 2.1.

References

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