CMU Mathematical Sciences
21-259 Sample Prerequisite Waiver Exam Booklet #2

Instructions:

- **On the Answer Sheet, clearly print your name in the Name field and your Andrew ID in the ID field. Darken the appropriate circle in the Version field (see the header above).**
- **Do not open the exam until told to do so.**
- **Read all problem statements carefully.**
- **No books, notes, calculators, or other electronic devices, including phones, earbuds/headphones, or tehcnology-enabled eyewear. Make sure your phone will not ring during the exam. Any use of technology is considered an Academic Integrity Violation.**
- **If you would like to leave the exam room before turning in your exam,**
  1. **bring your Exam Booklet and Answer Sheet to the front of the classroom and leave them on the table.**
  2. **demonstrate to the exam proctor that you are not taking any technology with you as you leave the room.**
- **The back side of each page, and the last page, can be used for scratch work. Do not put any scratch work on the Answer Sheet.**
- **If you need any additional paper for scratch work during the exam, please raise your hand.**
- **Time limit: 90 minutes.**
- **You may leave when you are finished if you finish before time is up.**
- **You must turn in ALL materials when you leave, including any scratch paper you were given.**
- **You must show your CMU ID card when you turn in your exam, Answer Sheet, and scratch paper.**
- **You must answer 10 of 15 questions correctly to earn the prerequisite waiver for 21-259.**

For each item, select the appropriate choice and darken the appropriate circle on your answer sheet completely. "None Of These" should be selected when none of the given choices is the correct answer.

Note: all angles are given in radian measure.

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1. Let $A=(1,0,-1)$, $B=(2,2,0)$, and $C=(0,1,3)$ be three points in space. Which of the following points lies in the *same plane* as A, B, and C?

   A. $(1,4,5)$

   B. $(3,1,-2)$

   C. None of these choices

   D. $(4,0,-8)$

   E. $(2,3,1)$

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2. Let C be the curve of intersection of the plane $x+y+z=2$ and the elliptic cylinder $\frac{x^2}{4}+\frac{y^2}{9}=1$. Which of the following is a smooth parametrization of the entire curve C?

   A. $\mathbf{r}(t)=\langle 2\cos t, 3\sin t, 2\cos t+3\sin t\rangle$

   B. $\mathbf{r}(t)=\langle 2\cos t, 3\sin t, 2-2\cos t-3\sin t\rangle$

   C. $\mathbf{r}(t)=\langle \cos t, \sin t, 2-\cos t-\sin t\rangle$

   D. None of these choices

   E. $\mathbf{r}(t)=\langle 2\cos t, 3\sin t, 2+2\cos t+3\sin t\rangle$

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3. Determine which of the following parametrizations is parametrized *with respect to arc length.*

   A. $\mathbf{r}(t)=\left\langle \frac{t^2}{2}, t, \frac{t^2}{2}\right\rangle$

   B. $\mathbf{r}(t)=\left\langle \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}, 0\right\rangle$

   C. $\mathbf{r}(t)=\langle \cos t, \sin t, t\rangle$

   D. None of these choices

   E. $\mathbf{r}(t)=\langle \cos(2t), \sin(2t), 0\rangle$

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4. Evaluate the limit $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4y^2}{x^8+y^4}$.

   A. None of these choices

   B. ∞

   C. $\frac{1}{2}$

   D. Does Not Exist

   E. 0

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5. Let $f(x,y) = 2\sqrt{x} - y^2$. Find the directional derivative of f at the point $(1,4)$ in the direction of the vector $\mathbf{v} = \langle 3, -4 \rangle$.

   A. None of these choices

   B. -5

   C. $\left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$

   D. $\langle 1, -8 \rangle$

   E. 7

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6. Find the equation of the tangent plane to the graph $z = f(x,y)$ at the point $(1,0,1)$.

$$f(x,y) = \sqrt{x^2+y^2} + \sin(xy)$$

   A. None of these choices

   B. $z = x + y + 1$

   C. $z = x + y$

   D. $z = x - y$

   E. $z = x + 2y$

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7. Let $f(x,y) = x^3 - 3xy + y^3$. Which of the following statements is true?

   A. $(0,0)$ is a local minimum of $f(x,y)$.

   B. $(0,0)$ is a local maximum of $f(x,y)$.

   C. $(0,0)$ is not a critical point of $f(x,y)$.

   D. $(0,0)$ is a saddle point of $f(x,y)$.

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8. Let C be the curve of intersection of the surfaces $xy = 1$ and $y^2 + z^2 = 1$. Determine the minimum value of $f(x,y,z) = yz + xy$ on C.

   A. $-\frac{1}{2}$

   B. 0

   C. None of these choices

   D. 1

   E. $\frac{1}{2}$

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9. Evaluate $\displaystyle\int_0^2 \int_{y\sqrt{3}}^{\sqrt{16-y^2}} \sqrt{x^2+y^2}\,dx\,dy$.

   A. $\dfrac{32\pi}{3}$

   B. None of these choices

   C. $\dfrac{64\pi}{3}$

   D. $\dfrac{128\pi}{9}$

   E. $\dfrac{64\pi}{9}$

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10. Let E be the solid region inside the sphere $x^2+y^2+z^2 \le 9$ and above the cone $z=\sqrt{x^2+y^2}$. Assume E has constant density. Find the centroid (the center of mass) of E.

    A. $\left(0,0,\dfrac{9(2+\sqrt{2})}{8}\right)$

    B. None of these choices

    C. $\left(0,0,\dfrac{9\sqrt{2}}{16}\right)$

    D. $\left(0,0,\dfrac{9(2+\sqrt{2})}{16}\right)$

    E. $\left(0,0,\dfrac{9(2-\sqrt{2})}{16}\right)$

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11. Let C be the curve parametrized by $\mathbf{r}(t)=\langle \cos t, \cos^2 t, \sin t\rangle$ for $0\le t\le 2\pi$. Evaluate

$$\int_C -z\,dx + z\,dy + x\,dz.$$

    A. π

    B. None of these choices

    C. 0

    D. 2π

    E. -2π

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12. Let C be the curve parametrized by $\mathbf{r}(t) = \langle t-1,\ e^{t^4},\ t^2+1 \rangle$, for $0 \le t \le 1$, and let

$$\mathbf{F}(x,y,z) = \left\langle e^{-x}\ln y,\ -\frac{e^{-x}}{y},\ z \right\rangle.$$

Calculate $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$.

A. 0

B. $\dfrac{3}{2}$

C. None of these choices

D. 1

E. $\dfrac{5}{2}$

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13. Let C be the positively oriented rectangle with vertices $(0,0)$, $(1,0)$, $(1,2)$, and $(0,2)$. Let

$$\mathbf{F}(x,y) = \left\langle -e^{xy} + x^2 y,\ \frac{x^3}{3} \right\rangle.$$

Compute $\displaystyle\oint_C \mathbf{F}\cdot d\mathbf{r}$.

A. $\dfrac{e^2-1}{2}$

B. $\dfrac{e^2-3}{2}$

C. e^2-3

D. None of these choices

E. $\dfrac{3-e^2}{2}$

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14. Let S be the surface defined by $z = 10 - x^2 - y^2$ with $z \ge 1$, oriented upward. Let

$$\mathbf{F}(x,y,z) = \langle 2xyz - 5y,\ x^2 z,\ e^x \cos(yz) \rangle.$$

Calculate $\displaystyle\iint_S \nabla \times \mathbf{F}\cdot d\mathbf{S}$.

A. 90π

B. None of these choices

C. 45π

D. 9π

E. 20π

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15. Let S be the closed surface consisting of the frustum of the cone $z^2 = x^2 + y^2$ for $1 \le z \le 2$, the bottom disk $z = 1$, $x^2 + y^2 \le 1$, and the top disk $z = 2$, $x^2 + y^2 \le 4$, oriented outward. Let

$$\mathbf{F}(x, y, z) = \langle 2y + e^z,\ \sin(xz^2),\ z(x^2 + y^2) \rangle.$$

Find the flux of $\mathbf{F}$ across S.

A. $\dfrac{31\pi}{10}$

B. $\dfrac{31\pi}{20}$

C. 0

D. None of these choices

E. $\dfrac{31\pi}{5}$

SCRATCH PAPER: You may detach this page from the exam, but this and all other scratch paper you use must be turned in when you leave. Raise your hand if you need any additional scratch paper during the exam.