CMU Mathematical Sciences
21-122: Integration and Approximation
Prerequisite Waiver Exam Topic List

Intended use of this document: This document provides a list of objectives that students are expected to know and demonstrate mastery of on the 21-122 Prerequisite Waiver Exam. This is a comprehensive list, and all questions on the exam will be represented by one or a combination of several of these objectives.

Note: the use of any and all technology, including calculators, is prohibited on the exam (without approved disability accommodations).

- Be able to use standard integration formulas to evaluate definite and indefinite integrals involving polynomials, rational functions, root functions, trigonometric functions (excluding hyperbolic trig functions), inverse trigonometric functions, exponential functions, and logarithmic functions.
- Be able to use u-substitution to rewrite indefinite and definite integrals in a form that can be evaluated.
- Be able to use integration by parts to evaluate indefinite and definite integrals.
- Be able to use trigonometric identities to simplify and evaluate integrals involving products of powers of trig functions and products of trig functions with different angles.
- Be able to use the technique of Trigonometric Substitution to simplify/rewrite integrals involving the radicals $\sqrt{a^2 - x^2}$, $\sqrt{x^2 + a^2}$, $\sqrt{x^2 - a^2}$.
- Be able to correctly perform a partial fraction decomposition of a rational integrand and use the decomposition to evaluate the integral of the rational function.
- Understand the meaning of convergence and divergence of an improper integral. Be able to identify discontinuities and asymptotes of given integrands that lead to improper integrals. Be able to evaluate improper integrals, including evaluating limits of indeterminate forms that may arise. This may require applications of L'Hospital's Rule.
- Know the values of p such that the integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges and diverges.
- Be able to use the Comparison Theorem correctly to determine the convergence/divergence behavior of an improper integral without evaluating the integral.
- Be able to use the Midpoint Rule and Trapezoid Rule to approximate the value of a particular integral numerically. Understand and be able to apply the theorems on error bounds for basic numerical integration rules.
- Be able to determine when sequences of real numbers converge or diverge. Be able to find the pattern representing the nth term a_n of a given sequence. Understand how the Squeeze Theorem and the Monotone Convergence Theorem can guarantee convergence of a given sequence.

- Understand the definitions related to infinite series, including convergence and divergence of the series. Know how the sequence of partial sums $\{s_n\}$ of the series can indicate convergence or divergence of the series, and how the sum of the series can be computed from the sequence of partial sums.
- Know the general form of a geometric series, be able to determine when a geometric series converges or diverges, and determine the sum of a convergent geometric series.
- Be able to apply the Test for Divergence to determine if a series diverges.
- Be able to apply the Integral Test to determine convergence or divergence of a series. Use the Remainder Estimate for the Integral Test to find upper and lower bounds on the nth remainder $R_n = s - s_n$ of a convergent series.
- Be able to apply the (Direct) Comparison Test and the Limit Comparison Test to determine whether a given series converges or diverges.
- Be able to apply the Alternating Series Test to determine when a given alternating series converges. Use the Alternating Series Estimation Theorem to find an upper bound on the nth remainder $R_n = s - s_n$ of a convergent alternating series.
- Be able to apply the Ratio Test and the Root Test to determine the convergence or divergence of a series. Understand the meaning of absolute convergence and conditional convergence.
- Understand the definition of a power series. Be able to determine the radius of convergence and the interval of convergence of a given power series using the Ratio or Root tests, being sure to check the endpoints of the interval of convergence if applying one of those tests. Know the three possible cases of the radius of convergence for a power series.
- Know how to represent $f(x) = (1-x)^{-1}$ as a power series and how to use this series to find power series representations of similar functions. Know how to combine power series by addition/subtraction and by multiplication. Be able to find the intersection of multiple intervals of convergence when combining power series.
- Be able to compute derivatives and integrals of a given power series to find power series representations of related functions.
- Know the definitions of Taylor and Maclaurin series for a general function $f(x)$ that is infinitely differentiable. Be able to find the Taylor/Maclaurin series of a given function, and be able to find the Taylor/Maclaurin series of a composite function based on an existing series.
- Be able to find a Taylor/Maclaurin polynomial of a given degree for a function. Be able to use Taylor's Theorem with Remainder to find an upper bound on the error in using a Taylor/Maclaurin polynomial to approximate a function value near the center. Know the conditions for convergence of a Taylor/Maclaurin series to the function it represents.
- Know how to find the first several terms of a binomial series representing $(1+x)^k$ for some real number k.

- Know how to use Newton's Method to find a root of a nonlinear function or a solution to a nonlinear equation. Understand how Newton's Method solves for the next approximation to the root from the previous approximation. Be able to set up Newton's Method for a given function $f(x)$. Be able to explain situations that might cause Newton's method to fail or diverge away from the root.

- Be able to find the arc length of a portion of a given curve using integration. Be able to find the area of a surface of revolution using integration.

- Understand the basics of differential equations, including how to find the order of a differential equation and how to verify that a given function is a solution to the differential equation. Be able to distinguish between general solutions and particular solutions of a differential equation. Understand what an initial value problem is.

- Be able to solve separable first-order differential equations and initial value problems. Be able to construct a separable differential equation model of a given scenario and solve the resulting problem.

- Be able to graph and algebraically represent functions as parametric curves. Be able to identify typical parametric curves such as those of a line or a circle. Be able to take a parametric curve and eliminate the parameter.

- Be able to calculate derivatives and definite integrals of parametric curves. Be able to find the arc length of a parametric curve.

- Be able to locate points in a plane via polar coordinates. Be able to convert points, functions, and equations from Cartesian to polar coordinates and vice-versa. Be able to sketch graphs of polar equations and identify symmetries in polar graphs.

- Be able to compute areas and arc lengths via integrals of polar curves.