

GENERAL TOPOLOGY BASIC EXAM: JANUARY 2026

Hand in solutions to *exactly five* of the following questions. All questions carry equal weight.

- (1) Define the following properties of a topological space X :
 - (a) X is T_3 .
 - (b) X is T_4 .
 - (c) X is paracompact.

Prove that every paracompact Hausdorff space is T_3 .
- (2) Let X be a topological space, let E be an equivalence relation on X , let Y be the set of equivalence classes, and let $\pi : X \rightarrow Y$ be the map which takes each $x \in X$ to its E -equivalence class $[x]$. Equip Y with the quotient topology (that is to say, the strongest topology for which π is continuous).
 Assume that the map π is open, that is to say $\pi[U]$ is open in Y for all open $U \subseteq X$. Prove that the following are equivalent:
 - (a) Y is Hausdorff.
 - (b) $\{(x, y) : xEy\}$ is closed in the product topology on $X \times X$.
- (3) We will say that subsets M and N of a space X are *separated* if and only if $cl(M) \cap N = M \cap cl(N) = \emptyset$.
 - (a) Prove that the following are equivalent for a subspace Y of a space X :
 - (i) Y is connected.
 - (ii) For all M, N such that $Y = M \cup N$ where M and N are separated subsets of X , either $M = \emptyset$ or $N = \emptyset$.
 - (b) Fill in the following outline of a proof that if A is a connected subset of a connected space X and C is a connected component of $X \setminus A$ then $X \setminus C$ is connected. You may assume when filling in a step that all the preceding steps have been completed, even if you did not fill them in. Assume that $X \setminus C = M \cup N$ where M and N are separated subsets of X .
 - (i) Without loss of generality, $A \cap M = \emptyset$.
 - (ii) $C \subseteq C \cup M \subseteq X \setminus A$.
 - (iii) If $C \cup M$ is connected then $M = \emptyset$.
 - (iv) Assume that $C \cup M = E \cup F$ where E and F are separated.
 Then:
 - (A) Without loss of generality, $C \cap E = \emptyset$.
 - (B) $E \subseteq M$.
 - (C) $X = E \cup F \cup N$.
 - (D) E and $F \cup N$ are separated.
 - (E) At least one of E and F is empty.
 - (v) $C \cup M$ is connected.
 - (vi) $M = \emptyset$.
 - (vii) $X \setminus C$ is connected.
- (4) Let S^1 be the unit circle in $\mathbb{R}^2$, that is $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ equipped with the subspace topology.

- (a) Outline a proof that for all $b \in S^1$, $\pi_1(S^1, b) \simeq (\mathbb{Z}, +)$. You don't need to give the details but should show that you understand the main ideas.
- (b) Let $h : S^1 \rightarrow S^1$ be a continuous map such that $h(-b) = -h(b)$ for all $b \in S^1$, where $-(x, y) = (-x, -y)$. Assume that $h(b_0) = b_0$, and let ψ be the group homomorphism from $\pi_1(S^1, b_0)$ to $\pi_1(S^1, b_0)$ induced by h . Prove that ψ maps any generator of $\pi_1(S^1, b_0)$ to an odd power of that generator.
- (5) A Lindelöf space is a space where every open cover has a countable subcover. Prove that if X is compact and Y is Lindelöf then $X \times Y$ is Lindelöf.
- (6) Let X be a topological space.
 - (a) State carefully what it means for X to be $T_{3.5}$ (equivalently a Tychonov space).
 - (b) Prove that every subspace of a compact Hausdorff space K is $T_{3.5}$.
 - (c) Given a $T_{3.5}$ space X , explain how to construct a compact Hausdorff space K such that X is homeomorphic to a subspace of K . You should define K clearly and give the main ideas in the proof that X embeds into K .