

TOPOLOGY BASIC EXAM: FALL 2025

- (1) Define the concept of a *connected subset* of a topological space X .
 - (a) Prove that if we define a binary relation on X by “ xEy if and only if there is a connected $C \subseteq X$ with $x, y \in C$ ” then E is an equivalence relation on X .
 - (b) Prove that the equivalence classes of E are connected closed sets.
 - (c) Prove that:
 - (i) $\mathbb{R} \setminus \mathbb{Q}$ is not a connected subset of $\mathbb{R}$.
 - (ii) $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is a connected subset of $\mathbb{R}^2$. Hint: Let $x, y \in \mathbb{R}^2 \setminus \mathbb{Q}^2$ with $x \neq y$, let L be the perpendicular bisector of the line segment xy , and consider sets of the form $xz \cup zy$ for $z \in L$.
- (2) Define the following properties of a topological space: regular, second countable, metrisable.
 - (a) Prove that the following are equivalent:
 - (i) X is regular.
 - (ii) For every point $x \in X$, every neighbourhood of x contains a closed neighbourhood of x .
 - (b) State the Urysohn metrisation theorem.
 - (c) Let $n > 0$. Let X be a second countable Hausdorff space such that for every $x \in X$ there is an open neighbourhood U of x such that U is homeomorphic to $\{x \in \mathbb{R}^n : \|x\| < 1\}$. Prove that X is metrisable.
- (3) Define the concepts of *net*, *subnet*, *ultranet*, *limit of a net*, *cluster point of a net*.
 - (a) Prove Tychonoff’s theorem using nets. You may assume that X is compact if and only if every net has a convergent subnet.
 - (b) Prove that the following are equivalent:
 - (i) X is T_1 .
 - (ii) For every point x in X , every net in X which is constant with value x has x as its only limit.
- (4) Define the concept of a *covering map* between two topological spaces.
 - (a) Prove that if $p : E \rightarrow X$ is a covering map, $\gamma : [0, 1] \rightarrow X$ is a path in X , $\gamma(0) = x$ and $p(e) = x$, there is a unique path $\delta : [0, 1] \rightarrow E$ in E such that $p\delta = \gamma$ and $\delta(0) = e$.
 - (b) Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and let $n > 0$. Let $f : S^1 \rightarrow S^1$ be given by $f(z) = z^n$. Prove that f is a covering map.
- (5) Define the concept of a Tychonoff ($T_{3.5}$) space.
 - (a) Describe the construction of the Stone-Ćech compactification $i : X \rightarrow \beta X$ of a Tychonoff space X (you don’t need to prove that it works).
 - (b) Prove that for every continuous map $f : X \rightarrow K$ from X to a compact Hausdorff space K , there is a unique continuous $g : \beta X \rightarrow K$ with $gi = f$.
 - (c) For which Tychonoff spaces X is i a homeomorphism between X and βX ?