Basic Exam in Set Theory
Friday, September 8, 2023

No books, no notes, no electronic devices.

Work alone.

Write your solutions in the blue books that are provided.

Write you name on each blue book you use.

Problem 1

Assume S is an infinite transitive set such that $S^{<\omega} \subseteq S$.

For each function $f : S \to S$, define

$$\mathcal{C}_f = \{X \mid X \preccurlyeq (S, \in, f)\}.$$

Reminders about notation

$(S, \in, f)$ is the structure $(S, \{(a, b) \in S^2 \mid a \in b\}, f)$.

$X \preccurlyeq (S, \in, f)$ means that

- $X \neq \emptyset$ and $f[X] \subseteq X$ so that $(X, \in, f \restriction X)$ is a structure, and
- $(X, \in f \restriction X)$ is an elementary substructure of $(S, \in, f)$.

Part 1 (15 points)

Suppose that, for each $a \in S$, we are given a function

$$f_a : S \to S.$$

Let

$$\mathcal{D} = \{X \subseteq S \mid \forall a \in X \ (\ X \in \mathcal{C}_{f_a}\ )\}.$$

Prove there exists a function

$$g : S \to S$$

such that

$$\mathcal{C}_g \subseteq \mathcal{D}.$$

Part 2 (10 points)

Suppose we are given a function

$$h : \mathcal{P}(S) \to S$$

such that, for every nonempty $X \subseteq S$,

$$h(X) \in X.$$

Prove there are $a \in S$ and $\mathcal{E} \subseteq \mathcal{P}(S)$ such that

(1) $h(X) = a$ for every $X \in \mathcal{E}$ and

(2) $\mathcal{E} \cap \mathcal{C}_f \neq \emptyset$ for every $f : S \to S$.

Suggestion: Use part 1 to solve part 2.

Problem 2

Weak Diamond (*WD*) is defined to be the principle:

$$\forall S : {}^{<\omega_1}2 \to 2 \quad \exists F : \omega_1 \to 2 \;\; \forall G : \omega_1 \to 2$$
$$\{\alpha < \omega_1 \mid S(G \restriction \alpha) = F(\alpha)\} \text{ is stationary in } \omega_1.$$

Part 1 (5 points)

Give the definition of $\diamondsuit$.

Part 2 (10 points)

Prove that $\diamondsuit$ implies WD.

Part 3 (10 points)

Prove that WD implies $2^{\aleph_0} < 2^{\aleph_1}$.

Suggestion: For contradiction, assume $2^{\aleph_0} = 2^{\aleph_1}$ and WD.

Problem 3

Suppose that $\varphi(\kappa, S, T)$ and $\psi(\kappa, S, T)$ are formulas such that

$$\mathrm{ZFC} \vdash \forall\, \kappa, S, T \;\; (\; \varphi(\kappa, S, T) \;\; \longleftrightarrow \;\; S \textit{ and } T \textit{ are trees on } \omega \times \kappa \;)$$

and

$$\mathrm{ZFC} \vdash \forall\, \kappa, S, T \;\; (\; \psi(\kappa, S, T) \;\; \longleftrightarrow \;\; (\; \varphi(\kappa, S, T) \;\; \wedge \;\; \mathrm{proj}([S]) \cap \mathrm{proj}([T]) = \emptyset \;)\;).$$

Suppose that $M \subseteq N$ are transitive class models of ZFC.

Part 1 (5 points)

Prove that $\varphi(\kappa, S, T)$ is absolute between M and N.

Part 2 (5 points)

Prove that $\psi(\kappa, S, T)$ is downward absolute from N to M.

Part 3 (15 points)

Prove that $\psi(\kappa, S, T)$ is upward absolute from M to N.

Additional instructions for all three parts below

You may cite without proof any of the main absoluteness theorems that were covered in 21-602. To do this, name the theorem or state it, then convince the reader that it applies each time you use it.

Problem 4 (25 points)

Prove that L is a class model of GCH.

Here is all that you may assume:

- ZFC theorems from 21-602 that are not specifically about L.
- There is a finite subtheory T of ZF − P so that the syntax and semantics of first order logic are absolute for transitive class models of T. In particular, the class relation

  $$\{(A, \ulcorner\varphi\urcorner, \overline{x}) \mid \overline{x} \in A^{<\omega} \text{ and } A \models \varphi(\overline{x})\}$$

  is absolute for transitive class models of T.
- L is a transitive class model of ZFC.

Suggestions

Start by writing a complete outline, then provide enough details for each step to convince the reader that you know the arguments.

Good first steps would be to provide the definition of the class function $\alpha \mapsto L_\alpha$ and to discuss its absoluteness.