Basic Exam in Set Theory, September 9, 2022, 6:30-9:30pm

Mechanics

You may use your device to view the 21-602 Google Drive folder during the exam. Work alone without consulting anything else or anyone else. Our Zoom session must run with your camera on without interruption from beginning to end. Once the exam has started, you may communicate with me via the chat function on Zoom. Should there be a technical issue, call me at 412-414-5194. Immediately after the exam, you will have a reasonable amount of time to create and submit a PDF file with your solutions.

Scoring

Each of the ten problems is worth 10 points. Partial credit might be awarded but the bar is high. 70 points or more is guaranteed to pass.

General instructions

You may use facts that were established in 21-602 as long as you cite and apply them correctly. To use a result from 21-602, you must also say why its hypotheses hold.

Most problems ask for an explanation or proof. What you write must convince me that 1) you have all the right ideas and 2) you see how they fit together and there are no gaps. You must avoid incomplete or vague writing. But you do not want to overexplain an obvious step while overlooking more complicated details or not leaving enough time for other problems. Find the right balance!

Specifically about notation

Given a formula $\varphi(\overline{v})$ and $\overline{x}$ from a set M, I would understand what you mean if you write either $M \models \varphi(\overline{x})$ or $\varphi(\overline{x})^M$ but you really mean

> $\mathrm{TRUTH}(M, r, \overline{x}) = 1$ *for the* $r \in \mathrm{FORMULA}$ *that is the code for* φ.

This formalism is not what you are being tested on here.

Problem 1

Assume V = L.

Let S be the set of ordinals $\beta < \omega_1$ for which there exists $\gamma < \omega_1$ such that L_γ satisfies the theory

ZFC - P + There are exactly three infinite cardinals

and

$$\beta = (\omega_2)^{L_\gamma}.$$

Prove that S is unbounded in ω_1.

Problem 2

Assume V = L and define S as in problem 1.

Prove that S is not stationary in ω_1.

Problem 3

Let α be the first ordinal such that L_α is a model of $\mathrm{ZFC} - \mathrm{P}$.

Let β be the second ordinal such that L_β is a model of $\mathrm{ZFC} - \mathrm{P}$.

Explain why $\mathcal{P}(\omega) \cap L_\beta \not\subseteq L_\alpha$.

Problem 4

Assume $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ is a $\diamondsuit$ sequence. Prove there exists a sequence

$$\langle B_\theta \mid \theta \in \lim(\omega_1) \rangle$$

that satisfies both of the following conditions.

(1) For every $\theta \in \lim(\omega_1)$, B_θ is an unbounded subset of θ.

(2) For every unbounded subset X of ω_1,

$$\{\theta \in \lim(\omega_1) \mid B_\theta \subseteq X\}$$

is a stationary subset of ω_1.

Problem 5

Assume that $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ is a $\diamondsuit$ sequence.

For each $X \subseteq \omega_1$, define

$$S_X = \{\alpha < \omega_1 \mid A_\alpha = X \cap \alpha\}.$$

Let T be the set of limit ordinals $\beta < \omega_1$ such that $\sup(A_\beta) = \beta$ but, for every limit ordinal $\alpha < \beta$, if $A_\alpha = A_\beta \cap \alpha$, then $\sup(A_\alpha) < \alpha$.

Prove that T is stationary and $T \cap S_X$ is nonstationary for all $X \subseteq \omega_1$.

Problem 6

Prove there is a sequence $\langle \mathcal{F}_\theta \mid \theta \in \lim(\omega_2) \rangle$ with the following property.

For every $\theta \in \lim(\omega_2)$,

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(1) $|\mathcal{F}_\theta| \leq \omega_2$ and

(2) for every $C \in \mathcal{F}_\theta$,

  (a) C is a club subset of θ,

  (b) $\mathrm{type}(C) \leq \omega_1$ and

  (c) for every $\eta \in \lim(C)$,

$$C \cap \eta \in \mathcal{F}_\eta.$$

Problem 7

True or false?

> *For every countable transitive model M of* ZFC − P*,*
> Σ^1_2 *formulas are downward absolute from* V *to* M*.*

Explain your answer.

Problem 8

Explain why the class of ordinal definable sets, OD, is Σ_2^{ZF}.

Your solution must include a formula written in a mix of mathematical notation and English, an explanation why it is a defining formula for OD, and an explanation why it is Σ_2. Convince me that you understand the answer by emphasizing the main points. (Take another look at the cover page.)

Problem 9

Suppose that ZFC is consistent.

Is there a bounded formula $\varphi(x)$ so that the theory

$$\mathrm{ZFC} + \mathrm{OD} = \{x \mid \varphi(x)\}$$

is also consistent?

Explain your answer.

Problem 10

True or false? Explain your answer.

> *If α is an ordinal such that V_α is a model of* ZF*,*
> *then α is uncountable.*