NOTE: THIS EDITION WAS THE FIRST TO BE "OPEN BOOK" IN A CERTAIN SENSE.
(THE GOOGLE DRIVE FOLDER MENTIONED ON THE COVER CONTAINED ALL COURSE MATERIALS.)
THE MECHANICS WERE DESCRIBED IN ADVANCE TO THOSE SIGNED UP FOR THE EXAM.
FUTURE BASIC EXAMS NEED NOT HAVE THE SAME RULES AND INSTRUCTIONS..

Basic Exam in Set Theory, January 26, 2022, 6:30-9:30pm

Mechanics

You may use your device to view the 21-602 Google Drive folder during the exam. Work alone without consulting anything else or anyone else.

Our Zoom session must run with your camera on without interruption from beginning to end. Once the exam has started, you may communicate with me via the chat function on Zoom. Should there be a technical issue, call me at XXX-XXX-XXXX. Immediately after the exam, you will have a reasonable amount of time to create and submit a PDF file with your solutions.

Scoring

Each part of each problem is worth 10 points. There are 13 parts. Partial credit might be awarded but the bar is high. 85 points is enough to pass.

General instructions

You may use facts that were established in 21-602 as long as you cite and apply them correctly. To use a result from 21-602, you must also say why its hypotheses hold.

Most problems ask for an explanation or proof. What you write must convince me that 1) you have all the right ideas and 2) you see how they fit together and there are no gaps. You must avoid incomplete or vague writing. But you do not want to overexplain an obvious step while overlooking more complicated details or not leaving enough time for other problems. Find the right balance!

The proofs I have in mind fit in the space provided, one page per part.

Specifically about notation

Given a formula $\varphi(\overline{v})$ and $\overline{x}$ from a set M, I would understand what you mean if you write either $M \models \varphi(\overline{x})$ or $\varphi(\overline{x})^M$ but you really mean

> $\mathrm{TRUTH}(M, r, \overline{x}) = 1$ *for the* $r \in \mathrm{FORMULA}$ *that is the code for* φ.

This formalism is not what you are being tested on here.

Problem 1

Assume V = L.

Let D be the set of $\delta < \omega_1$ such that

$$L_\delta \models \mathrm{ZFC} - \mathrm{P}$$

and

$$L_\delta \models \text{There are exactly three infinite cardinals.}$$

For each $\delta \in D$, define

$$\alpha_\delta = (\aleph_0)^{L_\delta},$$

$$\beta_\delta = (\aleph_1)^{L_\delta}$$

and

$$\gamma_\delta = (\aleph_2)^{L_\delta}.$$

Let

$$A = \{\alpha_\delta \mid \delta \in D\},$$

$$B = \{\beta_\delta \mid \delta \in D\}$$

and

$$C = \{\gamma_\delta \mid \delta \in D\}.$$

Part 1

In each column of the following table, mark the cell that corresponds to the unique correct answer and leave the remaining cells unmarked.

	A	B	C	D
club				
contains a club but not closed				
stationary and costationary				
unbounded but nonstationary				
bounded				

Problem 1, part 2

Prove your answer for column C of the previous table.

Problem 1, part 3

Explain why there are $\delta < \delta'$ in D so that $\beta_\delta = \beta_{\delta'}$ and $\gamma_\delta = \gamma_{\delta'}$.

Problem 1, part 4

Explain why there are $\delta < \delta'$ in D so that $\beta_\delta = \beta_{\delta'}$ and $\gamma_\delta < \gamma_{\delta'}$.

Problem 1, part 5

Explain why if $\delta < \delta'$ are the two least members of D, then $\delta < \beta_{\delta'}$.

Problem 2

Assume that $\langle A_\alpha \mid \alpha < \omega_1 \rangle$ is a $\diamondsuit$ sequence.

For each $X \subseteq \omega_1$, define

$$S_X = \{\alpha < \omega_1 \mid A_\alpha = X \cap \alpha\}.$$

Let T be the set of limit ordinals $\beta < \omega_1$ such that $\sup(A_\beta) = \beta$ but, for every limit ordinal $\alpha < \beta$, if $A_\alpha = A_\beta \cap \alpha$, then $\sup(A_\alpha) < \alpha$.

Prove that T is stationary and $T \cap S_X$ is nonstationary for all $X \subseteq \omega_1$.

Problem 3

True or false?

> *For every countable transitive model M of* $\mathrm{ZFC} - \mathrm{P}$,
> Σ^1_2 *formulas are downward absolute from* V *to* M.

Explain your answer.

Problem 4, part 1

Explain why the class function $\alpha \mapsto V_\alpha$ is Δ_2^{ZF}.

(Use a mix of mathematical notation and English to write two informal formulas, one Σ_2, the other Π_2, each of which defines the class function. Then add appropriate comments.)

Problem 4, part 2

Explain why the class function $\alpha \mapsto V_\alpha$ is not Δ_1^{ZFC}.

(Remember what is written on the front page. Do not be vague.)

Problem 4, part 3

Explain why the class of ordinal definable sets, OD, is Σ_2^{ZF}.

(Use a mix of mathematical notation and English to write an informal Σ_2 formula that defines the class. Then add appropriate comments.)

(Remember what is written on the front page about notation.)

Problem 5

Assume μ is a strongly inaccessible cardinal and $\langle \mathcal{F}_\alpha \mid \alpha < \mu \rangle$ is a sequence with the following properties:

1) For every $\alpha < \mu$,

$$|\mathcal{F}_\alpha| \leq |\alpha|.$$

2) For every $X \subseteq \mu$, there is a club $C \subseteq \mu$ such that, for every $\alpha \in C$,

$$X \cap \alpha \in \mathcal{F}_\alpha.$$

Prove that μ is not a measurable cardinal.

Problem 6

Assume that μ is a measurable cardinal. Let S be a stationary subset of μ. Prove there exists a strongly inaccessible cardinal $\lambda < \mu$ such that $S \cap \lambda$ is a stationary subset of λ.

Problem 7

True or false? Explain your answer.

> *If α is an ordinal such that V_α is a model of* ZF*,*
> *then α is uncountable.*