Basic Exam in Set Theory, February 8, 2021
Public version

Mechanics

[Omitted]

Instructions about using facts from 21-602

For problems 1–6, you may use facts that were proved in 21-602 if you 1) correctly and completely quote the fact you wish to use and 2) explain why the fact applies in your situation. In particular, do not repeat proofs of facts established in 21-602.

Problem 7 is different because it asks you to repeat the proof of a specific theorem from 21-602.

Problem 1

Let

$$\mathrm{P} = \{\alpha \mid \alpha \text{ is a positive ordinal}\}.$$

Define a class function

$$\mathrm{F} : \mathrm{P} \to \mathrm{P}$$

by letting $\mathrm{F}(\delta)$ be the least $\tau \leq \delta$ such that $\delta = \alpha + \tau$ for some $\alpha < \delta$.

Is the following true or false? Prove your answer.

$\mathrm{F}(\mathrm{F}(\delta)) = \mathrm{F}(\delta)$ for every $\delta \in \mathrm{P}$.

Problem 2

Let φ be the sentence in the language of set theory that says:

"Every set is countable."

Assume $\mathrm{V} = \mathrm{L}$.

Let

$$A = \{\, \delta < \omega_1 \mid L_\delta \models \mathrm{ZFC} - \mathrm{P} + \varphi \,\}.$$

Which of the following is true? Give an entirely convincing argument.

1) A is not stationary in ω_1.

2) Both A and $\omega_1 - A$ are stationary in ω_1.

3) A contains a club subset of ω_1 but is not club in ω_1.

4) A is club in ω_1.

Problem 3

Is the following true or false? Give an entirely convincing argument.

Every transitive model of $\mathrm{ZFC} - \mathrm{P}$ is Σ^1_2 correct.

Problem 4

Let μ be a measurable cardinal.

Is the following true or false? Give a detailed proof.

> There is a family $\mathcal{F} \subseteq \mathcal{P}(\mu)$ such that
> $$|\mathcal{F}| > \mu$$
> but, for every infinite cardinal $\lambda < \mu$,
> $$|\{X \cap \lambda \mid X \in \mathcal{F}\}| \leq \lambda.$$

Problem 5

Prove there is an uncountable cardinal λ such that $\mathrm{HOD}^{V_\lambda} = \mathrm{HOD} \cap V_\lambda$.

Problem 6

[Extension to problem 5, ultimately treated as "extra credit".]

Problem 7

In 21-602, you saw the proof of the Solovay splitting theorem:

> Let μ be a regular uncountable cardinal and S be a stationary subset of μ. Then there is a partition of S into μ many stationary subsets.

Repeat the proof.