Basic Exam in Set Theory for January 14, 2020

Problem 1 (10 points)

Let $\mathcal{F}$ be an uncountable collection of finite subsets of ω_1.

Prove there exists an uncountable $\mathcal{G} \subseteq \mathcal{F}$ and a finite set R such that, for all $X, Y \in \mathcal{G}$, if $X \neq Y$, then $X \cap Y = R$.

Problem 2 (20 points)

Let κ be an uncountable regular cardinal. Let $\lambda = \kappa^+$.

Prove there exists a sequence $\langle f_\alpha \mid \alpha < \lambda \rangle$ such that

(1) each f_α is a function from κ to κ,

(2) for all $\alpha < \beta < \lambda$,
$$\{\eta < \kappa \mid f_\alpha(\eta) < f_\beta(\eta)\}$$
contains a club in κ, and

(3) for all $\beta < \lambda$ and $g : \kappa \to \kappa$, if
$$\{\eta < \kappa \mid g(\eta) < f_\beta(\eta)\}$$
is stationary in κ, then there exists $\alpha < \beta$ such that
$$\{\eta < \kappa \mid g(\eta) = f_\alpha(\eta)\}$$
is stationary in κ.

Hint: If $\kappa \leq \alpha < \lambda$, then there is a wellordering W_α of κ with type α.

Problem 3 (10 points)

Part 1

Consider countable elementary substructures $X_0 \prec X_1 \prec H(\omega_1)$ with $X_0 \in X_1$. Say $X_0 \simeq M_0$ and $X_1 \simeq M_1$ where M_0 and M_1 are transitive sets. Is $M_0 \prec M_1$? Prove your answer.

Part 2

Consider countable elementary substructures $Y_0 \prec Y_1 \prec H(\omega_2)$ with $Y_0 \in Y_1$. Say $Y_0 \simeq N_0$ and $Y_1 \simeq N_1$ where N_0 and N_1 are transitive sets. Is $N_0 \prec N_1$? Prove your answer.

Problem 4 (20 points)

Suppose there is a sequence $\langle \mathcal{F}_\alpha \mid \alpha < \omega_1 \rangle$ such that

- for every $\alpha < \omega_1$, $\mathcal{F}_\alpha$ is a countable subfamily of $\mathcal{P}(\alpha)$ and
- for every $X \subseteq \omega_1$, $\{\alpha < \omega_1 \mid X \cap \alpha \in \mathcal{F}_\alpha\}$ is stationary in ω_1.

Prove $\diamondsuit$ holds.

Problem 5 (40 points)

Let $\omega \leq \kappa < \omega_1 = \lambda$. Assume that

$$\big(\,\kappa \text{ is a cardinal and } \kappa^+ = \lambda\,\big)^L.$$

Outline a proof that there exists a wellordering (A, R) such that

- A is a $\boldsymbol{\Delta^1_2}$ subset of ${}^\omega\omega$,
- R is a $\boldsymbol{\Delta^1_2}$ subset of ${}^\omega\omega \times {}^\omega\omega$ and
- $\operatorname{type}(A, R) = \omega_1$.

Delta^1_2 should be Sigma^1_2 everywhere in this problem.

Remark: $\boldsymbol{\Delta^1_2}$ is boldface, whereas Δ^1_2 is lightface. Recall that

$$\boldsymbol{\Delta^1_2} = \bigcup_{z \in {}^\omega\omega} \Delta^1_2(z).$$

Hint and remark about expectations: The solution to Problem 5 involves generalizations of facts about L to $L[S]$ for $S \subseteq \omega$. You may not assume anything about $L[S]$. Provide the definition, decide which facts must be generalized from L to $L[S]$, accurately state what you believe to be true, and outline the proofs. Highlight the main points and say enough to convince me you could provide the remaining details if pressed. Among other things, this is an opportunity to show you understand the basic theory of L.

In the literature, you might have seen $L(S)$ and $L[S]$. These have different definitions but turn out to be the same when $S \subseteq \omega$. Problem 5 assumes you have seen neither before. In particular, you cannot lose points for mismatching the definitions.

You must also show how to use the theory of $L[S]$ to solve Problem 5.

Problem 5 is worth a lot of points. Earn them!