Basic Exam in Set Theory
September 3, 2019

Note: You may request elaborations on notation but not hints.

Problem 1 (10 points)

Let λ be an uncountable regular cardinal. Let

$$\langle A_\alpha \mid \alpha < \lambda \rangle$$

and

$$\langle B_\alpha \mid \alpha < \lambda \rangle$$

be two sequences of subsets of λ such that

$$\{A_\alpha \mid \alpha < \lambda\} = \{B_\alpha \mid \alpha < \lambda\}.$$

Prove there exists a set C that is closed and unbounded in λ and

$$C \cap \triangle_{\alpha<\lambda} A_\alpha = C \cap \triangle_{\alpha<\lambda} B_\alpha.$$

Reminder about notation: $\triangle$ is the diagonal intersection operator.

Problem 2 (20 points)

Let λ be an uncountable cardinal. Prove the following are equivalent.

(1) λ is a strongly inaccessible cardinal.

(2) For every $0 < \kappa < \lambda$ and sequence $\langle A_\alpha \mid \alpha < \kappa \rangle$ of subsets of λ, there exists $\langle B_\alpha \mid \alpha < \kappa \rangle$ such that

  (a) $\bigcap_{\alpha<\kappa} B_\alpha$ has cardinality λ and

  (b) for every $\alpha < \kappa$, either $B_\alpha = A_\alpha$ or $B_\alpha = \lambda - A_\alpha$.

Remark on terminology: Please use the phrase

$\langle B_\alpha \mid \alpha < \kappa \rangle$ is a flip of $\langle A_\alpha \mid \alpha < \kappa \rangle$

to refer to property (2)(b) in your solution.

Hint for (1) implies (2):

Let $\mathcal{F}$ be the family of flips of $\vec{A}$. Prove that $\lambda = \bigcup_{\vec{B}\in\mathcal{F}} \bigcap_{\alpha<\kappa} B_\alpha$.

Problem 3 (40 points)

Assume $V = L$. Let κ be an infinite cardinal and $\lambda = \kappa^+$. For each ordinal α such that $\kappa < \alpha < \lambda$, let $h(\alpha)$ be the least $\eta > \alpha$ such that

$$L_\eta \models \text{ZFC} - \text{P} + \text{“There exists a surjection from } \kappa \text{ onto } \alpha.\text{”}$$

Define

$$\mathcal{F}_\alpha = \mathcal{P}(\alpha) \cap L_{h(\alpha)}$$

and

$$\mathcal{G}_\alpha = \mathcal{P}(\alpha) \cap L_{h(\alpha)+1}.$$

(1) Prove that $|\mathcal{G}_\alpha| = \kappa$ whenever $\kappa < \alpha < \lambda$.

(2) Consider an arbitrary $A \subseteq \lambda$.

(a) Prove there exists an ordinal α such that $\kappa < \alpha < \lambda$ and

$$A \cap \alpha \in \mathcal{F}_\alpha.$$

(b) Prove there is a club subset C of λ so that, for every $\alpha \in C$,

$$A \cap \alpha \in \mathcal{F}_\alpha.$$

(c) Prove there is a club subset C of λ so that, for every $\alpha \in C$,

$$A \cap \alpha \in \mathcal{F}_\alpha$$

and

$$C \cap \alpha \in \mathcal{G}_\alpha.$$

What you are allowed to use for Problem 3: You may cite the theorem that L is a model of ZFC + GCH and specific facts and lemmas about L and $<_L$ that went into the proof of this theorem in 21-602 in Fall, 2018. For example, you may state the Condensation Lemma and simply write, “This was proved in 21-602”. However, nothing about $\diamondsuit$ principles may be cited without definitions and proofs.

Hints and remarks regarding Part (2)

- Obviously, (2)(c) implies (2)(b) implies (2)(a).
- The proof I have in mind involves the cardinal $\mu = \lambda^+$ and certain elementary substructures $Y \prec H_\mu$.
- The proof that $\diamondsuit_\lambda$ holds in L, which was given in 21-602, shares ideas with the solution to Problem 3 but there are differences. The proof you saw of $\diamondsuit_\lambda$ is related but not the same!

Problem 4 (10 points)

Let M be a transitive class model of ZFC and T be a tree on ω such that $T \in M$. Prove that at least one of the following holds.

(1) $[T] \subseteq M$.

(2) There is a perfect subtree S of T such that $S \in M$.

Additional instructions for Problem 4: You may use machinery from the proof the Cantor Perfect Set Theorem but you must explain your notation.

Your solution must be sufficiently attentive to the difference between truth in V and truth in M. If you are claiming a statement is absolute, then you need to be precise about which statement is absolute and why it is absolute, citing results from 21-602 when appropriate.

Problem 5 (10 points)

Let M be a transitive class model of ZFC. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two structures of the same finite language, both of which belong to M. Assume that

$$M \models \text{The universe of } \mathfrak{A} \text{ is countable.}$$

Suppose that there is an elementary embedding from $\mathfrak{A}$ to $\mathfrak{B}$. Prove that there exists $\pi \in M$ such that π is an elementary embedding from $\mathfrak{A}$ to $\mathfrak{B}$.

Additional instruction for Problems 5: Your solution must be sufficiently attentive to the difference between truth in V and truth in M. If you are claiming a statement is absolute, then you need to be precise about which statement is absolute and why it is absolute, citing results from 21-602 when appropriate.

Problem 6 (10 points)

Let λ be a regular cardinal and

$$S \subseteq \{\alpha < \lambda \mid \alpha \text{ is a limit ordinal of uncountable cofinality}\}.$$

Assume that S is stationary in λ. Let

$$T = \{\alpha \in S \mid S \cap \alpha \text{ is not stationary in } \alpha\}.$$

Prove that T is stationary in λ.