SET THEORY BASIC EXAM: JANUARY 2017

Attempt four of the following six questions. All questions carry equal weight.

(1) Define the terms *cardinal*, *singular cardinal*, *regular cardinal*, *strong limit cardinal*. Prove that if κ is a singular strong limit cardinal then $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$. Prove that for any infinite cardinal κ, $\mathrm{cf}(2^\kappa) > \kappa$.

(2) State the Condensation Lemma for L, and give a brief outline of the proof.

Assuming that $V = L$, prove that:

(a) If X is countable with $X \prec L_{\omega_1}$, then $X = L_\alpha$ for some countable α.

(b) There is a countable X with $X \prec L_{\omega_2}$ which is not of the form L_α for any ordinal α.

(3) Define the terms *Aronszajn tree* and *Souslin tree*.

Assume that T is an ω_1-tree and there exists a function $f : T \to \omega$ such that for all s and t in T, $s <_T t \implies f(s) \neq f(t)$. Prove that:

(a) T is Aronszajn.

(b) T is not Souslin.

(4) Let S be a stationary set of countable limit ordinals, and let $(x_\alpha)_{\alpha \in S}$ be such that $x_\alpha \subseteq \alpha$ and x_α is cofinal in α with order type ω. Given $T \subseteq S$, say that a function g is *good for* T if $\mathrm{dom}(g) = T$, g is $1-1$ and $g(\alpha) \in x_\alpha$ for all $\alpha \in T$.

Prove that:

(a) If T is countable, there exists g which is good for T.

(b) If T is stationary, there exists no g which is good for T.

(c) If T is non-stationary, there exists g which is good for T.

(5) For $f, g \in {}^\omega\omega$ say that $f <^* g$ (f *is eventually dominated by* g) if there exists n such that $f(m) < g(m)$ for all $m > n$. A set $X \subseteq {}^\omega\omega$ is *unbounded* if there is no $g \in {}^\omega\omega$ such that $f <^* g$ for all $f \in X$, and *dominating* if for all $f \in {}^\omega\omega$ there exists $g \in X$ with $f <^* g$.

Let κ be the least cardinality of an unbounded set and λ the least cardinality of a dominating set. Prove that:

(a) κ is uncountable and regular.

(b) $\mathrm{cf}(\lambda) \geq \kappa$.

(6) State and prove the Reflection Theorem. Define the class HOD, and outline a proof that HOD is a transitive class model of ZFC set theory. Prove that $\aleph_\omega^V$ is a singular cardinal in HOD.