SET THEORY BASIC EXAM: SEPTEMBER 2016

Attempt four of the following six questions. All questions carry equal weight.

(1) Prove that $\diamondsuit_{\omega_1}$ implies the following guessing principle (diamond for functions): there is a sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ such that $f_\alpha : \alpha \to \alpha$, and for every $f : \omega_1 \to \omega_1$ there are stationarily many α such that $f \restriction \alpha = f_\alpha$. Outline some construction of a Souslin tree assuming $\diamondsuit_{\omega_1}$.

(2) Give the definition of the class HOD, and prove that it is a model of the Axiom of Replacement (you may assume the Reflection Theorem, but you must state it correctly).

(3) State Shoenfield's absoluteness theorem and outline the main steps in the proof. Prove that the assertion "there is a countable transitive model of ZFC" is Σ^1_2. We call this assertion ϕ. Prove that if M is a countable transitive model of ZFC with $On \cap M$ minimal then ϕ is false in M but true in V, and explain why this does not contradict Shoenfield's theorem.

(4) Prove that:

  (a) If κ is a singular cardinal and the function $\lambda \mapsto 2^\lambda$ is eventually constant for $\lambda < \kappa$ with eventual value μ, then $2^\kappa = \mu$.

  (b) The generalised continuum hypothesis (GCH) states that $2^\kappa = \kappa^+$ for all infinite cardinals κ. Assuming GCH prove that if κ and λ are infinite cardinals with $\lambda < cf(\kappa)$ then $\kappa^\lambda = \kappa$. What happens when $\lambda \geq cf(\kappa)$?

(5) Let $\theta > \omega_2$ be regular and let $<_\theta$ be a wellordering of H_θ. Prove that:

  (a) If $M \prec (H_\theta, \in, <_\theta)$ and M is countable then $M \cap \omega_1 \in \omega_1$, and furthermore $M \cap \omega_1 \in C$ for all $C \in M$ with C club in ω_1.

  (b) The following are equivalent for $S \subseteq \omega_1$:
  - S is stationary.
  - There is a countable $M \prec (H_\theta, \in, <_\theta)$ with $M \cap \omega_1 \in S$ and $S \in M$.

(6) Assume that $V = L$. For each $\alpha < \omega_1$, let $f(\alpha)$ be the least $\beta > \alpha$ (if it exists) such that α is countable in L_β. Prove that:

  (a) $f(\alpha)$ exists.

  (b) $f(\alpha)$ is a successor ordinal.

  (c) $f(\alpha) < \omega_1$.