SET THEORY BASIC EXAM: JANUARY 2015

Attempt four of the following six questions. All questions carry equal weight.

(1) State the combinatorial principle $\diamondsuit_{\omega_1}$. Prove that $\diamondsuit_{\omega_1}$ implies that $2^{\aleph_0} = \aleph_1$. Outline the main steps in the proof that if $V = L$ then $\diamondsuit_{\omega_1}$ holds.

(2) State the Reflection Theorem and outline the proof. Give the definition of the class HOD, and (assuming that it is a model of ZF) explain carefully why HOD is a model of AC.

(3) Prove that if f is a function with domain $\aleph_1$ such that $f(\alpha)$ is a finite subset of $\aleph_1$ for each α, then there is an uncountable set $S \subseteq \aleph_1$ such that $\beta \notin f(\alpha)$ whenever α and β are distinct elements of S.

(4) Prove that:
   (a) If κ is singular strong limit, $2^\kappa = \kappa^{\mathrm{cf}(\kappa)}$.
   (b) $\aleph_n^{\aleph_0} = \max\{\aleph_n, 2^{\aleph_0}\}$ for all finite n.

(5) Fill in the details of the following argument for the existence of an $\aleph_1$-Aronszajn tree.
   (a) There is a sequence $\langle f_\alpha : \alpha < \omega_1 \rangle$ such that f_α is an injective map from α to ω for all α, and $\{\gamma < \alpha : f_\alpha(\gamma) \neq f_\beta(\gamma)\}$ is finite for all α and β with $\alpha < \beta$. Hint: Construct the f_α inductively and maintain the hypothesis that $rge(f_\alpha)$ is coinfinite.
   (b) If T is the set of x such that $\mathrm{dom}(x) \in \omega_1$ and $\{\gamma \in \mathrm{dom}(x) : f_{\mathrm{dom}(x)}(\gamma) \neq x(\gamma)\}$ is finite, then T forms an ω_1-Aronszajn tree under end-extension.

(6) State and prove the Condensation Lemma. Use it to prove that if $V = L$ then GCH holds.