Sample Basic Qualifying Exam - Section Probability

Time: 2 hrs

1. Recite precisely the following definitions/facts/theorems/lemmas:
   (a) If $\mathcal{A}_k$ is a sequence of σ-fields then the asymptotic σ-field is defined as ...
   (b) Etemadi's strong law of large numbers
   (c) Lindebergh's CLT
   (d) Upcrossing inequality
   (e) Give three equivalent characterizations of uniform integrabilty
   (f) Cramer's Theorem (large deviations)

2. Give the statement and proof of **ONE** of the following theorems:
   (a) optional stopping thm for martingales with a.s. finite stopping times
   (b) classical CLT

   Solve **TWO** out of the three following problems:

3. The distribution μ with distribution function $F_\mu(x) = e^{-e^{-x}}$ is one example of the so-called extremal distributions.
   (a) Verify that F_μ is indeed a distribution function.
   (b) Let M_n be the running maximum of i.i.d. exponential variables with parameter $\lambda = 1$ i.e., $M_n := \max(X_1, X_2, ..., X_n)$ and $\forall x \geq 0,\ \mathbf{P}\big[\, X_k > x \,\big] = e^{-x}$. Show first $\overline{\lim}_{n\to\infty}(X_n/\log(n)) \leq 1$ a.s. and then $\lim_{n\to\infty} M_n/\log(n) \leq 1$ a.s.
   (c) In fact it can be seen that M_n is concentrated around $\log(n)$. Prove that in particular $M_n - \log(n)$ converges weakly to μ as $n \to \infty$. *Hint:* Don't use Fourier transforms.

4. Let T, S be stoppintg times w.r.t. a filtration $(\mathcal{F}_k)_{k\geq 0}$.
   (a) What is the relation between $\sigma(T)$ and $\mathcal{F}_T$? Explain.
   (b) Consider a random walk on the integers starting at 0. Let T be the hitting time of $[10, \infty)$ and S be the hitting time of $[5, \infty)$. Is the event $\{S = 15\}$ $\mathcal{F}_T$-measurable? (prove or disprove)
   (c) Show that $T \vee S$, the maximum of T and S, is a stopping time. Identify $\mathcal{F}_{T\vee S}$ in terms of $\mathcal{F}_S$ and $\mathcal{F}_T$. (After guessing, prove that your guess is correct.)

5. Let $X_1, X_2, ...$ be independent RVs and $S_n = X_1 + ... + X_n$. Suppose $P[X_k = -1] = P[X_k = 1] = (1 - 1/k^2)/2$ and $P[X_k = -k] = P[X_k = k] = (1/k^2)/2$.
   (a) Determine the asymptotics of the variance of S_n.
   (b) Based on this asymptotics conjecture (state but don't prove) a CLT for S_n.
   (c) Check whether the Lindeberg-Feller condition is satisfied.
   (d) Prove finally an appropriate CLT. Hint: Set $Y_k = \text{sign}(X_k)$ and note that $\sum_k P[X_k \neq Y_k] < \infty$. Then use Borel Cantelli. Recall that if $Z_n \xrightarrow{\text{w}} Z$ and $C_n \to 0$ a.s., then $(Z_n + C_n) \xrightarrow{\text{w}} Z$.