Basic examination: probability

Aug 30th, 2024

The exam is 180 minutes long

Problem 1 (8pts)**.** *Give definitions of*

- *Product probability space*
- *Convergence of a sequence of random variables* $(X_n)_{n=1}^{\infty}$ *to a random variable* X *almost surely*
- *Discrete time martingale*
- *Moment generating function*
- *Stopping time*

Problem 2 (8pts)**.** *State the following:*

- *Dynkin's* π-λ *theorem*
- *Jensen's inequality*
- *Kolmogorov's zero-one law*
- *The Tonelli (Fubini–Tonelli) theorem for non-negative functions on product spaces*
- *The Lévy continuity theorem*

Problem 3 (14pts)**.** *Prove the following simplified version of Wald's second identity. For every* n, *let* $S_n = y_1 + \cdots + y_n$, *where* $y_1, y_2, \ldots$ *are i.i.d with* $\mathbb{E}\, y_1 = 0$ *and* $\operatorname{Var} y_1 = 1$. *Further, let* T *be a stopping time (with respect to the filtration generated by* $y_1, y_2, \ldots$*) with* $\mathbb{P}\{T < M\} = 1$ *for some number* $M < \infty$. *Then* $\mathbb{E} S_T^2 = \mathbb{E} T$.

Problem 4 (14pts)**.** *Let* $(y_1, y_2, \ldots)$ *be a sequence of i.i.d non-negative random variables, with* $\mathbb{E}\, y_1 \leq 1$. *(a) Show that the sequence*

$$X_n = \prod_{i=1}^{n} y_i,\ n \geq 1; \quad X_0 := 1,$$

is a supermartingale with respect to the filtration generated by $y_1, y_2, \ldots$; *(b) Show that the sequence converges to zero almost surely unless* $y_1 = 1$ *a.s.*

Problem 5 (14pts)**.** *Let* $X_1, X_2, \ldots$ *be a sequence of mutually independent random variables of zero means and with the fourth moments uniformly bounded above by a constant* $M < \infty$. *Consider the sequence of averages*

$$S_n := \frac{1}{n} \sum_{i=1}^{n} X_i, \quad n \geq 1.$$

Prove that the sequence $(S_n)_{n \geq 1}$ *converges to zero a.s by estimating* $\mathbb{E}\, S_n^4$, $n \geq 1$.

Problem 6 (14pts)**.** *Let* X_1, X_2 *be i.i.d standard Gaussian variables. Is the difference* $|X_1| - |X_2|$ *a Gaussian variable? Hint: It may be helpful to recall that the characteristic function of a Gaussian variable with mean* μ *and variance* σ^2 *equals* $\exp(\mathbf{i} t\mu - \frac{1}{2}\sigma^2 t^2)$.

Problem 7 (14pts)**.** *Let* $X = (X_1, X_2, X_3)$ *be a random vector distributed on the unit Euclidean sphere* S^2 *according to the normalized surface area measure. Compute the probability density function of* X_3^2.

Problem 8 (14pts)**.** *Let* X *be a non-negative random variable, and assume that for* **every** *non-negative random variable* Y *with* $\mathbb{E}\, Y < \infty$ *(whether independent from* X *or not), we have* $\mathbb{E}\, XY < \infty$. *Prove that* X *is uniformly bounded, that is, there is a number* $M < \infty$ *such that* $\mathbb{P}\{X > M\} = 0$.