Problem 1 (10pts). Give definitions of
- Product probability measure for two probability spaces \((\Omega_1, \Sigma_1, P_1)\) and \((\Omega_2, \Sigma_2, P_2)\)
- Characteristic function of a random vector in \(\mathbb{R}^n\)
- Weak convergence of a sequence of random variables \((X_n)_{n=1}^{\infty}\) to a random variable \(X\)
- Discrete time martingale
- Mutual independence for a collection of sigma-fields

Problem 2 (15pts). State the following:
- The dominated convergence theorem
- Jensen’s inequality
- The Borel–Cantelli lemmas (first and second)
- The strong law of large numbers for a sequence of i.i.d random variables

Problem 3 (15pts). Prove the following concentration inequality of Azuma. Let \((\mathcal{F}_i)_{i=0}^{N}\) be a finite filtration on a probability space \((\Omega, \Sigma, P)\). Further, let \((X_i)_{i=0}^{N}\) be a martingale with respect to \((\mathcal{F}_i)_{i=0}^{N}\), and for each \(1 \leq i \leq N\), set
\[
d_i := \|X_i - X_{i-1}\|_{L^\infty} := \sup \{ t \geq 0 : P\{|X_i - X_{i-1}| \geq t\} > 0 \}.
\]
Then for every \(t > 0\) we have
\[
P\{|X_N - X_0| \geq t\} \leq 2 \exp \left( - \frac{t^2}{C \sum_{i=1}^{N} d_i^2} \right),
\]
where \(C > 0\) is a universal constant.

Problem 4 (15pts). Let \(X_1, X_2, \ldots\) be a sequence of independent identically distributed mean zero random variables such that for some number \(M > 0\), we have \(P\{|X_i| \geq M\} = 0, i = 1, 2, \ldots\). Apply Lindeberg’s replacement method to prove that the sequence of random variables
\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i
\]
converges in distribution to Gaussian random variable of mean zero and variance equal to \(\mathbb{E}X_i^2\) (the condition that the variables \(X_i\) are uniformly bounded is added here to simplify the proof).

Problem 5 (15pts). Let \(X_1, X_2, \ldots\) be a sequence of i.i.d random variables, each variable uniformly distributed on \([0, 1]\). Define events \(B_n := \{ X_n > \max_{1 \leq i \leq n-1} X_i \}, n \geq 2\).
- Prove that \(P(B_n) = 1/n\) for every \(n \geq 2\) (Hint: note that for every fixed permutation \(\pi\) of \(\{1, 2, \ldots, n\}\), the sequences \((X_1, X_2, \ldots, X_n)\) and \((X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)})\) are equidistributed).
- Prove that the events \(B_n, n \geq 2\), are pairwise independent (consider \(P(B_m | B_n)\) for \(m < n\)).
Apply the formula for the variance of a sum of pairwise independent variables and the Markov (Chebyshev) inequality to conclude that the sequence of random variables

\[ \frac{1}{\ln(n)} \sum_{i=2}^{n} 1_{B_i}, \quad n \geq 2, \]

converges to one in probability. Here, \( 1_{B_i} \) is the indicator of the event \( B_i \).

**Problem 6** (15pts). Let \((n_i)_{i=1}^{\infty}\) be a sequence of positive integers with \( \lim_{i} n_i = \infty \). Further, for each \( i \geq 1 \) let \( X_{i1}, X_{i2}, \ldots, X_{in_i} \) be mutually independent Bernoulli random variables, with \( \Pr\{X_{ij} = 1\} = p_{ij}, 1 \leq j \leq n_i \).

- Write down a formula for the characteristic functions of the random variables
  \[ \sum_{j=1}^{n_i} X_{ij} \sum_{j=1}^{n_i} p_{ij}, \quad i \geq 1. \]

- Apply the Lévy continuity theorem for the characteristic functions to prove that under the assumption \( \lim_{i \to \infty} \sum_{j=1}^{n_i} p_{ij} = +\infty \), the sequence
  \[ \sum_{j=1}^{n_i} X_{ij} \sum_{j=1}^{n_i} p_{ij}, \quad i \geq 1, \]
  converges to one in probability.

(You may need the approximation formulas \( \sin(\epsilon) = \epsilon + o(\epsilon) \), \( \cos(\epsilon) = 1 + O(\epsilon^2) \), \( 1 + \epsilon = \exp(\epsilon + o(\epsilon)) \)).

**Problem 7** (15pts). Let \( b_1, b_2, \ldots \) be a sequence of mutually independent Bernoulli\((1/2)\) random variables. Define variables \( X_1, X_2, \ldots \) inductively as follows:

\[ X_1 := 1; \quad X_k := \begin{cases} 2X_{k-1}, & \text{if } b_k = 1 \\ 1, & \text{otherwise} \end{cases} \quad k \geq 2. \]

Does the sequence

\[ \frac{X_k}{k^{1/2}}, \quad k \geq 1, \]

converge almost surely to a non-random limit? Justify.