Basic examination: probability

September 6th, 2023

The exam is 180 minutes long

Problem 1 (10pts). *Give definitions of*

- *Product probability measure for two probability spaces* $(\Omega_1, \Sigma_1, \mathbb{P}_1)$ *and* $(\Omega_2, \Sigma_2, \mathbb{P}_2)$
- *Characteristic function of a random vector in* $\mathbb{R}^n$
- *Weak convergence of a sequence of random variables* $(X_n)_{n=1}^{\infty}$ *to a random variable* X
- *Discrete time martingale*
- *Mutual independence for a collection of sigma-fields*

Problem 2 (15pts). *State the following:*

- *The dominated convergence theorem*
- *Jensen's inequality*
- *The Borel–Cantelli lemmas (first and second)*
- *The strong law of large numbers for a sequence of i.i.d random variables*

Problem 3 (15pts). *Prove the following concentration inequality of Azuma. Let* $(\mathcal{F}_i)_{i=0}^N$ *be a finite filtration on a probability space* $(\Omega, \Sigma, \mathbb{P})$. *Further, let* $(X_i)_{i=0}^N$ *be a martingale with respect to* $(\mathcal{F}_i)_{i=0}^N$, *and for each* $1 \le i \le N$, *set*

$$d_i := \|X_i - X_{i-1}\|_{L_\infty} := \sup\big\{t \ge 0 : \ \mathbb{P}\{|X_i - X_{i-1}| \ge t\} > 0\big\}.$$

Then for every $t > 0$ *we have*

$$\mathbb{P}\big\{\big|X_N - X_0\big| \ge t\big\} \le 2\exp\bigg(-\frac{t^2}{C\sum_{i=1}^N d_i^2}\bigg),$$

where $C > 0$ *is a universal constant.*

Problem 4 (15pts). *Let* $X_1, X_2, \dots$ *be a sequence of independent identically distributed mean zero random variables such that for some number* $M > 0$, *we have* $\mathbb{P}\{|X_i| \ge M\} = 0$, $i = 1, 2, \dots$. *Apply Lindeberg's replacement method to prove that the sequence of random variables*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i$$

converges in distribution to Gaussian random variable of mean zero and variance equal to $\mathbb{E}\,X_1^2$ *(the condition that the variables* X_i *are uniformly bounded is added here to simplify the proof).*

Problem 5 (15pts). *Let* $X_1, X_2, \dots$ *be a sequence of i.i.d random variables, each variable uniformly distributed on* $[0, 1]$. *Define events* $\mathcal{B}_n := \big\{X_n > \max\limits_{1 \le i \le n-1} X_i\big\}$, $n \ge 2$.

- *Prove that* $\mathbb{P}(\mathcal{B}_n) = 1/n$ *for every* $n \ge 2$ *(Hint: note that for every fixed permutation* π *of* $\{1, 2, \dots, n\}$, *the sequences* $(X_1, X_2, \dots, X_n)$ *and* $(X_{\pi(1)}, X_{\pi(2)}, \dots, X_{\pi(n)})$ *are equidistributed).*
- *Prove that the events* $\mathcal{B}_n$, $n \ge 2$, *are pairwise independent (consider* $\mathbb{P}(\mathcal{B}_m \,|\, \mathcal{B}_n)$ *for* $m < n$*).*

- *Apply the formula for the variance of a sum of pairwise independent variables and the Markov (Chebyshev) inequality to conclude that the sequence of random variables*

$$\frac{1}{\ln(n)} \sum_{i=2}^{n} \mathbf{1}_{\mathcal{B}_i}, \quad n \geq 2,$$

converges to one in probability. Here, $\mathbf{1}_{\mathcal{B}_i}$ *is the indicator of the event* $\mathcal{B}_i$*.*

Problem 6 (15pts)**.** *Let* $(n_i)_{i=1}^{\infty}$ *be a sequence of positive integers with* $\lim\limits_{i} n_i = \infty$*. Further, for each* $i \geq 1$ *let* $X_{i\,1}, X_{i\,2}, \ldots, X_{i\,n_i}$ *be mutually independent Bernoulli random variables, with* $\mathbb{P}\{X_{ij} = 1\} = p_{ij}$*,* $1 \leq j \leq n_i$*.*

- *Write down a formula for the characteristic functions of the random variables*

$$\frac{\sum_{j=1}^{n_i} X_{ij}}{\sum_{j=1}^{n_i} p_{ij}}, \quad i \geq 1.$$

- *Apply the Lévy continuity theorem for the characteristic functions to prove that under the assumption* $\lim\limits_{i \to \infty} \sum_{j=1}^{n_i} p_{ij} = +\infty$*, the sequence*

$$\frac{\sum_{j=1}^{n_i} X_{ij}}{\sum_{j=1}^{n_i} p_{ij}}, \quad i \geq 1,$$

converges to one in probability.

(You may need the approximation formulas $\sin(\varepsilon) = \varepsilon + o(\varepsilon)$*,* $\cos(\varepsilon) = 1 + O(\varepsilon^2)$*,* $1 + \varepsilon = \exp(\varepsilon + o(\varepsilon))$*).*

Problem 7 (15pts)**.** *Let* $b_1, b_2, \ldots$ *be a sequence of mutually independent Bernoulli(1/2) random variables. Define variables* $X_1, X_2, \ldots$ *inductively as follows:*

$$X_1 := 1; \quad X_k := \begin{cases} 2X_{k-1}, & \text{if } b_k = 1 \\ 1, & \text{otherwise} \end{cases} \qquad k \geq 2.$$

Does the sequence

$$\frac{X_k}{k^{100}}, \quad k \geq 1,$$

converge almost surely to a non-random limit? Justify.