2 Sep 2022

Basic examination: Probability

180 min.

20pts 1. State the following definitions, theorems and provide full proofs (if asked). Be precise.
   a) The Borel-Cantelli lemmas with proofs.
   b) Gaussian random vector in \( \mathbb{R}^n \), show it has independent components if and only if they are uncorrelated.
   c) Lindeberg’s central limit theorem in \( \mathbb{R} \).
   d) The characteristic function \( \phi_X \) of a random variable \( X \). Show that \( \phi_X''(0) \) exists if and only if \( E[X^2] < \infty \). Lévy’s continuity theorem.
   e) A martingale sequence, Doob’s decomposition, the quadratic variation process.

16pts 2. Let \( X_1, X_2, \ldots \) be integrable random variables with the same distribution. Let \( Y_k = X_k 1_{|X_k| \leq k} \) for \( k \geq 1 \).
   a) Suppose that \( \frac{Y_1 + \cdots + Y_n}{n} \) converges a.s. to some \( a \in \mathbb{R} \). Show that then \( \frac{X_1 + \cdots + X_n}{n} \) also converges a.s. to \( a \).
   b) Show that \( \sum_{k=1}^{\infty} \frac{1}{k^2} \text{Var}(Y_k) < \infty \).

16pts 3. Show that for a random variable \( X \) with characteristic function \( \phi \) and every \( t > 0 \), we have
\[
P( |X| > 2/t ) \leq \frac{1}{t} \int_{-t}^{t} [1 - \phi(x)] \, dx.
\]

16pts 4. Let \( (a_k)_{k \geq 1} \) be a bounded sequence in \( \mathbb{R} \). Let \( X_1, X_2, \ldots \) be independent random variables with \( P( X_k = \pm a_k ) = \frac{1}{2} \) for \( k \geq 1 \). Show that \( Y_n = \frac{\sum_{k=1}^{n} X_k}{\sqrt{\text{Var}(\sum_{k=1}^{n} X_k)}} \) converges in distribution.

16pts 5. Let \( M = (M_n)_{n \geq 0} \) be a martingale with \( M_0 = 0 \) such that \( E[M_n^2] < \infty \) for each \( n \). Show that on the event \( \{ |M|_\infty = +\infty \} \), we have \( \frac{M_n}{(M_0^2)} \to 0 \).

16pts 6. Let \( U_1, U_2, \ldots \) be i.i.d. random variables uniform on \([-1, 3]\). We define \( S_n = U_1 + \cdots + U_n \) and \( \tau = \inf \{ n \geq 1, U_n \geq 0 \} \). Find \( E[S_\tau] \) and \( E[(S_\tau - \tau)^2] \). (If you choose to apply Wald’s identities, please state them precisely along with proofs.)