2 Sep 2022 **CMU**

Basic examination: Probability

180 min.

20pts **1.** State the following definitions, theorems and provide full proofs (if asked). Be precise.

a) The Borel-Cantelli lemmas with proofs.

b) Gaussian random vector in $\mathbb{R}^n$, show it has independent components if and only if they are uncorrelated.

c) Lindeberg's central limit theorem in $\mathbb{R}$.

d) The characteristic function ϕ_X of a random variable X. Show that $\phi_X''(0)$ exists if and only if $\mathbb{E}X^2 < \infty$. Lévy's continuity theorem.

e) A martingale sequence, Doob's decomposition, the quadratic variation process.

16pts **2.** Let $X_1, X_2, \ldots$ be integrable random variables with the same distribution. Let $Y_k = X_k \mathbf{1}_{|X_k| \leq k}$ for $k \geq 1$.

(a) Suppose that $\frac{Y_1+\cdots+Y_n}{n}$ converges a.s. to some $a \in \mathbb{R}$. Show that then $\frac{X_1+\cdots+X_n}{n}$ also converges a.s. to a.

(b) Show that $\sum_{k=1}^{\infty} \frac{1}{k^2} \operatorname{Var}(Y_k) < \infty$.

16pts **3.** Show that for a random variable X with characteristic function ϕ and every $t > 0$, we have

$$\mathbb{P}\left(|X| > 2/t\right) \leq \frac{1}{t} \int_{-t}^{t} [1 - \phi(x)] \mathrm{d}x.$$

16pts **4.** Let $(a_k)_{k \geq 1}$ be a bounded sequence in $\mathbb{R}$. Let $X_1, X_2, \ldots$ be independent random variables with $\mathbb{P}\left(X_k = \pm a_k\right) = \frac{1}{2}$ for $k \geq 1$. Show that $Y_n = \frac{\sum_{k=1}^n X_k}{\sqrt{\operatorname{Var}(\sum_{k=1}^n X_k)}}$ converges in distribution.

16pts **5.** Let $M = (M_n)_{n \geq 0}$ be a martingale with $M_0 = 0$ such that $\mathbb{E}M_n^2 < \infty$ for each n. Show that on the event $\{\langle M \rangle_\infty = +\infty\}$, we have $\frac{M_n}{\langle M \rangle_n} \to 0$.

16pts **6.** Let $U_1, U_2, \ldots$ be i.i.d. random variables uniform on $[-1, 3]$. We define $S_n = U_1 + \cdots + U_n$ and $\tau = \inf\{n \geq 1,\ U_n \geq 0\}$. Find $\mathbb{E}S_\tau$ and $\mathbb{E}(S_\tau - \tau)^2$. (If you choose to apply Wald's identities, please state them precisely along with proofs.)