27 Jan 2022 **CMU**

Basic examination: Probability

<u>*180 min.*</u>

16pts **1.** State the following definitions, theorems and provide full proofs (if asked). Be precise.

a) a random variable, the law (distribution) of a random variable, the cumulative distribution function of a random variable,

b) π-system, λ-system, Dynkin's lemma,

c) show that the cumulative distribution function determines the law,

d) convergence in distribution, the (vanilla) central limit theorem,

e) a martingale sequence, Doob's convergence theorem.

10pts **2.** Let X and Y be independent standard Gaussian random variables. Let $\theta \in [0, 2\pi)$.

a) What is the density of the random vector $V = \begin{bmatrix} (\cos\theta)X + (\sin\theta)Y \\ -(\sin\theta)X + (\cos\theta)Y \end{bmatrix}$?

b) Are the components of V independent?

c) Find $\mathbb{P}(X < 100Y)$ and $\mathbb{E}[((\cos\theta)X + (\sin\theta)Y)^{10}]$.

16pts **3.** Suppose that a random variable X with variance one has the following property: $\frac{X+X'}{\sqrt{2}}$ has the same distribution as X, where X' is an independent copy of X. Show that X is standard Gaussian.

16pts **4.** Let $(X_n)_{n=0}^{\infty}$ be a martingale with $X_0 = 0$ and bounded increments: there is a constant $C > 0$ such that for every n, $|X_n - X_{n-1}| \leq C$. Let $u > 0$ and $\tau = \inf\{n \geq 0,\ X_n > u\}$.

(i) Show that τ is a stopping time and $X_{\tau\wedge n} \leq u + C$ for every n.

(ii) Show that on the event $\{\tau = +\infty\}$, we have "$\lim X_n$ exists and is finite".

(iii) Show the following dichotomy

$$\mathbb{P}\Big(\{\liminf X_n = -\infty,\ \limsup X_n = +\infty\} \cup \{\lim X_n \text{ exists and is finite}\}\Big) = 1.$$

10pts **5.** Let $(X_k)_{k=0}^n$, $(Y_k)_{k=0}^n$ be two martingale sequences (adapted to the same filtration) such that $\mathbb{E}X_n^2 < \infty$, $\mathbb{E}Y_n^2 < \infty$.

(i) Show that $\mathbb{E}X_k^2 < \infty$, $\mathbb{E}Y_k^2 < \infty$ for each $0 \leq k \leq n$.

(ii) Show that

$$\mathbb{E}X_nY_n - \mathbb{E}X_0Y_0 = \sum_{k=1}^{n} \mathbb{E}\Big[(X_k - X_{k-1})(Y_k - Y_{k-1})\Big].$$

16pts **6.** Let $\{X_{n,k}\}_{n\geq 1, 1\leq k\leq n}$ be a family of Bernoulli random variables such that for every $n \geq 1$, the variables $X_{n,1}, \ldots, X_{n,n}$ are independent and $\mathbb{E}\sum_{k=1}^n X_{n,k} \xrightarrow[n\to\infty]{} \lambda$ for some $\lambda \in (0, \infty)$ and $\max_{1\leq k\leq n} \mathbb{E}X_{n,k} \xrightarrow[n\to\infty]{} 0$. Show that $X_{n,1} + \cdots + X_{n,n}$ converges in distribution to a Poisson random variable with parameter λ.

16pts **7.** Let $X = (X_1, \ldots, X_n)$ be a random vector uniform in the cube $[-1, 1]^n$. Show that for every unit vector $(a_1, \ldots, a_n)$ in $\mathbb{R}^n$ and every $t > 0$, we have

$$\mathbb{P}\left(\left|\sum_{j=1}^{n} a_j X_j\right| > t\right) \leq 2\exp\left\{-\frac{3}{2}t^2\right\}.$$

Show that the constant $\frac{3}{2}$ in the exponent is best possible, that is the statement fails if $\frac{3}{2}$ is replaced with any $c > \frac{3}{2}$.