1. State the following definitions, theorems and provide full proofs (if asked). Be precise.
   a) a random variable, the law (distribution) of a random variable, the cumulative distribution function of a random variable,
   b) \( \pi \)-system, \( \lambda \)-system, Dynkin’s lemma,
   c) show that the cumulative distribution function determines the law,
   d) convergence in distribution, the (vanilla) central limit theorem,
   e) a martingale sequence, Doob’s convergence theorem.

2. Let \( X \) and \( Y \) be independent standard Gaussian random variables. Let \( \theta \in [0, 2\pi) \).
   a) What is the density of the random vector \( V \)?
   b) Are the components of \( V \) independent?
   c) Find \( \mathbb{P}(X < 100Y) \) and \( \mathbb{E}[(\cos \theta)X + (\sin \theta)Y] \).
   d) Find \( \mathbb{E}[(\cos \theta)X + (\sin \theta)Y]^{10} \).

3. Suppose that a random variable \( X \) with variance one has the following property: \( \frac{X + X'}{\sqrt{2}} \) has the same distribution as \( X \), where \( X' \) is an independent copy of \( X \). Show that \( X \) is standard Gaussian.

4. Let \( (X_n)_{n=0}^{\infty} \) be a martingale with \( X_0 = 0 \) and bounded increments: there is a constant \( C > 0 \) such that for every \( n \), \( |X_n - X_{n-1}| \leq C \). Let \( u > 0 \) and \( \tau = \inf\{n \geq 0, X_n > u\} \).
   (i) Show that \( \tau \) is a stopping time and \( X_{\tau \wedge n} \leq u + C \) for every \( n \).
   (ii) Show that on the event \( \{\tau = +\infty\} \), we have “\( \lim X_n \) exists and is finite”.
   (iii) Show the following dichotomy
   \( \mathbb{P}(\{\lim \inf X_n = -\infty, \lim \sup X_n = +\infty\} \cup \{\lim X_n \text{ exists and is finite}\}) = 1. \)

5. Let \( (X_k)_{k=0}^{n}, (Y_k)_{k=0}^{n} \) be two martingale sequences (adapted to the same filtration) such that \( \mathbb{E}X_n^2 < \infty, \mathbb{E}Y_n^2 < \infty \).
   (i) Show that \( \mathbb{E}X_k^2 < \infty, \mathbb{E}Y_k^2 < \infty \) for each \( 0 \leq k \leq n \).
   (ii) Show that
   \[ \mathbb{E}X_nY_n - \mathbb{E}X_n\mathbb{E}Y_n = \sum_{k=1}^{n} \mathbb{E}(X_k - X_{k-1})(Y_k - Y_{k-1}). \]

6. Let \( (X_{n,k})_{n \geq 1, 1 \leq k \leq n} \) be a family of Bernoulli random variables such that for every \( n \geq 1 \), the variables \( X_{n,1}, \ldots, X_{n,n} \) are independent and \( \mathbb{E}\sum_{k=1}^{n} X_{n,k} \xrightarrow{n \to \infty} \lambda \) for some \( \lambda \in (0, \infty) \) and \( \max_{1 \leq k \leq n} \mathbb{E}X_{n,k} \xrightarrow{n \to \infty} 0. \) Show that \( X_{n,1} + \cdots + X_{n,n} \) converges in distribution to a Poisson random variable with parameter \( \lambda \).

7. Let \( X = (X_1, \ldots, X_n) \) be a random vector uniform in the cube \([-1, 1]^n\). Show that for every unit vector \( (a_1, \ldots, a_n) \) in \( \mathbb{R}^n \) and every \( t > 0 \), we have
   \[ \mathbb{P}\left( \left| \sum_{j=1}^{n} a_j X_j \right| > t \right) \leq 2 \exp\left( -\frac{3}{2} t^2 \right). \]
   Show that the constant \( \frac{3}{2} \) in the exponent is best possible, that is the statement fails if \( \frac{3}{2} \) is replaced with any \( c > \frac{3}{2} \).