30 Aug 2021 **CMU**

Basic examination: Probability

180 min.

16pts **1.** State the following definitions and theorems. Be precise.

a) a discrete random variable, a continuous random variable, the cumulative distribution function (CDF),

b) the characteristic function of a random variable, Lévy's continuity theorem

c) a.s. convergence, the strong law of large numbers

d) Doob's convergence theorem, the law of the iterated logarithm (for random signs)

e) is there an atom-less random variable which is *not* continuous but whose CDF is continuous?

16pts **2.** Let $(X_n)_{n\geq 0}$ be a submartingale. Show that for every $t > 0$ and positive integer n, we have

$$\mathbb{P}\left(\max_{0\leq k\leq n} X_k \geq t\right) \leq \frac{1}{t}\mathbb{E}X_n^+.$$

Let $p > 1$. If additionally (X_n) is nonnegative, show that

$$\left(\mathbb{E}\max_{0\leq k\leq n} X_k^p\right)^{1/p} \leq \frac{p}{p-1}\left(\mathbb{E}X_n^p\right)^{1/p}.$$

10pts **3.** Show that there is a positive constant δ such that for every random variable X and every interval I in $\mathbb{R}$ of length at most δ, we have

$$\mathbb{P}\left(X \in I\right) \leq 2\int_{|t|\leq 1}|\phi_X(t)|\mathrm{d}t,$$

where ϕ_X is the characteristic function of X.

In the next two questions $\varepsilon_1, \varepsilon_2, \dots$ are i.i.d. symmetric random signs, $\mathbb{P}\left(\varepsilon_j = \pm 1\right) = \frac{1}{2}$.

16pts **4.** Let $\alpha \in \mathbb{R}$ and set

$$Z_n = \exp\left\{-n\alpha^2/2 + \alpha\sum_{j=1}^n \varepsilon_j\right\}.$$

Show that $(Z_n)_{n\geq 1}$ is a supermartingale. Does it converge a.s.? Does it converge in L_1? If yes, find the limit.

16pts **5.** Let X and Y be independent random variables. Show that if X is continuous, then $X+Y$ is also continuous. Let $0 < \lambda < \frac{1}{2}$. Show that $\sum_{k=1}^\infty \lambda^k\varepsilon_k$ converges a.s. to a random variable X which is *singular*, that is $\mathbb{P}\left(X = a\right) = 0$ for every $a \in \mathbb{R}$ and there is a set S of Lebesgue measure zero with $\mathbb{P}\left(X \in S\right) = 1$. Are there independent random variables U and V such that each is singular, but $U + V$ is continuous?

16pts **6.** Let G be a Gaussian vector in $\mathbb{R}^n$ with $\mathbb{E}G = 0$. Show that $\mathbb{P}\left(\|G\| \geq \sqrt{\mathbb{E}\|G\|^2}\right) > c_0$, for some universal constant $c_0 > 0$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

10pts **7.** Let X be a square integrable continuous random variable with density f on $\mathbb{R}$. Show that

$$\mathrm{Var}(X)\sup_{x\in\mathbb{R}}|f(x)|^2 \geq \frac{1}{12}.$$

Is the constant $\frac{1}{12}$ in this inequality best possible?