4 Feb 2021 **CMU**

Basic examination: Probability

180 min.

16pts **1.** State the following definitions and theorems. Be precise.

a) σ-algebra, probability measure,

b) first and second Borel-Cantelli lemmas,

c) Fatou's lemma, Lebesgue's dominated and monotone convergence theorems,

d) conditional expectation, matringale sequence $(X_n)_{n\geq 0}$.

10pts **2.** State and prove the "vanilla" central limit theorem (i.e. for sums of i.i.d. random variables).

16pts **3.** Show that if $Y_1, Y_2, \ldots$ is a sequence of random variables such that for every $\varepsilon > 0$, we have $\sum_{n=1}^{\infty} \mathbb{P}(|Y_n| > \varepsilon) < \infty$, then Y_n converges to 0 a.s. Let $X_1, X_2, \ldots$ be independent random variables such that each one has mean zero and there is a constant C such that $\mathbb{E}|X_n|^4 \leq C$ for every n. Show that $\frac{1}{n}(X_1 + \cdots + X_n)$ converges to 0 a.s.

16pts **4.** Let $\varepsilon_1, \varepsilon_2, \ldots$ be i.i.d. symmetric random signs, $\mathbb{P}(\varepsilon_j = \pm 1) = \frac{1}{2}$. Let $X_0 = 0, X_n = \varepsilon_1 + \cdots + \varepsilon_n$. Let $\tau = \inf\{n \geq 1, X_n = 1\}$, the first time the symmetric random walk (X_n) starting at 0 visits 1.

a) Show that τ is a stopping time.

b) Fix $\lambda > 0$ and let $M_n = (\cosh \lambda)^{-n} e^{\lambda X_n}$, $n \geq 0$. Show that (M_n) is a martingale.

c) Considering $M_{\tau \wedge n}$ and using Doob's optional sampling lemma, or otherwise, show that $\mathbb{E}(\cosh \lambda)^{-\tau} = e^{-\lambda}$.

d) Deduce that $\mathbb{P}(\tau < \infty) = 1$.

e) Deduce that $\mathbb{P}(\tau = 2k - 1) = (-1)^{k+1}\binom{1/2}{k}$, $k \geq 1$.

16pts **5.** Let G be a Gaussian vector in $\mathbb{R}^n$ with $\mathbb{E}G = 0$. Show that $\mathbb{P}\left(\|G\| \geq \sqrt{\mathbb{E}\|G\|^2}\right) > c_0$, for some universal constant $c_0 > 0$, where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

16pts **6.** Show that if Y is a mean 0 random variable with $|Y| \leq a$, then $\mathbb{E}e^{\lambda Y} \leq e^{\lambda^2 a^2/2}$ for every $\lambda \in \mathbb{R}$ (hint: for a convex function f, $f(y) \leq \frac{a-y}{2a} f(-a) + \frac{a+y}{2a} f(a)$, $|y| \leq a$). Let $(M_n)_{n\geq 0}$ be a martingale with $M_0 = 0$ and $|M_k - M_{k-1}| \leq a_k$ for every $k \geq 1$ for some positive constants $a_1, a_2, \ldots$. Show that for every $n \geq 1$ and $t > 0$,

$$\mathbb{P}\left(\max_{k\leq n} M_k \geq t\right) \leq \exp\left\{-\frac{t^2}{2\sum_{k=1}^n a_k^2}\right\}.$$

10pts **7.** Let $X_1, \ldots, X_n$ be independent symmetric random variables ($-X_j$ has the same distribution as X_j) such that $\mathbb{P}(X_j = 0) = 0$ for every j. Show that for every $t > 0$,

$$\mathbb{P}\left(\frac{X_1 + \cdots + X_n}{\sqrt{X_1^2 + \cdots + X_n^2}} \geq t\right) \leq e^{-t^2/2}.$$