10 Sep 2020 **CMU**

Basic examination: Probability

180 min.

16pts **1.** State the following definitions and theorems. Be precise.

a) π-system, λ-system, Dynkin's lemma, an example of a π-system generating $\mathcal{B}(\mathbb{R}^n)$.

b) independent family of σ-algebras, tail σ-algebra, Kolmogorov's $0-1$ law

c) convergence in probability, a.s., in L_p, in distribution

d) standard Gaussian vector in $\mathbb{R}^n$, arbitrary Gaussian vector in $\mathbb{R}^n$, Lindeberg's central limit theorem (in $\mathbb{R}$)

e) matringale sequence $(X_n)_{n\geq 0}$, Doob's maximal inequality in L_p, $p>1$.

8pts **2.** State and prove the first and second Borel-Cantelli lemmas.

16pts **3.** Let $g_1, g_2, \dots$ be i.i.d. standard Gaussian random variables. Let $(a_{ij})_{i<j}$ be in ℓ_2, that is $\sum_{i<j} a_{ij}^2 < \infty$. Let $X_n = \frac{1}{n}\sum_{i=1}^n g_i^2 + \frac{1}{\sqrt{n}}\sum_{1\leq i<j\leq n} a_{ij} g_i g_j$. Show that the sequence $(X_n)_{n\geq 1}$ converges to 1 a.s.

18pts **4.** Let X be a random variable with characteristic function ϕ_X. Justify the following statements.

a) For every $a \in \mathbb{R}$, $\mathbb{P}(X=a) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{-iat}\phi_X(t)\mathrm{d}t$.

b) If Y is an independent copy of X, then $\mathbb{P}(X=Y) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\phi_X(t)|^2\mathrm{d}t$.

c) $\mathbb{P}(X=Y) = \sum_{x\in\mathbb{R}} \mathbb{P}(X=x)^2$.

d) X has no atoms if and only if $\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\phi_X(t)|^2\mathrm{d}t = 0$.

e) If $\lim_{t\to\infty}\phi_X(t) = 0$, then X has no atoms.

f) If X is continuous, then $\lim_{t\to\infty}\phi_X(t) = 0$.

16pts **5.** Let $X_1, X_2, \dots$ be i.i.d. nonnegative random variables with mean 1 such that $\mathbb{P}(X_1 = 1) < 1$. Show that the sequence $(M_n)_{n\geq 1}$, $M_n = X_1 \cdot \ldots \cdot X_n$ converges a.s. Does it converge in L_1?

16pts **6.** Let $X, X_1, X_2, \dots$ be i.i.d. random variables such that $\mathbb{E}e^{tX} < \infty$ for every $t \in \mathbb{R}$ and $\frac{\mathbb{E}X_1 e^{tX}}{\mathbb{E}e^{tX}} \to \infty$ as $t \to +\infty$. Show that for every $a > \mathbb{E}X$, we have

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(X_1+\dots+X_n \geq an) = -\sup_{t>0}\left(ta - \log\mathbb{E}e^{tX}\right).$$

(Do not invoke Cramér's theorem here. The point of this question is to give a self-contained proof of it under the above simplifying assumptions.)

10pts **7.** Let $z_1, \dots, z_n$ be complex numbers. Prove that there exists a subset I of $\{1,\dots,n\}$ such that

$$\left|\sum_{k\in I} z_k\right| \geq \frac{1}{\pi}\sum_{k=1}^n |z_k|.$$