Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Probability
Spring 2020

Time allowed: 180 minutes.

1. Recite precisely the following definitions/facts/theorems/lemmas:

   (a) Give the definitions of the following convergences: (i) almost surely, (ii) in probability, (iii) in $\mathcal{L}_1$, (iv) weak (= in distribution). Specify all relations between these convergences.

   (b) Let (X_n) be a nonnegative submartingale. Will it converge (i) almost surely, (ii) in probability, (iii) in $\mathcal{L}_1$, (iv) weakly to some (finite) random variable X_∞? If needed, formulate additional (as sharp as possible) conditions on (X_n) that yield these convergences.

   (c) Kolmogorov's three-series theorem on convergence of sums of IRVs.

   (d) Doob's maximal $\mathcal{L}^p$ inequalities, $p > 1$.

   (e) Theorem on equivalence between weak convergence and convergence of characteristic functions.

2. Let (X_n) be IID Gaussian RVs with mean 0 and variance 1 and $h = h(t)$ be some strictly increasing function on $(0, \infty)$. Obtain conditions on $h = (h(t))$ so that

$$\limsup_{n\to\infty} \frac{X_n}{h(n)} = 1, \quad (a.s.).$$

3. Let (M_n) be a strictly positive UI martingale in the form:

$$M_n = \prod_{k=1}^{n} X_k, \quad M_0 = 1,$$

   where (X_n) are IRVs. Find all $p > 0$ such that $\mathbb{E}\left(\max_n M_n^p\right) < \infty$.

4. Let (X_n) be bounded IID RVs with mean $\mu = \mathbb{E}(X_1) \neq 0$ and variance $\sigma^2 = \mathbb{E}((X_1 - \mu)^2) > 0$. Obtain necessary and sufficient conditions on the sequence of real numbers (a_n) that are equivalent to the weak convergence of $\sum_n a_n X_n$.

5. Let (X_n) be Exp IID RVs, that is, their density function has the form:

$$f(t) = e^{-t}, \quad t \geq 0.$$

   Let $S_n = X_1 + \cdots + X_n$ and $Y_n = \mathbb{E}\left(X_n | S_n > \frac{n}{2}\right)$ be the conditional expectation of X_n given the event $\left\{S_n > \frac{n}{2}\right\}$. Will the sequence (Y_n) converge? It yes, then compute the limit.

6. Let (X_n) be non-negative IID RVs. Suppose that

$$\frac{X_1 + \ldots X_n}{n} \to \mu < \infty, \quad n \to \infty, \quad (a.s.).$$

   Can we assert that $\mathbb{E}(X_1) = \mu$?

   Remark. Be careful. We are not given that $\mathbb{E}(X_1) < \infty$.

7. Let $(X_{n,m})$ be IID random variables with values in non-negative integers such that

$$\mu = \mathbb{E}[X_{1,1}] > 1 \quad \text{and} \quad \sigma^2 = \mathbb{E}[(X_{1,1} - \mu)^2] < \infty.$$

   Define random variables (Z_n), recursively, as

$$Z_0 = 1,$$
$$Z_{n+1} = \sum_{m=1}^{Z_n} X_{n+1,m}.$$

   Show that

$$M_n = \frac{Z_n}{\mu^n} \to M_\infty \text{ in } \mathcal{L}_2$$

   and compute the first and second moments of M_∞.

8. Let (X_n) be a symmetric random walk on integers with $X_0 = 0$. Let $a \in \mathbb{Z}_+$. Among all stopping times τ with $\mathbb{E}[\tau] \leq a^2$, find the one that maximizes $\mathbb{E}(|X_\tau|)$.