Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Probability
Fall 2019

Time allowed: 180 minutes.

1. Recite precisely the following definitions/facts/theorems/lemmas:
   (a) π-systems and uniqueness of extensions of σ-finite measures (Dynkin's lemma).
   (b) Convergence theorems (Monotone, Fatou, Dominated, Bounded).
   (c) Doob's Upcrossing Lemma and "Forward" Convergence Theorem for submartingales.
   (d) Doob's maximal $\mathcal{L}^p$ inequalities, $p > 1$.
   (e) Theorem on equivalence between weak convergence and convergence of characteristic functions.

2. Write the proof of Kolmogorov's $0-1$ law using Martingale Convergence Theorem.

3. Let (M_n) be a martingale with $M_0 = 0$ and suppose that
$$|M_n - M_{n+1}| \leq c_n,$$
for some $c_n \geq 0$. Prove that
$$\mathbb{P}(\sup_n M_n \geq x) \leq e^{-\frac{x^2}{2a^2}}, \quad x > 0,$$
where
$$a^2 = \sum_n c_n^2.$$

4. Let (X_n) be IID Gaussian RVs with mean 0 and variance 1 and denote
$$S_n = X_1 + \cdots + X_n.$$
Compute $\mathbb{E}\left(X_n | S_n > 0\right)$, the conditional expectation of X_n given the event $\{S_n > 0\}$.

5. Let (X_n) be Cauchy IID RVs, that is, their density function has the form:

$$f(t) = \frac{1}{\pi(1+t^2)}, \quad t \in \mathbb{R}.$$

Obtain necessary and sufficient conditions on a sequence of real numbers (a_n) such that $\sum_n a_n X_n$ converges a.s.

6. Let (M_n) be a non-negative UI martingale. Is it possible to find a random variable $Y \in \mathcal{L}_1$ such that

$$|M_n| \leq Y, \quad n \geq 1?$$

Answer the same question, assuming that $M_n = \prod_{k=1}^n X_k$ for IRVs (X_n). In both cases, either prove or give a counter-example.

7. Let (X_n) be IID Gaussian RVs with mean 0 and variance 1. Denote

$$Y_n = \frac{1}{n}\sum_{k=1}^n X_k, \quad Y^* = \sup_n |Y_n|.$$

Prove that

$$\mathbb{P}\left(Y^* \geq c\right) \leq e^{-\frac{1}{2}c^2}, \quad c \geq 0.$$

8. Let (X_n) be Exp IID RVs, that is, their density function has the form:

$$f(t) = e^{-t}, \quad t \geq 0.$$

Let $S_n = X_1 + \cdots + X_n$ and

$$\tau = \inf\{n \geq 1 : \ S_n \geq 1\}.$$

For $\lambda, \mu > 0$, compute $\mathbb{E}\left(e^{-\lambda\tau + (1-\mu)S_\tau}\right)$.