Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Probability
Fall 2018

Time allowed: 180 minutes. Justify your answers.

1. Recite precisely the following definitions/facts/theorems/lemmas:
   (a) Dynkin's lemma on π-systems.
   (b) First and second Borel-Cantelli lemmas.
   (c) Kolmogorov's three-series theorem on convergence of sums of IRVs.
   (d) Let (X_n) be a non-negative submartingale. Will it converge (i) almost surely, (ii) in probability, (iii) in $\mathcal{L}_1$, (iv) weakly to some random variable X_∞? If needed, formulate additional (as sharp as possible) conditions on (X_n) under which these convergences take place.

2. Let $X_1, \dots, X_N$ be IID Gaussian RVs with mean 0 and variance 1 and denote $S_N = \sum_{1 \le n \le N} X_n$ and $Y = I(S_N > 0)$. Compute the function $f = f(x)$ such that

$$f(Y) = \mathbb{E}\left(X_1 \middle| Y \right).$$

3. Let (X_n) be IID bounded (non-constant) RVs and suppose that a sequence of real numbers (a_n) is chosen so that the series $\sum_n a_n X_n$ converges almost surely.
   (a) Will the series $\sum_n a_n X_n$ converge in $\mathcal{L}_1$?
   (b) Is it true that $\sum_n |a_n| < \infty$?.

4. Let (X_n) be a martingale bounded in $\mathcal{L}_2$ and $(\mathcal{G}_n)$ be the filtration generated by the absolute values of (X_n):

$$\mathcal{G}_n = \sigma(|X_1|, |X_2|, \dots, |X_n|), \quad n \ge 1.$$

   Will the sequence $Y_n = \mathbb{E}\left(X_n \middle| \mathcal{G}_n \right)$, $n \ge 1$, converge (a) a.s.? (b) in $\mathcal{L}_2$?

5. Let (X_n) be IID RVs with uniform distribution on $(-1, 1)$. Denote

$$S_n = X_1 + \cdots + X_n, \quad S_0 = 0.$$

Show that there is a constant $a > 0$ such that

$$\limsup_n \mathbb{P}\left(\max_{k\leq n} S_k \geq c\sqrt{n}\right) \leq e^{-ac^2}, \quad \forall c > 0.$$

Write your best possible estimate for a.

6. Let (X_n) be IID RVs with the distribution function

$$\mathbb{P}\left(X_1 \leq x\right) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{dy}{1+y^2}, \quad x \in \mathbb{R}.$$

There are a constant $p > 0$ and a random variable $Y \neq 0$ such that the sequence

$$Y_n \triangleq \frac{1}{n^p}\sum_{1\leq k\leq n} X_k, \quad n \geq 1,$$

converges weakly to Y. Compute p and the distribution function of Y.

7. Let (X_n) be a symmetric random walk on integers starting at 0. For an integer $a > 0$ define the hitting time

$$\tau = \inf\{n \geq 0 : \ X_n = a\}.$$

Compute the Laplace transform $L(\lambda) = \mathbb{E}\left(e^{-\lambda\tau}\right)$, $\lambda \geq 0$ and the mean $\mathbb{E}\left(\tau\right)$.

8. Let (X_n) be IID non-negative RVs, each having density $\phi = \phi(x)$ with respect to the Lebesgue measure and expectation

$$\mathbb{E}\left(X_1\right) = \int_0^\infty x\phi(x)dx = 1.$$

Let $Y_0 = 1$ and $Y_n = \prod_{k=1}^n X_k$, $n \geq 1$.

   (a) Obtain a (deterministic) integral equation, which solution yields

   $$a(\phi) = \mathbb{P}\left(\max_{n\geq 0} Y_n \geq 2\right).$$

   (b) Write your best estimate for $a^* = \sup_\phi a(\phi)$.