1. Recite precisely the following definitions/facts/theorems/lemmas:
   (a) Dynkin’s lemma on $\pi$-systems.
   (b) First and second Borel-Cantelli lemmas.
   (c) Kolmogorov’s three-series theorem on convergence of sums of IRVs.
   (d) Let $(X_n)$ be a non-negative submartingale. Will it converge (i) almost surely, (ii) in probability, (iii) in $L_1$, (iv) weakly to some random variable $X_\infty$? If needed, formulate additional (as sharp as possible) conditions on $(X_n)$ under which these convergences take place.

2. Let $X_1, \ldots, X_N$ be IID Gaussian RVs with mean 0 and variance 1 and denote $S_N = \sum_{1\leq n\leq N} X_n$ and $Y = I(S_N > 0)$. Compute the function $f = f(x)$ such that $f(Y) = \mathbb{E}(X_1|Y)$.

3. Let $(X_n)$ be IID bounded (non-constant) RVs and suppose that a sequence of real numbers $(a_n)$ is chosen so that the series $\sum_n a_n X_n$ converges almost surely.
   (a) Will the series $\sum_n a_n X_n$ converge in $L_1$?
   (b) Is it true that $\sum_n |a_n| < \infty$?

4. Let $(X_n)$ be a martingale bounded in $L_2$ and $(G_n)$ be the filtration generated by the absolute values of $(X_n)$:
   $$ G_n = \sigma(|X_1|, |X_2|, \ldots, |X_n|), \quad n \geq 1. $$
   Will the sequence $Y_n = \mathbb{E}(X_n|G_n)$, $n \geq 1$, converge (a) a.s.? (b) in $L_2$?
5. Let \((X_n)\) be IID RVs with uniform distribution on \((-1, 1)\). Denote 
\[ S_n = X_1 + \cdots + X_n, \quad S_0 = 0. \]
Show that there is a constant \(a > 0\) such that
\[ \limsup_n \mathbb{P} \left( \max_{k \leq n} S_k \geq c\sqrt{n} \right) \leq e^{-ac^2}, \quad \forall c > 0. \]
Write your best possible estimate for \(a\).

6. Let \((X_n)\) be IID RVs with the distribution function
\[ \mathbb{P}(X_1 \leq x) = \frac{1}{\pi} \int_{-\infty}^{x} \frac{dy}{1 + y^2}, \quad x \in \mathbb{R}. \]
There are a constant \(p > 0\) and a random variable \(Y \neq 0\) such that the sequence
\[ Y_n \triangleq \frac{1}{n^p} \sum_{1 \leq k \leq n} X_k, \quad n \geq 1, \]
converges weakly to \(Y\). Compute \(p\) and the distribution function of \(Y\).

7. Let \((X_n)\) be a symmetric random walk on integers starting at 0. For an integer \(a > 0\) define the hitting time
\[ \tau = \inf \{ n \geq 0 : X_n = a \}. \]
Compute the Laplace transform \(L(\lambda) = \mathbb{E} (e^{-\lambda \tau}), \lambda \geq 0\) and the mean \(\mathbb{E} (\tau)\).

8. Let \((X_n)\) be IID non-negative RVs, each having density \(\phi = \phi(x)\) with respect to the Lebesgue measure and expectation
\[ \mathbb{E} (X_1) = \int_{0}^{\infty} x\phi(x)dx = 1. \]
Let \(Y_0 = 1\) and \(Y_n = \prod_{k=1}^{n} X_k, \ n \geq 1.\)
   (a) Obtain a (deterministic) integral equation, which solution yields
   \[ a(\phi) = \mathbb{P} \left( \max_{n \geq 0} Y_n \geq 2 \right). \]
   (b) Write your best estimate for \(a^* = \sup_{\phi} a(\phi)\).