Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Probability
Spring 2018

Time allowed: 180 minutes.

1. Recite precisely the following definitions/facts/theorems/lemmas:
   (a) Tail σ-algebra. Kolmogorov's $0-1$ law.
   (b) Kolmogorov's three-series theorem on convergence of sums of IRVs.
   (c) Doob's maximal $\mathcal{L}_p$ inequalities for martingales.
   (d) Method of characteristic functions in weak convergence.

2. Let (X_n) be IID RVs with uniform distribution on $[0,1]$. For each of the items below describe all sequence of real numbers (a_n) such that $\sum_n a_n X_n$ converges (i) almost surely, (ii) weakly, (iii) in $\mathcal{L}_1$, (iv) in $\mathcal{L}_2$.

3. Let (M_n) be a martingale bounded in $\mathcal{L}_2$, that is, $\sup_n \mathbb{E}\left[M_n^2\right] < \infty$. Is it possible to find a random variable $Y \in \mathcal{L}_2$ such that

$$|M_n| \leq Y, \quad n \geq 1?$$

   Justify your answer. Answer same question but with $\mathcal{L}_2$ replaced by $\mathcal{L}_1$.

4. Let (X_n) be independent non-negative random variables, each of mean 1 and set $Y_n \triangleq \prod_{k=1}^n X_k, \quad n \geq 1$. Will the sequence (Y_n) cross the barrier $a > 1$ with probability one? If not, then what is the maximal possible value for such probability? Justify the answer. In particular, construct an explicit example where the maximal value is attained.

5. Let (X_n) be IID Gaussian RVs with mean 0 and variance 1. Denote

$$S_n = X_1 + \cdots + X_n, \quad S_0 = 0.$$

   Prove that

$$\mathbb{P}\left[\max_{k \leq n} S_k \geq c\right] \leq e^{-\frac{c^2}{2n}}, \quad \forall c > 0.$$

6. Let (X_n) be sequence of RVs such that $X_n \to X$ (a.s) and $|X_n| \leq Y \in \mathcal{L}_1$. Let $(\mathcal{F}_n)$ be a filtration. Show that $\mathbb{E}\left(X_n|\mathcal{F}_n\right) \to \mathbb{E}\left(X|\mathcal{F}_\infty\right)$ (a.s) and in $\mathcal{L}_1$.

7. Let (X_n) be a random walk on integers starting at 0 with the probability $0 < p < 1$ to go up and the probability $q = 1 - p > p$ to go down. For a positive integer a define the stopping time

$$\tau = \inf\{n \geq 0 : \; X_n \geq a\}.$$

   Compute $\mathbb{P}(\tau < \infty)$.

8. A gambler throws dice until she gets N sixes in a row. The cost of one throw is q and the reward at the end is A. Find the relationship between q, A and N for the game to *fair*, that is, for the expected payoff to be 0.