Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Probability
Fall 2017

Time allowed: 180 minutes.

1. Recite precisely the following definitions/facts/theorems/lemmas:
   (a) Tail σ-algebra. Kolmogorov's $0-1$ law.
   (b) Kolmogorov's three-series theorem on convergence of sums of IRVs.
   (c) Doob's maximal $\mathcal{L}_p$ inequalities for martingales.
   (d) Method of characteristic functions in weak convergence.

2. Let (M_n) be a martingale bounded in $\mathcal{L}_1$, that is, $\sup_n \mathbb{E}\left[|M_n|\right] < \infty$.
   (a) Will (M_n) converge in distribution?
   (b) Will $\sup_n |M_n| < \infty$ (a.s)?
   (c) Will (M_n) be uniformly integrable?
   (d) Is it possible to find a random variable M_∞ such that $M_n = \mathbb{E}\left[M_\infty \,|\, \mathcal{F}_n\right]$?

   Justify your answers.

3. Let (X_n) be IRVs with zero mean such that $|X_n| \leq K$ for a positive constant K. Let
$$S_n = X_1 + \cdots + X_n, \quad S_0 = 0,$$
   and suppose that (S_n) converges (a.s.) to a RV Y.
   (a) Can we assert that $S_n = \mathbb{E}\left[Y \,|\, X_1, \ldots, X_n\right]$?
   (b) Is it true that $\mathbb{E}\left[\sup_n S_n^2\right] < \infty$?

   Justify your answers.

4. Let (X_n) be IID Gaussian RVs with mean 0 and variance 1. Denote

$$S_n = X_1 + \cdots + X_n, \quad S_0 = 0.$$

Prove that

$$\mathbb{P}\left[\max_{k \le n} S_k \ge c\right] \le e^{-\frac{c^2}{2n}}, \quad \forall c > 0.$$

5. Let (X_n) be IID RVs with uniform distribution on $[0,1]$. We denote

$$M_n = 2^n \prod_{k \le n} X_k, \quad n \ge 1.$$

Show that there is a RV Y such that $M_n \to Y$ (a.s.) and compute Y.

6. Let (X_n) be IID RVs with uniform distribution on $[0,1]$. Compute the characteristic function $\phi_n = \phi_n(\theta)$ of

$$Y_n = \mathbb{E}\left[X_1 \middle| X_1 + \cdots + X_n\right].$$

Show that $\phi_n(\theta) \to \phi(\theta)$, $\theta \in \mathbf{R}$, and compute $\phi = \phi(\theta)$.

7. A gambler throws dice until he gets N identical outcomes in a row. The cost of one throw is q and the reward at the end is A. Find the relationship between q, A and N for the game to be *fair*, that is, for the expected payoff to be 0.

8. Let (X_n) be IID RVs with uniform distribution on $[0,1]$. Denote

$$S_n = X_1 + \cdots + X_n, \quad S_0 = 0,$$

and

$$\tau = \min\{n \ge 0 : \ S_n > 1\}.$$

Compute $\mathbb{E}[\tau]$ and $\mathbb{E}[S_\tau]$.