Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Probability
Spring 2017

Time allowed: 180 minutes.

1. Recite precisely the following definitions/facts/theorems/lemmas:
   (a) First and second Borel-Cantelli lemmas.
   (b) Tail σ-algebra. Kolmogorov's $0-1$ law.
   (c) Give the definitions of the following convergences: (i) almost surely, (ii) in probability, (iii) in $\mathcal{L}_1$, (iv) weak (= in distribution). Specify all relations between these convergences.
   (d) Let (X_n) be a nonnegative martingale. Will it converge (i) almost surely, (ii) in probability, (iii) in $\mathcal{L}_1$, (iv) weakly to some random variable X_∞? If needed, formulate additional (ideally, necessary and sufficient) conditions on (X_n) under which these convergences take place.
   (e) Method of characteristic functions in weak convergence.

2. At time 0, an urn contains 1 black ball and 1 white ball. At each time $1, 2, 3, \dots,$ a ball is chosen at random from the urn and is replaced together with a new ball of the same color. Just after time n, there are therefore $n+2$ balls in the urn, of which $B_n + 1$ are black, where B_n is the number of black balls chosen by time n.

   Let $M_n = (B_n+1)/(n+2)$, the proportion of black balls in the urn just after time n. Prove that (M_n) converges a.s. to a RV Θ and find the distribution of Θ.

3. Let (X_n) be positive IID RVs with same continuous distribution function and denote $M_n = \max_{0\le k\le n} X_k$. Will the series
$$\sum_n X_n 1_{\{X_n = M_n\}}$$
   converge?

4. Let (X^n) be IRVs such that $|X^n| \leq 1$ and $\sum_n X^n$ converges in distribution. Will this series also converge almost surely?

5. Let (M_n) be a martingale with $M_0 = 0$ and suppose that

$$|M_n - M_{n+1}| \leq c_n,$$

for some $c_n \geq 0$. Prove that

$$\mathbb{P}(\sup_n M_n \geq x) \leq e^{-\frac{x^2}{2a^2}}, \quad x > 0,$$

where

$$a^2 = \sum_n c_n^2.$$

6. Let (Y_n) are IID Bernoulli's RVs taking values 1 and -1 with probability $1/2$. For an integer $n \geq 1$ define

$$X_n \triangleq Y_0 Y_1 \dots Y_n.$$

What is the minimal value of $\mathbb{P}(\{Y_0 = 1\} \cap A)$, where we are minimizing over the events A such that

$$A \in \cap_{n \geq 1} \sigma(X_n, X_{n+1}, \dots) \quad \text{and} \quad \mathbb{P}(A) \geq 1/8.$$

7. Let X and Y be IID RVs such that $X + \frac{1}{2}Y$ has uniform distribution on $[-1, 1]$. Compute the characteristic function $\phi = \phi(\theta)$ of X. Will the integral

$$I \triangleq \int_{\mathbb{R}} |\phi(\theta)| d\theta$$

be finite?

8. Let (X_n) be a symmetric random walk on integers starting at 0. Among all stopping times τ find the one that maximizes $\mathbb{E}[|X_\tau| e^{-\lambda \tau}]$, where $\lambda > 0$. The answer may not be too explicit. Do the best you can.