Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Probability
Fall 2016

Time allowed: 180 minutes.

1. Recite precisely the following definitions/facts/theorems/lemmas:

   (a) Give the definitions of the following convergences: (i) almost surely, (ii) in probability, (iii) in $\mathcal{L}_1$, (iv) weak (= in distribution). Specify all relations between these convergences.

   (b) Doob's maximal $\mathcal{L}_p$ inequalities for martingales.

   (c) Let (X_n) be a non-negative supermartingale. Will it converge (i) almost surely, (ii) in probability, (iii) in $\mathcal{L}_1$, (iv) weakly to some random variable X_∞? If needed, formulate additional (ideally, necessary and sufficient) conditions on (X_n) under which these convergences take place.

   (d) Method of characteristic functions in weak convergence.

2. Let (X_n) be IID RVs such that

$$\mathbb{E}\left[X_1\right] = 0, \quad 0 < \mathbb{E}\left[X_1^2\right] < \infty.$$

   For each of the items (i)–(iv) describe all sequences of real numbers (a_n) such that $\sum_n a_n X_n$ converges (i) almost surely, (ii) in probability, (iii) in $\mathcal{L}_1$, (iv) in $\mathcal{L}_2$.

3. Let (X_n) be IID RVs in $\mathcal{L}_1$ such that $X_1 \geq 0$, $\mathbb{E}\left[X_1\right] = 1$ and $X_1 \neq 1$. Set

$$Y_n \triangleq \prod_{k=1}^{n} X_k.$$

   Can we find a filtration $(\mathcal{F}_n)$ and a random variable Y_∞ such that $Y_n = \mathbb{E}[Y_\infty | \mathcal{F}_n]$?

4. Let (X_n) be IID RVs in $\mathcal{L}_2$ with common zero mean. Denote

$$Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k, \quad Y^* = \sup_n |Y_n|.$$

Show that for $p > 1$ there is a constant $c = c(p)$ (dependent only on p) such that

$$\mathbb{E}\left[(Y^*)^p\right] \le c(p)\mathbb{E}\left[|X_1|^p\right].$$

5. A dice is thrown repeatedly until the sequence "6,6, ...,6" of length N appears. Compute the expected value of the sum of outcomes.

6. Let (X_n) be a symmetric random walk on integers starting at 0 and let $T > 0$. Among all stopping times τ with $\mathbb{E}[\tau] \le a^2$, where $a \in \mathbb{Z}_+$, find the one that maximizes $\mathbb{E}[|X_\tau|]$.