Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Probability
Spring 2016

Time allowed: 120 minutes.

1. Let (X_n) be IID random variables taking values in $[-1,1]$ and having the common mean $\mu = \mathbb{E}[X_n] = 0$ and the variance $\sigma^2 = \mathbb{E}[X_n^2] > 0$. Let (a_n) be a sequence in $(-1,1)$. Define
$$Y_n \triangleq \prod_{k=1}^{n}(1 + a_k X_k), \quad n \geq 1.$$
Will (Y_n) converge (i) almost surely, (ii) in probability, (iii) in $\mathcal{L}_1$, (iv) weakly to some random variable Y_∞? Will $Y_n = \mathbb{E}[Y_\infty | \sigma(X_1, \ldots, X_n)]$? If needed, formulate additional (ideally, necessary and sufficient) conditions on the sequence (a_n) under which the answers are affirmative.

2. Let X and Y be random variables in $\mathcal{L}_2(\Omega, \mathcal{F}, \mathbb{P})$ such that
$$\mathbb{E}[X] = \mathbb{E}[Y] = \mathbb{E}[XY] = 0,$$
$$\mathbb{E}\left[X^2\right] = \mathbb{E}\left[Y^2\right] = 1,$$
and $\mathcal{A}$ be a sub-σ-algebra of $\mathcal{F}$.

   (a) Show that
$$\mathbb{E}[\mathbb{E}[X | \mathcal{A}]\, \mathbb{E}[Y | \mathcal{A}]] \leq \frac{1}{2}.$$

   (b) Assume in addition that X and Y are IID RVs. Find a sub-σ-algebra $\mathcal{A}$ of $\mathcal{F}$ such that
$$\mathbb{E}[\mathbb{E}[X | \mathcal{A}]\, \mathbb{E}[Y | \mathcal{A}]] = \frac{1}{2}.$$

3. Let (X_n) be IID RVs in $\mathcal{L}_1$ and denote by $\mu = \mathbb{E}[X_1]$ their common first moment. Let τ be a stopping time with $\mathbb{E}[\tau] < \infty$ and $S_n = \sum_{k=1}^n X_k$. Show that $S_\tau \in \mathcal{L}_1$ and compute $\mathbb{E}[S_\tau]$.

4. Let (X_n) be IID RVs in $\mathcal{L}_2$ with common zero mean. Denote

$$Y_n = \frac{1}{n}\sum_{k=1}^{n} X_k, \quad Y^* = \sup_n Y_n.$$

Show that

$$\mathbb{E}\left[(Y^*)^2\right] \leq 4\mathbb{E}\left[X_1^2\right].$$

5. A dice is thrown repeatedly. Compute the expected time of getting the sequence "6,6, ... ,6" of length N.