Department of Mathematical Sciences
Carnegie Mellon University

Basic Examination
Probability
Spring 2015

Time allowed: 120 minutes.

1. Let (Y_n) be IID RVs taking values 1 and -1 with equal probabilities. Compute the function
$$f(\mathbf{x}) = \mathbb{P}[\sum_n x_n Y_n \text{ converges}]$$
defined on sequences $\mathbf{x} = (x_n)$ of real numbers.

2. Let (X_n) be independent RVs. Establish the relationships between the following statements:
   (a) $\sum_n X_n$ converges in distribution;
   (b) $\sum_n X_n$ converges in probability;
   (c) $\sum_n X_n$ converges almost surely.

3. Let $X_0, X_1, \ldots, X_n$ be random variables having a joint normal distribution. Assume that $\mathbb{E}[X_j] = 0$ and denote $\Gamma_{ij} = \mathbb{E}[X_i X_j]$. Compute $\mathbb{E}[X_0 | X_1, \ldots, X_n]$.

4. Let (X_n) be IID RVs with zero mean, $S_n = \sum_{k=1}^n X_k$, and
$$\tau = \inf\{n : \; S_n > 0\}.$$
Will τ have a finite first moment?

5. Let (X_n) be a random walk on integers starting at 0 with the probability p to go up and the probability q to go down; $p > q$. For an integer $a > 0$ define the hitting time
$$\tau = \min\{n \geq 0 : \; X_n = a\}.$$
Compute $\mathbb{E}[\tau^2]$.