Department of Mathematical Sciences
Carnegie Mellon University

# Basic Examination
# Probability
# Fall 2014

**Time allowed: 120 minutes.**

1. Let $(X_n)$ be independent non-negative random variables, each of mean 1. Define
$$Y_n \triangleq \prod_{k=1}^{n} X_k, \quad n \geq 1.$$
Will $(Y_n)$ converge (i) almost surely, (ii) in probability, (iii) in $\mathcal{L}_1$, (iv) weakly to some random variable $Y_\infty$? If needed, formulate additional (ideally, necessary and sufficient) conditions on $(X_n)$ under which these convergences take place.

2. The events $A$, $B$, and $C$ are pairwise independent and cannot occur simultaneously. Assuming that these events have same probability $p$, compute the probability that at least one of them occurs. Determine the largest possible value of $p$.

3. Let $(X_n)$ be IID RVs each with $N(0,1)$ (standard normal) distribution. Let $a$ and $b$ be real numbers. Denote
$$S_n \triangleq \sum_{k=1}^{n} X_k, \quad Y_n \triangleq \exp(aS_n - bn), \quad n \geq 1.$$
Determine the values of $a$ and $b$ for which
   (a) $(Y_n)$ converges to 0 almost surely;
   (b) $(Y_n)$ converges to 0 in $\mathcal{L}^p$ for $p \geq 1$.

4. Let $(X_n)$ be IID random variables such that $\mathbb{P}[X_n = -1] = \mathbb{P}[X_n = 1] = 1/2$. For $p \geq 0$ set
$$Y_n(p) \triangleq \sum_{k=1}^{n} \frac{X_k}{k^p}, \quad n \geq 1.$$
Determine the values of $p$ when there is a filtration $(\mathcal{F}_n)$ and a random variable $Y_\infty(p)$ such that $Y_n(p) = \mathbb{E}[Y_\infty(p)|\mathcal{F}_n]$.

5. Consider a random walk $(X_n)$ on the integers starting at 0 with the probability $0 < p < 1$ to go up and the probability $q = 1 - p > p$ to go down. For the random variable $X^* = \sup_{n \geq 0} X_n$, compute the distribution function and the mean.