Probabilistic Combinatorics (Fall 2024)

Monday, September 9, 2024 (in-person exam)

Solve any four of the five problems below.

1. Prove that there is an absolute constant $C > 0$ so that for every $n \times n$ matrix with distinct real entries, one can permute its rows so that no column in the permuted matrix contains an increasing subsequence of length at least $C\sqrt{n}$.

   Sketch. Randomly permute and compute expected number of increasing subsequences of length $C\sqrt{n}$. Use Markov and apply the union bound. $\square$

2. Let $f(n,p)$ be the minimum number of colors needed to edge-color K_n so that no K_p receives fewer than p colors. Show that $f(n,4) = O(n^{2/3})$.

   Sketch. Local lemma. $\square$

3. For $1 \le i \le n$, let $v_i = (x_i, y_i)$ be n two-dimensional vectors, where each x_i and y_i is an integer whose absolute value does not exceed $2^{n/2}/(100\sqrt{n})$. Show that there are two disjoint sets $I, J \subset \{1, 2, \ldots, n\}$ such that
$$\sum_{i \in I} v_i = \sum_{j \in J} v_j \, .$$

   Sketch. Choose random sums: $X = \sum_{i=1}^{n} \epsilon_i x_i$ and $Y = \sum_{i=1}^{n} \epsilon_i y_i$, with ϵ_i equal to 0 or 1, with equal probability and independently over i. Use Cheyshev's inequality.

4. Let $\mathcal{F}$ be a collection of subsets of $[n]$ with the following two properties:
   - (intersecting) $A, B \in \mathcal{F}$ implies $A \cap B \neq \phi$, and
   - $A, B \in \mathcal{F}$ implies $A \cup B \neq [n]$.

   i) Show that $|\mathcal{F}| \le 2^{n-2}$.
   ii) Is there an example family showing tightness in the inequality? If so, provide one.

   Sketch. FKG. $\square$

5. We have a collection of n bins. A sequence of $2n$ balls arrive one at a time. When ball i arrives a bin A_i is chosen uniformly at random. If A_i is **not** an empty bin then the i^{th} ball is placed in A_i. If bin A_i is empty then a second bin B_i is chosen at random and the i^{th} ball is placed in that bin (regardless of the number balls already in the bin).

   Let Y be the number of empty bins after all $2n$ balls have been placed in binds. Find a real number α such that $Y = (\alpha \pm o(1))n$ with high probability. (You do not need to prove concentration; simply find the number α.)

   Sketch. Differential equations method. Note that the expected one-step change in Y is $-(Y/n)^2$. Then introduce $y(t)$ that gives the trajectory. This satisfies $dy/dt = -y^2$ and $y(0) = 1$. So $y(t) = 1/(1+t)$. $\square$